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AND CONNECTIONS

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STRUCTURAL MEMBERS AND CONNECTIONS

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PREFACE

This volume is one of a series designed to provide the engineer and the student with a reference work covering thoroughly the design and construction of the principal kinds and types of modern civil engineering structures. An effort has been made to give such a complete treatment of the elementary theory that the books may also be used for home study.

The titles of the six volumes comprising this series are as follows:

Foundations, Abutments and Footings
Structural Members and Connections
Stresses in Framed Structures
Steel and Timber Structures
Reinforced Concrete and Masonry Structures
Movable and Long-span Steel Bridges

Each volume is a unit in itself, as references are not made from one volume to another by section and article numbers. This arrangement allows the use of any one of the volumes as a text in schools and colleges without the use of any of the other volumes.

Data and details have been collected from many sources and credit is given in the body of the books for all material so obtained. A few chapters, however, throughout the six volumes have been taken without special mention, and with but few changes, from Hool and Johnson's Handbook of Building Construction.

The Editors-in-Chief wish to express their appreciation of the spirit of cooperation shown by the Associate Editors and the Publishers. This spirit of cooperation has made the task of the Editors-in-Chief one of pleasure and satisfaction.

G. A. H.
W. S. K.

MADISON, WIS.
September, 1923.

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STRUCTURAL MEMBERS
AND
CONNECTIONS

SECTION 1

GENERAL THEORY

DEFINITIONS

1. Structure.—A *structure* is a part, or an assemblage of parts, constructed to support certain definite loads. Structures are acted upon by external forces and these external forces are held in equilibrium by internal forces, called *stresses*.

2. Member.—A *member* or piece of structure is single unit of the structure, as a beam, a column, or a web member of a truss.

3. Beam.—A *beam* is a structural member which is ordinarily subject to bending and is usually a horizontal member carrying vertical loads. In a framed floor, beams are members upon which rest directly the floor plank, slab, or arch.

A *simple beam* is one which rests on supports at the ends. A *cantilever beam* is a beam having one end rigidly fixed and the other end free. Extending a simple beam beyond either support gives a combination of a simple beam and a cantilever beam. A beam with both ends free and balanced over a support is also called a cantilever beam. A *restrained beam* is one which is more or less fixed at one or both points of support. A *built-in* or *fixed beam* is a beam rigidly fixed at both ends. A *continuous beam* is one having more than two points of support.

4. Girder.—A *girder* is a beam which receives its load in concentrations. In a framed floor it supports one or more cross beams which in turn carry the flooring. The term "girder" is also applied to any large heavy beam, especially a built-up steel beam or plate girder. In Bethlehem steel sections the terms "beam" and "girder" are used to denote rolled sections of different proportions.

5. Column.—A *column*, *strut* or *post* is a structural member which is compressed endwise. A strut is usually considered of smaller dimensions than either a column or post.

6. Tie.—A *tie* is a structural member which tends to lengthen under stress.

7. Truss.—A *truss* is a framed or jointed structure. It is composed of straight members which are connected only at their intersections, so that if the loads are applied at these intersections the stress in each member is in the direction of its length. Each member of a truss is either a tie or a strut.

8. Force.—*Force* is that which tends to change the state of motion of a body, or it is that which causes a body to change its shape if it is held in place by other forces.

9. Outer Forces.—The *external* or *outer forces* acting upon a structure consist of the applied loads and the supporting forces, called *reactions*.

10. Inner Forces.—The *internal* or *inner forces* in a structure are the stresses in the different members which are brought into action by the outer forces and hold the outer forces in equilibrium.

11. Dead Load.—*Dead load* is the weight of a structure itself plus any permanent loads. In design, the weight of the structure must be assumed; and the design corrected later if the assumed weight is very much in error.

12. Live Load.—*Live load* is any moving or variable load which may come upon the structure—as, for example, the weight of people or merchandise on a floor, or the weight of snow and the pressure of wind on a roof. The *total load* or dead load plus live load must be used in design. In addition the dynamic effect or impact of the live load must often be considered.

13. Statical Moment of an Area.[1]—The statical moment of an area about a given axis is the moment of each element of this area about the given axis.

14. Center of Gravity of an Area.[1]—The center of gravity of an area is the point at which the entire area must be concentrated, in order that the product of the area times the distance from this point to a given axis may be equal to the statical moment of the area about the given axis. (The statical moment of an area is zero for an axis through its center.)

15. Moment of Inertia.[1]—The moment of inertia of an area with respect to any axis is the sum of the products formed by multiplying each element of the area by the square of its distance from the given axis.

16. Moments of Inertia for Parallel Axes.[1]—The moment of inertia of an area with respect to any axis equals the moment of inertia with respect to parallel axes through the center of gravity plus the product of the area and the square of the distance between the axes.

17. Principal Axes and Principal Moments of Inertia.[1]—Through the center of gravity of a cross section there is always a pair of axes about one of which the moment of inertia is a maximum and about the other a minimum. These moments of inertia are called *principal moments of inertia*, and the axes about which they are taken are called *principal axes*. The maximum and minimum values of moments of inertia occur for axes which are 90 deg. apart.

18. Radius of Gyration.[1]—The radius of gyration of an area is the distance from a given axis to a point at which the entire area of the section must be applied, in order that the product of the area times the square of this distance to the given axis may be equal to the moment of inertia of the section about the given axis.

[1] See also Appendix *B*.

STRESS AND DEFORMATION

BY C. A. WILLSON

19. Stress.—If we consider a body subjected to the action of external forces to be cut by a plane section, an internal force will be transmitted across this section which will tend to hold the body in equilibrium. This force is called *stress*, and the material of which the body is composed is said to be stressed. The stress may be uniformly distributed over the area of the section or the stress per unit of area may vary in several different ways, depending upon the arrangement of the external forces acting upon the body. The stress per unit of area is called the *unit stress* or the *intensity of stress.*

Let P represent the value of a force acting upon a rod of cross-section A, and let p represent the value of the unit stress. Then the *total stress* on a section will be equal to the value of the force or P. If the stress is uniformly distributed, the *unit stress* may be found readily by dividing the total stress by the area of cross-section, or

$$p = \frac{P}{A}$$

With P expressed in pounds and A expressed in square inches, the resulting value of p will be expressed in pounds per square inch.

20. Deformation or Strain.—When a body is stressed under the action of external forces, an accompanying change of shape occurs. This change of shape is called *deformation* or *strain*, and the body is said to be deformed or strained.

If a body originally of length l is elongated to a length $l + \delta l$ when acted upon by an external force, its total deformation is δl and its deformation per unit of length, or its *unit deformation*, is

$$\frac{\delta l}{l} = \delta$$

With δl expressed in inches or a fractional part thereof and l expressed in inches, the resulting value of δ will be expressed as a fractional part of an inch per inch.

21. Elastic Limit.—The *elastic limit* is the unit stress at the limit of proportionality of stress and deformation. From zero load up to the elastic limit of the material, equal increments of stress produce equal increments of deformation. This relation is known as *Hooke's Law.* Above the elastic limit the ratio of deformation to stress is greater than that below the elastic limit and is increasing in value instead of remaining constant. For any stress below the elastic limit the material will return to its original form and dimensions upon removal of the load. When stressed above the elastic limit, the material will not fully recover its shape upon removal of the load but a permanent change or *set* will be produced.

22. Modulus of Elasticity.—The *modulus of elasticity* is found by dividing any unit stress below the elastic limit by the deformation corresponding to that point. Let E equal the modulus of elasticity, f the unit stress, and δ the unit deformation. Then

$$E = \frac{f}{\delta}$$

The modulus of elasticity is often called *Young's modulus.* The moduli of elasticity in tension and compression are practically the same for all elastic materials.

23. Kinds of Stress.—Any one of several different kinds of stress, or a combination of two or more kinds of stress, may be produced in a body, depending upon the arrangement of the external forces. These stresses are as follows: Tension, compression, flexure, shear, and torsion. Tension and compression are called direct stresses because they act perpendicularly to the section under consideration. Our knowledge of the mechanical properties of the materials of construction is derived principally from tension tests. Compression as it is considered in this chapter is limited to its application to short prisms.

24. Direct Stress.—When a body is subjected to the action of a tensile force, it is elongated in the direction of the force and its cross-section in a plane perpendicular thereto is reduced. Conversely, when a body is subjected to the action of a compressive force, it is shortened in the direction of the force and its cross-section increased. The ratio of the change in length of each lateral dimension to that of the longitudinal dimension is called *Poisson's ratio* and may be denoted by the Greek letter λ. The value of Poisson's ratio for concrete varies from $\frac{1}{12}$ to $\frac{2}{12}$; for steel, glass and stone it is about $\frac{3}{12}$; for copper and brass it is about $\frac{4}{12}$; and for lead it is about $\frac{5}{12}$.

25. Stress—Deformation Diagrams.—As the load is being applied in a tensile or compression test, simultaneous readings of load and deformation are

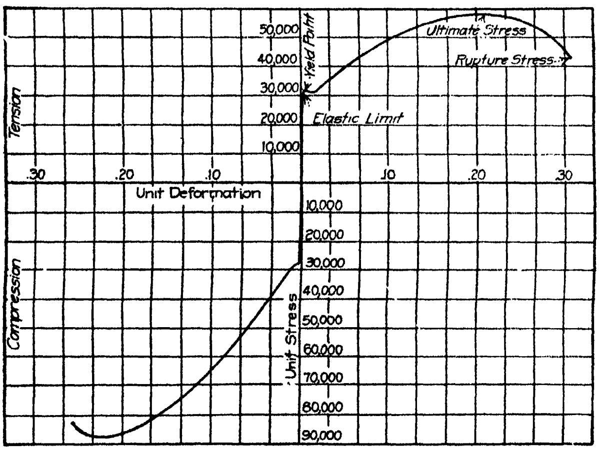


Fig. 1.—Stress-deformation diagram for steel.

made and recorded. The loads are reduced to unit stresses by dividing by the area of cross-section of the specimen, and the total deformations are reduced to unit deformations by dividing by the length over which the deformations are being measured. If the unit stresses are represented by vertical ordinates and the unit deformations are represented by horizontal abscissae, and if points are plotted which correspond to the readings taken, a curve drawn through these points will be a graphical representation of the action of the specimen under test. Such a diagram is called a *stress-deformation diagram* in which the ordinates

represent unit stresses and the abscissae represent the corresponding unit deformations. Typical stress-deformation diagrams for steel, cast iron, wood and concrete are shown in Figs. 1, 2, 3, and 4.

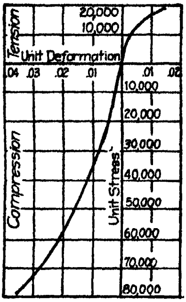


FIG. 2.—Stress-deformation diagram for cast iron.

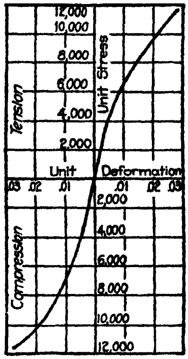


FIG. 3.—Stress-deformation diagram for timber.

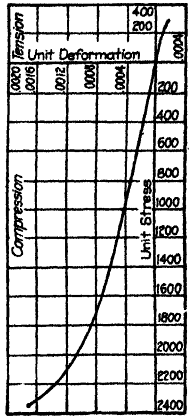


FIG. 4.—Stress-deformation diagram for concrete.

26. Information Obtained from Tension Tests.—The facts usually obtained from a tension test are as follows: Elastic limit, modulus of elasticity, ultimate strength, percentage of elongation, and percentage of reduction of area.

26*a*. Elastic Limit.—The elastic limit as defined in Art. 21 is the unit stress at the limit of proportionality of stress and deformation, or, it is the unit stress on a stress-deformation diagram where the curve departs from a straight line, as illustrated in Fig. 1. Often it is difficult to determine exactly the location of the point where the curve first begins to depart from a straight line. For this reason, in the commercial testing of wrought iron and structural steel it is customary to locate the point where the deformation increases rapidly. This point is called the *yield point* or the *apparent elastic limit.* In many cases it is so close to the true elastic limit that it may be used as such, but in other cases it is considerably beyond that limit and should be used then only with a full appreciation of the distinction between the two points.

26*b*. Modulus of Elasticity.—The stress-deformation diagram for a material like steel is straight up to the yield point. Therefore the modulus of elasticity is constant up to this point from the point of zero load (see Fig. 1).

The stress-deformation diagrams for materials like cast iron, timber, and concrete, are curved almost from the origin. Therefore, the modulus of elasticity is variable and must be determined for the assumed working stress by dividing this stress by the corresponding deformation (see Figs. 2, 3 and 4).

26*c*. Ultimate Strength.—The *ultimate strength* of a material is found by dividing the maximum load by the original area of cross-section. Just preceding the failure of a ductile material there is a localized elongation and reduction of area, or "necking-down" as it is sometimes called. Therefore, at

failure, the area of cross-section may be much less than the original area of cross-section. Nevertheless, in finding the ultimate strength the *maximum load* and the *original* area of cross-section are used.

26*d*. Percentage of Elongation.—In order to reduce the results of different tensile tests to a comparative basis, the American Society for Testing Materials has adopted 8 in. as the standard gage length. After failure has occurred the two parts of the specimen are fitted together and the final length between two points originally 8 in. apart is measured. The *percentage of elongation* is found by dividing the increase in length by the original length.

26*e*. Percentage of Reduction of Area.—The *percentage of reduction of area* is found by dividing the difference between the area of the original cross-section and the area of the cross-section at the break after failure has occurred by the area of the original cross-section.

27. Information Obtained from Compression Tests.—Plastic materials like wrought iron, soft and medium steel, lead, copper and zinc simply flow when acted upon by a compressive force without giving any preliminary indication of approaching failure. Brittle materials like cast iron, hard or tempered steel, brick, stone and concrete fail by crushing to a powder, by crumbling to pieces, or by shearing along planes which make certain definite angles with the direction of the compressive force.

In most cases compression tests of a material are made to determine its resistance to compression and its elastic properties under compression, while the general mechanical properties of the material are usually determined by means of tensile tests since they can be made more easily than satisfactory compression tests. When testing highly elastic materials in compression, it is customary to get the elastic limit, yield-point, modulus of elasticity and, if possible, the ultimate strength. When testing brittle materials it is customary to get the unit stress at the first crack or other sign of approaching failure, the elastic limit, if there is one, and the ultimate strength.

28. Shear and Torsion.—Shearing stress exists at a section when the two parts of a body in contact at the section tend to move tangential to the plane of the section but in opposite directions. The external forces must be an infinitely small distance apart in order to develop pure shearing stress unaccompanied by the tension and compression caused by flexure. Since, in testing or in actual practice there must be, necessarily, some distance between the lines of action of the forces producing the shearing stresses, the element of flexure is never entirely eliminated. However, in the following discussion we shall assume that we may subject a body to pure shearing stress unaccompanied by tension or compression except that which is produced in a particle due to the shear itself.

Torsion is twisting stress. It is seldom of importance in structural design, although it may occur in such members as spandrel beams with rigidly connected slabs.

29. Shear Tests.—Since the transverse shearing strength of materials becomes of major importance only in the design of splices, connections, and very short beams, especially wooden beams, we are concerned with the transverse shearing strength of only a few metals and the various kinds of structural timber. The external forces must be only an infinitesimal distance apart in order to produce pure shearing stress on a given plane. This condition is never realized

in practice nor can it be attained fully in testing. However, the cross-bending effect may be reduced and the ideal conditions may be approached in testing by reducing the cross-section of the specimen in the plane of the shear by grooves and by reinforcing the specimen on each side of this section.

30. Axial and Combined Stresses.—When a force acts parallel to the axis of a member and at the center of gravity of its cross-section, it produces what is called *axial stress*. Such stress is uniformly distributed over the cross-section. A force parallel to the axis of a member but not acting along this axis is called an *eccentric force*. It is equivalent to an axial force of like amount and a couple whose moment is equal to the product of the force by the normal distance from the force to the axis of the member. Thus an eccentric force as described above produces *combined stresses*. The axial stresses may be considered separately from those due to moment, and the resulting stresses added to obtain the total stress at any point. For cases of combined stresses which are not parallel, as horizontal and vertical shear, or shear and direct stress, the combined stress must be figured by methods given in the chapter on "Simple and Cantilever Beams."

31. Bending Stress and Modulus of Rupture.—*Bending stresses* are stresses induced by loads perpendicular to the member. *Modulus of rupture* is the maximum bending stress computed on the assumption that elastic conditions exist until failure. Bending stress is discussed in the chapter on "Simple and Cantilever Beams."

32. Stiffness.—*Stiffness* is a term used with reference to the rigidity of structural members. In columns or struts it refers to their lateral stability; i.e., by a stiff column is meant one with a small ratio of length to least radius of gyration, as compared to a slender column. In the case of beams, stiffness refers to lack of deflection rather than to strength.

33. Bond Stress.—The combined action of steel and concrete is dependent upon the grip of concrete upon steel, called *bond*. Denoting the allowable bond stress per square inch by u, the load which a rod can take from the concrete per lineal inch is $u\pi d$ for a round rod, and $4ud$ for a square rod. The allowable stress in the rod is $f_s \frac{\pi d^2}{4}$ for round rods and $f_s d^2$ for square rods. The length of embedment of a straight rod necessary to develop its allowable strength is therefore $\frac{f_s d}{4u}$ (in inches) for both round and square rods. For given stresses the necessary length of embedment is easily computed. For example, let $f_s = 10{,}000$ lb. per sq. in. and $u = 80$, then $l = \frac{10{,}000d}{(4)(80)} = 31 +$ diameters.

34. Shrinkage and Temperature Stresses.—Shrinkage is a function of materials which are poured in a semi-liquid state and then harden by cooling or by chemical action. Such materials are cast iron and concrete. A cast-iron member should be designed so that in cooling it will not shrink unequally and cause stresses which may crack it. For this reason adjacent parts should be made of nearly equal thickness, and filets should be used at all angles and corners.

Concrete shrinks when setting in air and expands when setting under water. If the ends of a concrete structure be rigidly fixed, stress will be developed equal to that required to change the length by the amount of the deformation which would occur if the ends were free, or $f = \delta E$.

All bodies change in length with changes in temperature, expanding with heat and contracting with cold. The *coefficient of expansion* is the change in length, per unit of length, per degree change in temperature. The total change in length of a body for a given change of temperature may be found by multiplying this coefficient by the length and the change of temperature in degrees. The fact that the coefficient of expansion is practically alike for both steel and concrete is an important factor in their combined use. As in the case of shrinkage stresses, a tendency to change of length in a member fixed at the ends induces stress equal to that which would cause the computed change in length; that is $f = \delta E$. This may be an important factor to consider in almost any form of steel or concrete construction. In wood construction there is usually sufficient play at columns to take up any expansion.

35. Impact.—Experience has shown that the stress produced in a member by a load which is applied in the nature of a blow is greater than that produced by the same load applied gradually. This excess of stress caused by the suddenness of the application is called *impact.* In many types of buildings and roof trusses the moving or variable load is relatively small compared to the dead load and hence this dynamic effect becomes unimportant. However, in the case of highway bridges which must carry fast moving motor cars and trucks or the slow but heavy road rollers and tractors and also in the case of railroad bridges which must carry rapidly moving trains, this dynamic effect becomes of a great deal of importance.

Provision for impact may be made by arbitrarily increasing the live load stress in accordance with some empirical formula which supposedly represents the effect of impact or by decreasing the live load working stress. The former method is the one usually followed in the design of railroad bridges and seems to be the more rational of the two methods.

The results of a very extensive series of experiments on impact are those conducted by the American Railway Engineering Association and described in its Bull. **125,** 1910. The formula suggested there is

$$I = \frac{S}{1 + \frac{L^2}{20,000}}$$

in which S is the live-load stress and L is the length of the span in feet which is loaded to produce the maximum live-load stress.

In Part III of "Modern Framed Structures," the authors recommend that the formula given above be modified to read

$$I = \frac{S}{1 + \frac{L^2}{30,000}}$$

in order to be in closer agreement with the experimental results.

In the series of experiments mentioned above it was found that with track and rolling stock in good condition, the chief cause of impact is the unbalanced condition of the drivers of the ordinary locomotive.

36. Repeated Stresses.—It has been found from numerous experiments that metals may be made to fail at stresses less than the ultimate strength, or at stresses even less than the elastic limit, when loads are repeated many thousands or many millions of times.

The word *fatigue* is sometimes used to denote failures due to these causes but the term *repeated stress failure* or *alternating stress failure* describes more clearly what actually occurs. The term *alternating stress* or *reverse stress* is used to denote the case in which the character of the stress changes from tension to compression. *Repeated stress* is a broader term including the preceding case as well as that in which the stress remains compressive or tensile throughout the test.

For stresses well within the elastic limit and for the relatively small number of repetitions of stress received by a bridge or other structure which the structural engineer is called upon to design, the element of repetition has little or no effect. It is common practice to make a liberal allowance for impact and to neglect the effect of repeated stress entirely.

37. Work and Resilience.—When a body is acted upon by a force and is deformed by it, the force is said to do *work.* This work is generally expressed in inch pounds. Upon removal of the force from an elastic body, this stored up energy may be recovered as mechanical work, provided that the material has not been stressed beyond its elastic limit. This ability of a body to give back the energy expended in deforming it is called *resilience.* The work expended in deforming a unit volume of any material to the elastic limit is called its *modulus of resilience.* It is the limiting value of the energy which can be recovered as mechanical work without loss. Beyond the elastic limit, part of the energy is used in permanently deforming the material and is lost in the form of heat. The work expended in deforming a unit volume of material when the unit stress is f, is the product of the unit deformation $\frac{f}{E}$, and the average unit force $\frac{f}{2}$, or the work is

$$W = \frac{f^2}{2E}$$

and is expressed in inch pounds per cubic inch when f and E are expressed in pounds per sq. in.

38. Working Stress and Factor of Safety.—The unit stress to which a material is to be subjected in practice, and for which the members of a structure are designed, is called the allowable unit stress or the *working stress.* The working stress is arbitrarily chosen and recommended by engineers, or others in authority, as the maximum value to which, in their judgment, a given material should be stressed in practice. The ratio of the ultimate strength of a material to the working stress is called the *factor of safety.*

39. Influences Governing Choice of Working Stress.—Several different influences must be given consideration in determining the value of the working stress which should be used under a given set of conditions. Among these influences are the following: Reliability of the material, kind of loading, position of member, maintenance, type of failure, and consequence of failure.

39*a*. Reliability of the Material.—Steel is perhaps the most reliable structural material due to its production on a large scale and almost entirely by manufacturing organizations of experience. A person who is familiar with this material can obtain by inspection and a few simple tests, a good notion of the quality of a given product. Concrete is not so reliable because of the variations in the composition of the aggregate, and the lack of standardization of the methods

of production. Also more elaborate equipment and more time are required for the testing of concrete than for the testing of steel.

39b. Kind of Loading.—A lower stress should be used if the load is to be applied in the nature of a blow than if the load is to be applied gradually, in order properly to care for the stress due to impact (see Art. 35). If the stress is apt to be repeated, or alternated several millions of times during the life of the member, a lower working stress should be used than would otherwise be the case.

39c. Position of Member.—The other actions to which a material is exposed in addition to the load which it is to carry, will have its influence upon the working stress which should be used. Certain kinds of brick and stone are not very resistant to the action of weathering forces. Unprotected wood will decay when exposed to moisture, air and moderate warmth, and due allowance for this action must be made unless some means of preservation is provided. Wood may be subjected to the attacks of insects, marine borers and fire. The corrosion of steel due to water and to the action of injurious gases given off by passing locomotives must be considered in the design of steel structures.

39d. Maintenance.—In the case of a bridge or other structure which is to be inspected frequently and repaired whenever necessary, a higher working stress may be allowed than in a case where this service is not to be provided.

39e. Type of Failure.—If there is apt to be some warning or sign of approaching failure, a higher working stress can be used than would otherwise be justified.

39f. Consequences of Failure.—Obviously, if failure of a structure is to be accompanied by loss of life, every precaution consistent with good engineering practice should be observed in order that failure may be avoided.

SIMPLE AND CANTILEVER BEAMS

By George A. Hool and L. D. Norsworthy

40. Loading.—The load carried by a beam consists of its own weight, known as the *dead load*, and certain temporary loads known as the *live load*. Dead load is present at all times, while live load may be removed from the beam at will. In some cases, these loadings are referred to as *fixed* and *movable* loads, respectively.

Loads may be *concentrated* (that is, applied at a single point or over a very short distance), or they may be *uniform*—that is, the loads cover all or a part of the beam, and, throughout the portion covered, the amount of load per unit of length is the same.

41. Effect of a Load on a Beam.—Whenever a beam carries a load of any kind, whether it is due solely to its own weight or due to external loads placed upon it, two things tend to happen, as follows: (1) The beam bends as shown in an exaggerated manner in Fig. 5a; (2) the beam tends to break off or shear at various sections, as shown in Fig. 5b.

The effect of the bending of the beam is to lengthen the lower fibers and shorten the upper fibers, or, in other words, to put the lower fibers in tension and the upper fibers in compression. It is evident that as we proceed from the top of the beam to the bottom we will reach a point where the fibers will be neither

in tension nor compression. All these fibers that remain unstressed will form a surface such as *MNOP*, Fig. 6, called the *neutral surface*. The intersection of the neutral surface with the plane of bending, the line *AB* of Fig. 6, is called the *neutral line* or *elastic curve*. Any section of the beam, such as *CD*, intersects the neutral surface in a line *EF*; this line is called the *neutral axis* of the section.

A beam not only tends to fail by shear on any transverse plane, but it also tends to split along its neutral surface.

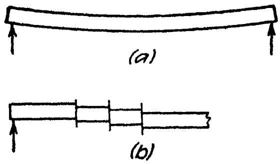


FIG. 5.

FIG. 6.

42. Reactions.—The reactions are the forces acting at the supports which balance the loads and so hold the beam in equilibrium.

43. Principles of Statics Used in Finding Reactions.

43a. Definitions.—*Statics* is the science which treats of forces in equilibrium.

Forces are said to be *concurrent* when their lines of action meet in a point; *non-concurrent* when their lines of action do not meet in this manner.

When a number of forces act upon a beam and the beam does not move, then the forces considered are said to be in *equilibrium*. Any one of the forces balances all the other forces and it is called the *equilibrant* of those other forces.

A single force which would produce the same effect as a number of forces is called the *resultant* of those forces. The process of finding the single force is called *composition*.

It is evident from the above that the equilibrant and resultant of a number of forces are equal in magnitude, act along the same line, but are opposite in direction.

Any number of forces whose combined effect is the same as that of a single force are called *components* of that force. The process of finding the components is called *resolution*.

The *moment of a force* with respect to a point is the measure of the tendency of the force to produce rotation about that point. It is equal to the magnitude of the force multiplied by the perpendicular distance of its line of action from the given point. The point about which the moment is taken is called the *origin* (or *center*) *of moments*, and the perpendicular distance from the origin to the line of action is called the *lever arm* (or *arm*) of the force. When a force tends to cause rotation in the direction of the hands of a clock, the moment is usually considered *positive*, and in the opposite direction, *negative*.

A *couple* consists of two equal and parallel forces, opposite in direction, and having different lines of action. The perpendicular distance between the lines of action of the two forces is called the *arm* of the couple. The *moment of a couple* about any point in the plane of the couple is equal to the algebraic sum of the moments of the two forces, composing the couple, about that point. (Algebraic sum of the moments means the sum of the moments of the forces, considering

positive moments *plus* and negative moments *minus*.) It can be shown that the moment of a couple is equal to one of the forces multiplied by the arm.

43b. Composition of Two Concurrent Forces.—In Fig. 7 let forces F_1 and F_2 which are concurrent forces acting at the point O, be represented in magnitude and direction by OA and OB respectively. From B draw BC parallel to OA, and from A draw AC parallel to OB. Join the point of intersection C with O. The line OC represents the magnitude of a single force R which would produce the same effect as the forces F_1 and F_2. Thus R is the resultant of F_1

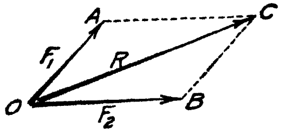


FIG. 7.

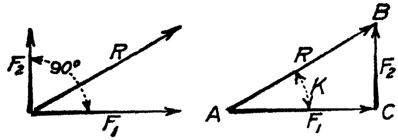


FIG. 8.

and F_2. A force equal and opposite in direction to R and with the same line of action would be the equilibrant of F_1 and F_2, since it would hold them in equilibrium. F_1 and F_2 are components of R.

It is not necessary to construct the entire parallelogram since either triangle OAC or OBC will suffice. Either of these triangles is called a force triangle and either one, if constructed, is sufficient to give the value of the resultant and the equilibrant of forces F_1 and F_2.

It is convenient to solve the force triangle algebraically where the angle between the lines of action of two forces is 90 deg. In Fig. 8 the angle between the lines of action of F_1 and F_2 is 90 deg. It is required to find the value of the resultant R. Since ABC is a right triangle

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$$

or

$$R = \sqrt{F_1^2 + F_2^2}$$

The direction of the resultant R is decided by the angle K. K may be determined as follows:

$$\tan K = \frac{BC}{AC} = \frac{F_2}{F_1}$$

43c. Resolution of a Force into Components.—If the resultant R is given at the point O, Fig. 8A, and it is desired to obtain two components of R parallel to the lines $o'\ a'$ and $o'\ b'$, then OC is first drawn equal in magnitude and parallel to R, OB is drawn from O parallel to $o'b'$, and CB is drawn from C parallel to $o'a'$ and the lengths of the lines OB and BC, when scaled from the drawing, give the magnitudes of the two components desired.

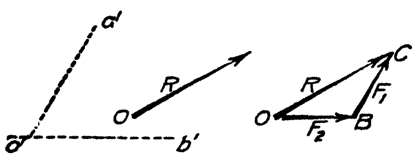


FIG. 8A.

When components are required making 90 deg. with each other, the magnitude of these forces may easily be determined algebraically. Thus, if R in Fig. 8 is known and the components F_1 and F_2 are required,

$$F_1 = R \cos K$$
$$F_2 = R \sin K$$

43d. Composition and Equilibrium of Non-concurrent Forces.—The resultant of any number of non-concurrent forces may be found in the following manner: Resolve each force algebraically into components F_x and F_y, parallel respectively to X and Y axes. Then the magnitude of R is given by the equation

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

and the angle it makes with the X axis is given by

$$\tan\theta = \frac{\Sigma F_y}{\Sigma F_x}$$

Its line of action is found by placing its moment about any point equal to the algebraic sum of the moments of the forces with respect to the same point. If the moment arm of the resultant is denoted by a, and the moment arms of the several forces by a_1, a_2, etc., then

$$Ra = F_1a_1 + F_2a_2 + \text{etc.}$$

If a force is applied equal and opposite to R and in the same line of action, the system of forces will be in equilibrium. Let ΣM represent the algebraic sum of the moments about any point. For equilibrium, then

$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma M = 0$$

In practice it is common to use horizontal and vertical axes, for which case the above equations may be written:

$$\Sigma H = 0 \quad \Sigma V = 0 \quad \Sigma M = 0$$

Problems in the equilibrium of non-concurrent forces may be solved if the number of unknowns is not greater than three. Three independent equations may be written, employing the three algebraic conditions above stated, and solving these equations simultaneously in any given case gives the three unknowns. It is often convenient to use two moment equations and either $\Sigma H = 0$ or $\Sigma V = 0$. A new moment center must be taken each time $\Sigma M = 0$ is used.

The three unknowns usually desired may be classed under three general cases; namely, where the following unknowns are required: (1) point of application, direction and magnitude of one force (that is, the force is wholly unknown); (2) magnitudes of two forces and the direction of one of these forces; and (3) magnitude of the three forces. The first case is nothing more than the finding of the resultant of a system of non-concurrent forces.

A special case in the solution of non-concurrent forces occurs when all the forces considered are parallel. Then the number of independent equations reduces to two and it is possible, therefore, to determine but two unknowns, namely: (a) point of application and magnitude of one force; and (b) magnitude of two forces.

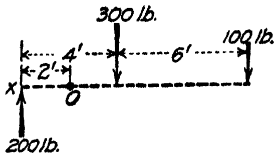


Fig. 9.

Illustrative Problem.—Find the resultant of the three vertical forces shown in Fig. 9. Since the forces are all vertical, $\Sigma H = 0$, and the resultant must also act in a vertical direction. Consider downward forces positive and upward forces negative. The magnitude of the resultant may be found as follows:

$$R = 300 + 100 - 200$$
$$= 200 \text{ lb., acting down (since the result is positive).}$$

It will be noticed that a force equal and opposite to R would make the forces in equilibrium.

It is now necessary to find the point of application of the resultant R. By the point of application in this case is meant a point on the line of action of the resultant.

The algebraic sum of the moment about the point o is equal to $(300)(2) + (100)(8) + (200)(2) = 1{,}800$ ft.-lb. The resulting force is 200 lb. and the problem resolves itself into finding how far from the point o the 200 lb. should be placed to have the same effect as the three loads shown, or, in other words, how far away from o a load equal and opposite to the 200-lb. resultant should be placed in order to cause equilibrium. Thus, $\Sigma M = 0$ may be used to find this distance.

$$\frac{1{,}800 \text{ ft.-lb.}}{200 \text{ lb.}} = 9 \text{ ft. to the right of } o.$$

It should be noted that the computations would have been more simple if the point x had been selected instead of the point o, that is, the work would have been simplified by taking the origin on the line of action of one of the forces. The computations for that case would be arranged as follows:

$$\frac{(300)(4) + (100)(10)}{200} = 11 \text{ ft. to the right of } x.$$

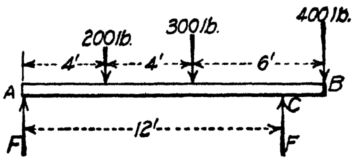


FIG. 10.

Illustrative Problem.—The beam AB (Fig. 10) is 14 ft. long and loaded as shown. It is simply supported at A and C. (a) Determine the supporting forces due to the three given loads. (b) Determine the supporting forces, including the weight of the beam which is 50 lb. per lin. ft.

(a) $R = 200 + 300 + 400 = 900$ lb., acting down.

$$F + F_1 = R = 900 \text{ lb.}$$

Origin at A:

$$(220)(4) + (300)(8) + (400)(14) = 12F_1$$
$$F_1 = 733 \text{ lb.}$$
$$F = 900 - 733 = 167 \text{ lb.}$$

Answers $\begin{cases} F = 167 \text{ lb.} \\ F_1 = 733 \text{ lb.} \end{cases}$

(b) Wt. of beam $= (50)(14) = 700$ lb.

$$R = 900 + 700 = 1{,}600 \text{ lb.}$$
$$(200)(4) + (300)(8) + (400)(14) + (700)(7) = 12F_1$$
$$F_1 = 1{,}142 \text{ lb.}$$
$$F = 1{,}600 - 1{,}142 = 458 \text{ lb.}$$

Answers $\begin{cases} F = 458 \text{ lb.} \\ F_1 = 1{,}142 \text{ lb} \end{cases}$

44. Shear.—Consider the forces acting on a beam to be resolved into horizontal and vertical components. Then the shear at any section is the algebraic sum of the vertical forces acting on either side of the section, and is the force which tends to cause the part of the beam on one side of the section to slide by the part on the other side. This tendency is opposed by the resistance of the transverse shearing.

When the resultant force acts upward on the left of the section, the shear is called *positive*, and when it acts downward on the same side of the section, it is called *negative*. Since $\Sigma V = 0$ when we consider the forces on both sides of the section, then the resultant of the forces on the right of the section must be equal and opposite in direction to the resultant of the forces on the left of the section. Thus, it makes no difference which side of the section we consider, the shear is *positive* when the resultant on the left is upward and when the resultant on the right is downward. Also the shear is *negative* when the resultant on the left is downward and when the resultant on the right is upward.

At the section ab, Fig. 11, the shear, since there are no loads between the section and the left support, equals the left reaction and is positive. This is

true of any section between the left support and the section cd. The shear to the right of cd is negative and is equal to the right hand reaction.

45. Bending Moment.—The bending moment (or moment) at any section of a beam is the algebraic sum of the moments of the forces acting on either side of the section about an axis through the center of gravity of the section, and is the moment which measures the tendency of the outer forces to cause the portion of the beam lying on one side of the section to rotate about the section. This tendency to bend the beam is opposed by internal fiber stresses of tension and compression.

When the resultant moment on the left of the section is clockwise, the moment is called *positive*, and when it is counter-clockwise on the same side of the section, it is called *negative*. Since $\Sigma M = 0$ when we consider the forces on both sides of the section, then the resultant moment of the forces on the left of the section is equal and opposite to the resultant moment of the forces on the right of the section. Thus, it makes no difference which side of the section we consider, the moment is *positive* when the resultant moment of the forces on the left is clockwise and when the resultant moment of the forces on the right is counter-clockwise. Also, the moment is *negative* when the resultant moment of the forces on the left is counter-clockwise and when the resultant moment of the forces on the right is clockwise.

At the section ab, Fig. 11, the moment is $\frac{P}{2}(x)$. It increases uniformly from the left support where it is zero to the section cd where it is $\left(\frac{P}{2}\right)\left(\frac{L}{2}\right) = \frac{PL}{4}$.

Positive bending moment causes compression in the upper fibers of a beam, and tension in the lower fibers. The reverse is true for negative bending moment.

46. Shear and Moment Diagrams.—The variation in the shear or bending

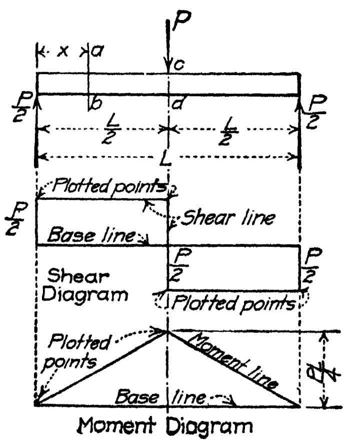


Fig. 11.

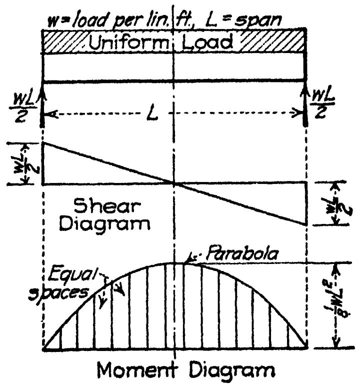


Fig 12.

moment from section to section for fixed loads may be well represented by means of diagrams, called shear and moment diagrams. The diagrams are constructed by laying off a *base-line* equal to the length of the beam and marking off on this

line the positions of the loads and the reactions. Positive shear and moment at given points should be represented above the base-line and negative shear or moment below this line. Points are plotted vertically above or below given points on the base-line, and the distance these plotted points are from the base-line should represent to some scale the magnitude of the shear or moment at these

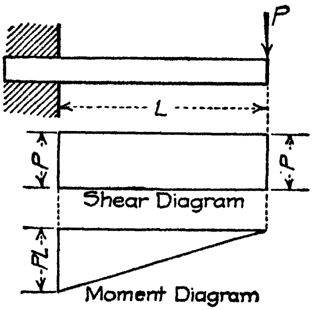


Fig. 13.

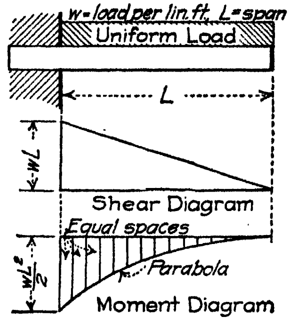


Fig. 14.

given points on the beam. The line joining the points plotted in this way is called the shear or moment line, depending upon whether a shear or moment diagram is being drawn.

To illustrate, in Fig. 15, the ordinate *ab* represents the value of the shear at the point *b* of the beam and the ordinate *cd* represents the value of the moment at the point *d*.

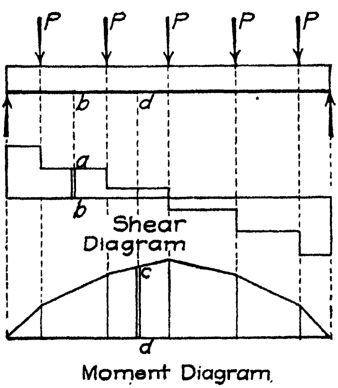


Fig. 15.

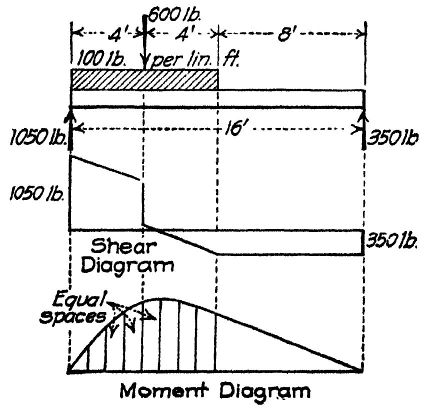


Fig. 16.

In shear diagrams for uniform loading, ordinates need only be erected at the ends of the beam and at the points of support. If concentrated loads are also applied to the beam, ordinates must also be plotted at their points of application.

In moment diagrams for uniform loading, ordinates should be erected and points plotted at the reactions and every foot or two along the beam. If con-

centrated loads are also applied to the beam, ordinates must also be plotted at their points of application.

If the shear or moment lines are not completely determined by the above rules, additional points should be taken.

A *cantilever* beam is a beam having one end fixed and the other end free (see Art. 3, p. 1). The reaction at the fixed end is indeterminate, but the shear or bending moment at a given section may be easily found by considering the loads between the section and the free end.

Shear and moment diagrams for both simple and cantilever beams with various loadings are shown in Figs. 11 to 16 inclusive. In all cases the weight of the beam is neglected.

47. Maximum Shear.—It is always desirable in proportioning beams to know the greatest or maximum value of the shear in a given case. The following rules apply:

1. In cantilevers fixed in a wall, the maximum shear occurs at the wall.
2. In simple beams, the maximum shear occurs at the section next to one of the supports.

These rules can be verified by examining the shear diagrams in Figs. 11 to 16 inclusive.

48. Maximum Moment.—By comparing the corresponding shear and moment diagrams in Figs. 11 to 16 inclusive, it will be found that the maximum moment occurs where the shear changes sign; that is, where the shear line crosses the base-line. This could also be shown algebraically.[1]

[1] Consider the beam carrying the general loading shown in Fig. $A(a)$. Remove the portion of the beam contained between two vertical planes ab and cd at a distance dx apart and indicate the forces acting on this portion of the beam. These forces are shown in Fig. $A(b)$, where dM and dV represent respectively the change in bending moment and shear across the section. Taking moments about a point in the plane ab, we have

$$+M - (M + dM) + V dx + w \frac{dx^2}{2} = 0$$

Since dx is a small distance, dx^2 is therefore infinitely small compared to the other terms and may be neglected. The above expression may then be written,

$$\frac{dM}{dx} = V \tag{1}$$

That is, the rate of change of moment at any point is equal to the external shear at that point.

In works on the calculus it is shown that a maximum or a minimum value of a function occurs when its rate of change is equal to zero. By these methods it can be shown that the bending moment in a beam, such as Fig. $A(a)$, is a maximum when its rate of change is zero. But eq. (1) shows that the rate of change in moment is equal to the shear at that point. Therefore, the bending moment in a beam is a maximum when the shear is equal to zero.

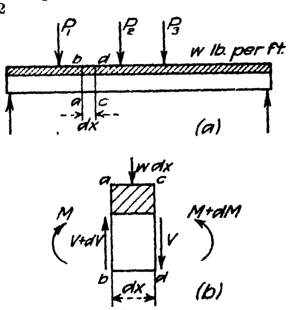


FIG. A.

The laws governing the change in the shear along a beam may be determined by taking a summation of vertical forces shown on Fig. $A(b)$. Thus

$$V + dV - V - w\,dx = 0$$

from which

$$\frac{dV}{dx} = w \tag{2}$$

That is, for uniform loads, the rate of change of shear is equal to the load per unit of length along the beam. A study of the shear diagrams given in the preceding articles will verify this statement.

For concentrated loads, a summation of vertical forces taken at a point between loads gives $\frac{dV}{dx} = 0$, that is, the shear is constant between loads. When a concentrated load is included between the planes *ab* and *cd* of Fig. *A*(*a*), we find $\frac{dV}{dx} = P$, where P is the load at the section. That is, at a point where a load is concentrated, the shear changes by the amount of the load.

By the help of this principle it is necessary to construct only the shear line and observe from it where the shear changes sign; then compute the bending moment for that section.

Illustrative Problem.—Construct shear and moment diagrams for a 20-ft. beam supported at the ends and loaded as shown in Fig. 17. Also, find the maximum shear and maximum moment, and the sections where they occur.

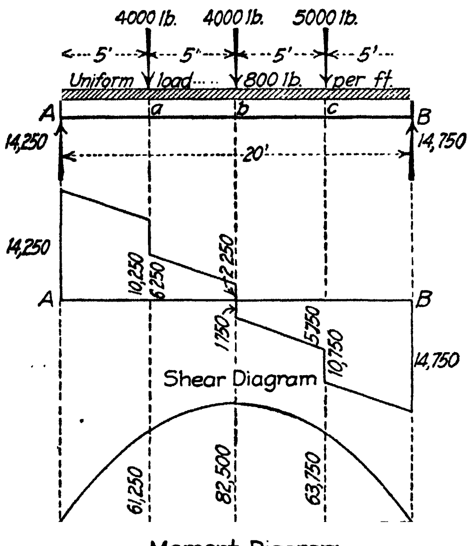


FIG. 17.

$$\text{Reaction } A = \frac{5{,}000)(5) + (4{,}000)(10 + 15)}{20} + 8{,}000$$
$= 14{,}250$ lb.

Reaction $B = 13{,}000 + 16{,}000 - 14{,}250$
$= 14{,}750$ lb.

Shear at $A = 0$

Shear at section just to right of $A = 14{,}250$

Shear at a $\begin{cases} \text{to left} = 14{,}250 - (800)(5) = 10{,}250 \\ \text{to right} = 10{,}250 - 4{,}000 = 6{,}250 \end{cases}$

Shear at b $\begin{cases} \text{to left} = 6{,}250 - (800)(5) = 2{,}250 \\ \text{to right} = 2{,}250 - 4{,}000 = -1{,}750 \end{cases}$

Shear at c $\begin{cases} \text{to left} = -1{,}750 - (800)(5) = -5{,}750 \\ \text{to right} = -5{,}750 - 5{,}000 = -10{,}750. \end{cases}$

Shear at section just to left of $B = -14{,}750 -10{,}750 - (800)(5) = -14{,}750$ (check)

Shear at $B = 0$.

We shall determine the moment at points A, a, b, c and B. Moments should also be found at sections 2 ft. apart on this beam to completely determine the moment curve.

Moment at $A = 0$
Moment at $a = (14{,}250)(5) - (800)(5)(5/2) = 61{,}250$
Moment at $b = (14{,}250)(10) - (8{,}000 + 4{,}000)(5) = 82{,}500$
Moment at $c = (14{,}750)(5) - (800)(5)(5/2) = 63{,}750$
Moment at $B = 0$

The maximum shear $= -14{,}750$ lb. at a section just to the left of the right support.

The shear changes sign at section b, consequently the moment is a maximum at that point $= 82{,}500$ ft.-lb.

In some cases the shear does not change sign at the point of application of a concentrated load and in such a case the position of the section, where the bending moment is a maximum, must be scaled or computed from the shear diagram to the nearest one-tenth of a foot.

49. Maximum Shear and Moment Due to Moving Loads.

49a. A Single Concentrated Moving Load.—For a single concentrated moving load the maximum positive live shear on a simple beam at any section as A, Fig. 18, occurs when the load is just to the *right* of the section. This statement is readily verified by considering how the shear varies at the section as a load passes across the beam from the right to the left support. The left reaction, and consequently the positive shear, is increased as the load P is moved from the right support up to the section, being greatest when the load is just to the right of the section. Now move the load to the left of A. The shear is equal to the difference between the left reaction and the load P and, since a load is always greater than either reaction (the load being equal to the sum of the reactions), the shear with the load to the left of A is negative, proving that the positive shear is a maximum with the load just to the right of the section. In practice the load is always placed *at* the section. This same line of reasoning might be followed through for negative shear, moving a load from the left abutment to the section and considering how the shear varies to the right of the section. The maximum negative shear is found to occur when the load is just to the *left* of the section.

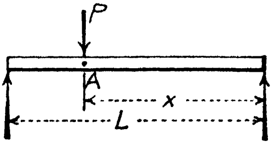


FIG. 18.

The value of the maximum positive shear for the load P is $P\frac{x}{L}$, and the maximum negative shear is $P\frac{L-x}{L}$.

The maximum live moment at A occurs with the load at A, for a movement to either side reduces the opposite abutment reaction and consequently the moment. The maximum moment is $P\frac{x}{L}(L - x)$.

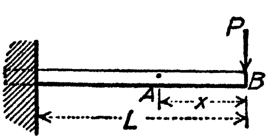


FIG. 19.

At any point on a cantilever beam, such as at A, Fig. 19, the shear is a maximum when the load is anywhere to the right of the point. When the load is on the left, the shear is zero. The moment is a maximum at the section when the load is at B and equals $(P)(x)$. When the load is to the left of A, the moment is zero.

49b. Moving Uniform Load.—For a moving uniform load the maximum positive live shear on a simple beam at any section as A, Fig. 20, occurs when the right hand section of the beam is loaded up to the point considered. This is seen to be true when we consider that adding a load to the right of A increases the left reaction and therefore the positive shear, while adding a load to the left of A increases the left reaction by an amount less than the load which is added, and hence decreases the positive shear. The maximum positive shear at A in Fig. 21 for a uniform load of w lb. per ft. $= \frac{1}{2} w \frac{x^2}{L}$.

From reasoning similar to the above, the maximum negative shear at any section as A, Fig. 20, is found by loading to the left of the point. Maximum

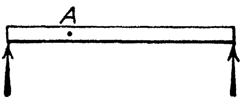


FIG. 20.

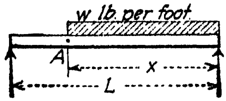


FIG. 21.

negative shear at A, Fig. 22, for a uniform load of w lb. per ft. $= \frac{1}{2} w \frac{(L - x)^2}{L}$ (considering the right hand reaction).

The maximum moment at any section as A occurs when the beam is *fully loaded*, for the addition of a load anywhere on the beam will add a positive moment at the section. For a load of w lb. per ft., the maximum

$$M = \frac{wL}{2}(L - x) - \frac{w(L - x)^2}{2} = \frac{w}{2}(L - x)(L - L + x) = \frac{w}{2}(x)(L - x).$$

If the section is at the center of the beam, the

$$\text{maximum } M = \tfrac{1}{8} wL^2$$

The above formulas for maximum moment give results in foot pounds, since w represents the load in pounds per foot and L the span of the beam in feet. To

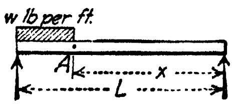


FIG. 22.

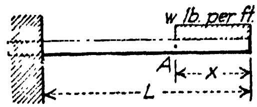


FIG. 23.

get inch pounds, multiply by 12 or insert for w in the formulas the load in pounds per inch and for L the span of the beam in inches.

At any point on a cantilever beam, such as at A, Fig. 23, the maximum shear occurs for either a full load over the entire length, or for full load on the portion of the beam between the section and the free end, and equals wx. The moment is always negative and the maximum moment occurs for the same loading giving maximum shear; i.e.,

$$\text{maximum } M = \frac{wx^2}{2}$$

49c. Concentrated Load Systems.—The methods of finding maximum shear and maximum moments due to concentrated load systems are explained fully in the chapter on "Moments and Shears in Beams and Trusses" in the volume on "Stresses in Framed Structures."

50. Common Theory of Flexure for Homogeneous Beams.—Having considered the external loads and their effect on a beam, there remains to be investigated the stresses set up in the beam itself which hold these loads in equilibrium and prevent the beam from failing. Considering the stresses due to bending first, it is shown in Art. **41** that in a simple beam the fibers above the neutral axis are in compression and those below in tension, while the fibers at the neutral axis remained unstressed. The problem now is to find the stress on the extreme top and bottom fibers of the beam when these stresses do not exceed the elastic limit of the material of which the beam is composed. It is also necessary to make *the following two assumptions which are borne out by tests on actual beams when the stresses do not exceed the elastic limit:*

1. That all transverse plane sections of the beam which are planes before bending, remain planes and normal to the longitudinal fibers after bending.

2. That the stress varies directly as the deformation and therefore as its distance from the neutral axis, and that the moduli of elasticity for the material are equal for tension and compression.

As stated in Art. **52**, the assumptions made above are not exact, but for all ordinary cases of bending the approximate are the true conditions. The formulas developed in the following discussion when applied to the types of beams generally encountered in practice, give results which enable the engineer to design beams which are entirely adequate for the purpose for which they are intended.

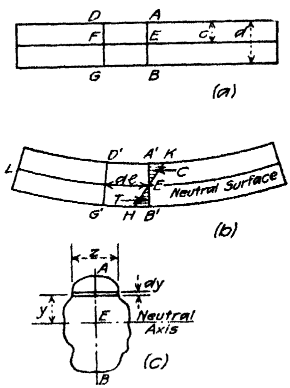


FIG. 24.

50a. To Find the Position of the Neutral Axis.—Let the shaded triangle in Fig. 24*b* represent the intensities of tensile and compressive fiber stresses in the section AB induced by the bending of the beam. Considering the portion EL as a free body it can be seen that the internal stresses of tension and compression are the only horizontal forces acting. Hence for equilibrium, the total tension must equal the total compression and have the opposite sign, that is

$$T = C \tag{1}$$

where T and C are respectively the total tension and compression below and above the neutral axis.

Let

zdy = the area of a very small strip of the cross-section of the beam parallel to the neutral axis, as shown in Fig. 24*c*.

z = the variable width of the section.

f_t and f_c = the intensity of stress, either tension or compression respectively, on a fiber at unit distance from the neutral axis.

$f_t y$ and $f_c y$ = intensity of stress on any fiber at a distance y from the neutral axis according to the second assumption.

Then

$f_c y_1 dA$ = stress on any elemental strip of the cross-section at a distance y_1 above the neutral axis.

and

$f_t y_2 dA$ = stress on any elemental strip of the cross-section at a distance y_2 below the neutral axis.

The total compression above the neutral axis is then $C = \Sigma_E^A f_c y_1 dA$ and the total tension below the neutral axis is $T = \Sigma_E^B f_t y_2 dA$, where the letters above and below the summation signs indicate the portion of the section over which the summations are made. Equation (1) then becomes

$$\Sigma_E^B f_t y_2 dA - \Sigma_E^A f_c y_1 dA = 0 \qquad (2)$$

According to assumption (2), the deformation of fibers varies directly as their distances from the neutral axis. Let e_1 and e_2 represent the deformation of fibers at unit distance above and below the neutral axis. Hence

$$e_1 = \frac{f_c}{E_c}\,dl \text{ and } e_2 = \frac{f_t}{E_t}\,dl$$

where E_c and E_t = moduli of elasticity for compression and tension respectively, and dl = length of fiber. Since these deformations are equal, we have $\frac{f_c}{E_c} = \frac{f_t}{E_t}$. According to assumption (2), the moduli E_c and E_t are equal and hence $f_c = f_t$. Substituting this equality in eq. (2), and replacing y_1 and y_2 by the general value y, we have

$$\Sigma_E^A y dA = 0 \qquad (3)$$

Equation(3) is an expression for the statical moment[1] of the area of the section of Fig. 24c about the neutral axis. But the statical moment of an area is zero only for an axis through its center of gravity.[2] Therefore, the neutral axis for bending passes through the center of gravity of the given section.

50b. The Extreme Fiber Stress in a Beam.—Let

f_o = intensity of stress (either tension or compression) on a fiber at a unit distance from the neutral axis.
f = intensity of stress on the most extreme fiber.
c = distance from most extreme-fiber to neutral axis.
I = moment of inertia[3] of cross-section of the beam about the neutral axis.
M_R = moment of resistance of internal fiber stresses taken about the neutral axis.
M = bending moment due to external forces.

Other notation same as in Art. 50a.

Consider again the beam shown in Fig. 24. In order that equilibirum may exist on any section, as AB, Fig. 24b, the moment of the external forces acting on the portion of the beam from L to E must be balanced by the moment of the internal fiber stresses taken about the neutral axis of the section. That is,

$$M = M_R \qquad (4)$$

The total stress on any fiber of area dA at a distance y from the neutral axis of Fig. 24c is $f_o y dA$, and the moment of resistance of this stress about the neutral axis is $f_o y^2 dA$. For the entire section we have

$$M_R = \Sigma_B^A f_o y^2 dA$$

[1] See definition in Art. 13. [3] See Art. 14. [2] See definition in Art. 15.

But

Hence

$$M_R = f_o I$$

By definition, $f_o = \frac{f}{c}$. Therefore,

$$M_R = \frac{fI}{c}$$

Substituting this value of M_R in eq. (4) we have

$$M = \frac{fI}{c} \tag{5}$$

which may also be written

$$f = \frac{Mc}{I} \tag{6}$$

In these equations, $\frac{I}{c}$ = moment of inertia of section divided by distance from neutral axis to extreme fiber, is known as the *section modulus*, and is usually denoted by S. Equations (5) and (6) may then be written

$$\left.\begin{aligned} M &= fS \\ f &= \frac{M}{S} \end{aligned}\right\} \tag{7}$$

Equations (5) (6) and (7) are the general formulas for the determination of moment carrying capacity, or fiber stress due to a given moment. They are the fundamental formulas for design of beams for bending.

The manner in which the stresses due to bending are distributed across the section is interesting and instructive. At a point distance y from the neutral axis of Fig. 24c the total fiber stress is $F = f_y z dy = \frac{My}{I} z dy$, where $z dy$ is the area of a strip parallel to the neutral axis.

The distribution of stress across the section is best shown by means of curves. To plot these curves consider the web of the beam divided into strips whose

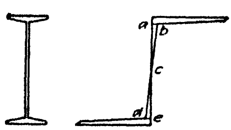


FIG. 25.

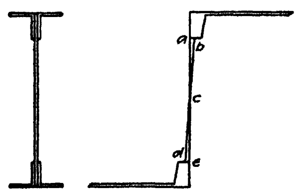


FIG. 26.

depth is unity, or say 1 in. Determine the area of this strip and multiply by the fiber stress as given by eq. (6). When plotted these values form a curve which shows the total fiber stress per unit of depth of the beam.

For a beam of rectangular section, this curve is similar to the fiber stress variation diagram. Figure 25 shows the curve for an I-beam and Fig. 26 shows the curve for a built-up beam or plate girder. In these figures, the portions of the total stress diagrams above and below ab and de show the proportional part of the total stress carried by the flanges, while the areas $abcde$ show the part

carried by the web between the flanges. It is therefore evident from these diagrams that the material in beam sections should be concentrated near the top and bottom of the beam section, for it is at these places that the products $f_y z dy$ are a maximum.

Illustrative Problem.—Determine the extreme fiber stress in a rectangular wooden beam, 6 in. wide and 10 in. deep, due to a bending moment of 8,000 ft.-lb.

For a rectangular section, $I = \frac{bd^3}{12}$ and $c = \frac{d}{2}$ or $\frac{I}{c} = S = \frac{bd^2}{6}$. Substituting 6 in. for b and 10 in. for d,

$$S = \frac{6(10)^2}{6} = 100 \text{ in.}^3$$

and

$$M = 12(8{,}000) \text{ ft.-lb.} = 96{,}000 \text{ in.-lb.}$$

Substituting these values in eq. (7) of the preceding article, we get

$$f = \frac{96{,}000}{100} = 960 \text{ lb. per sq. in.}$$

Illustrative Problem.— Proportion a rectangular beam for a bending moment of 60,000 in.-lb. on the assumption that the extreme fiber stress is not to exceed 1,200 lb. per sq. in.

From eq. (7) of the preceding article $S = \frac{M}{f}$, and from the preceding problem $S = \frac{bd^2}{6}$ for a rectangular section.

Therefore

$$bd^2 = \frac{6M}{f} = \frac{6(50{,}000)}{1{,}200} = 250 \text{ in.}^3$$

If we assume $d = 8$

$$b = \frac{250}{(8)^2} = 3.91 \text{ in.}$$

A 4 × 8-in. beam is satisfactory.

Illustrative Problem.—Determine the extreme fiber stress in a 15-in. 42.9-lb. I-beam due to a bending moment of 900,000 in.-lb.

From a steel handbook we find that $I = 441.8$ in.^{4} and $S = 59.8$ in.^{3} and we know that $c = \frac{d}{2} = 7.5$ in. The extreme fiber stress f may be obtained by substituting the values of M, c and I in eq. (6) of the preceding article or it may be obtained by substituting the values of M and S in eq. (7). Using the latter equation, we get

$$f = \frac{900{,}000}{58.9} = 15{,}300 \text{ lb. per sq. in.}$$

51. Shearing Stresses in a Homogeneous Beam.—In Art. **44** methods have been given for the determination of the total shearing force on any section due to external loads. It now remains to determine how this shear is distributed over the cross-section of the beam.

Figure 27a shows a beam deflected under any set of applied loads. It is assumed that the beam is composed of a single piece of material. Suppose now, that the given beam is composed of several pieces laid flatwise, as shown in Fig. 27b. When this beam is deflected by any set of applied loads, it will be found that the several pieces composing the beam will slip over each other as shown, greatly exaggerated, in Fig. 27b. This same tendency for horizontal layers to slide over each other is also present in the beam of Fig. 27a, but it is prevented by *horizontal shearing stresses* set up in the web of the beam. Methods will now be developed for the determination of these shearing stresses.

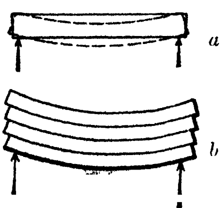


FIG. 27.

51a. Relative Intensity of Vertical and Horizontal Shearing Stresses. As stated in Art. 44 and shown in Figs. 11 to 17 inclusive, shearing forces exist on the vertical sections of a beam under any set of applied loads, and in the preceding article it has been shown that horizontal shearing forces must also exist. In determining the distribution of shearing stresses over the cross-section of the beam, attention must be paid to the conditions of equilibrium existing on the particles composing the web of the beam when acted upon by the vertical and horizontal shearing stresses mentioned above.

Consider two sections of a beam, as shown in Fig. 28a. Let the distance between these sections be so small that the shears on the two sections may be assumed to be equal. Take out any small element of the web and indicate all forces, as shown in Fig. 28b. These forces are the fiber stresses of intensity f due to bending, also considered as equal; a shearing stress of intensity v acting vertically on the faces AC and BD due to the vertical external shear on the section; and a shearing stress of intensity v_1 acting horizontally on the faces AB and CD.

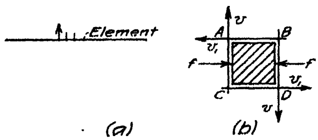


FIG. 28.

Since the element is in equilibrium, moments about any convenient point, as corner A Fig. 28b, must be equal to zero. Assuming the depth of the element perpendicular to the plane of the paper to be unity, and noting that the total stress on any face, as BD, is vBD, we have

$$(v)(BD)(AB) - (v_1)(CD)(AC) = 0$$

But $AB = CD$ and $AC = BD$. Hence

$$v = v_1 \tag{1}$$

That is, the intensities of vertical and horizontal shearing stresses on the faces of any element in the web of a beam are equal.

51b. Intensity of Horizontal Shearing Stress.—The intensity of horizontal shearing stress at any point in the web of a beam can be determined

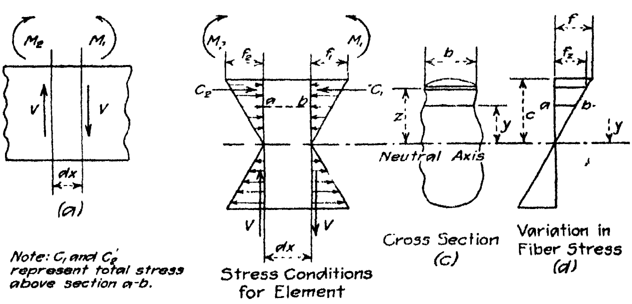


FIG. 29.

by a consideration of the stress conditions on two vertical sections separated by a very small horizontal distance. Let Fig. 29a show these sections, and let M_1 and M_2 represent the bending moments at these sections. Assume that M_1

is greater than M_2. Figure 29*b* shows the internal stresses acting on these sections.

Let C_1 and C_2 represent the total stresses acting above a line *a-b* on the two sections, as shown in Fig. 29*b*, and assume, as for moments, that C_1 is greater than C_2. The shearing stress intensity on the area *a-b* is then equal to C_1-C_2 divided by the area on the line *a-b*, or

$$v_1 = \frac{C_1 - C_2}{bdx} \tag{2}$$

where v_1 = the horizontal shearing stress intensity on a section distance y above the neutral axis, and b and dx are the width and length of the area *a-b* of Fig. 29.

In general, the total stress above any line, as *a-b* may be found from the conditions shown in Fig. 29*d*. If C represents this total stress, we have

$$C = \Sigma_y^c f_z bdz$$

But

$$f_z = \frac{f}{c} z$$

Hence

$$C = \Sigma_y^c \frac{f}{c} zbdz$$

From eq. (6) of Art. **50***b*, $f = \frac{Mc}{I}$. Therefore

$$C = \frac{M}{I} \Sigma_y^c zbdz.$$

In this expression, the term $\Sigma_y^c zbdz$ is the *statical moment* taken about the neutral axis, of the area of the section above the surface *a-b*. Call this term Q.
Then

$$C = \frac{M}{I} Q$$

Values of C for the two sections shown in Fig. 29*b* may be determined by substituting proper values of M in the above equation, from which

$$C_1 = M_1 \frac{Q}{I} \text{ and } C_2 = M_2 \frac{Q}{I}$$

Placing these values of C_1 and C_2 in eq. (2), we have

$$v_1 = \left(\frac{M_1 - M_2}{dx}\right) \cdot \frac{Q}{bI}$$

If the two sections of Fig. 29*b* are taken an infinitesimal distance dx apart, we may write $M_1 - M_2 = dM$, where dM is the change in moment.
Then

$$v_1 = \frac{Q}{bI} \cdot \frac{dM}{dx}$$

From p. 17, $\frac{dM}{dx} = V$, where V = external shear on the section. Therefore

$$v_1 = \frac{QV}{bI} \tag{3}$$

Since we have shown in Art. **51***a* that the vertical and horizontal shearing stress intensities are equal, eq. (3) is a general expression for intensity of horizontal

and vertical shearing stress intensity on any section at a distance y from the neutral axis.

In eq. (3)

$v = v_1$ = intensity of horizontal or vertical shearing stress intensity at any surface distance y from the neutral axis.

Q = statical moment, taken about the neutral axis, of the area of the section outside the shear plane in question.

V = external vertical shear at the section in question.

b = width of beam section at the shear area in question.

I = moment of inertia of the beam section.

51c. Variation in Shearing Stress Across a Section.—The variation in shearing stress intensity for cross-sections in common use for beams will now be determined by means of eq. (3) of Art. 51b.

Rectangular Section.—Let it be required to find the general expression for the shearing stress intensity at a distance y above the neutral axis of the section shown in Fig. 30. Assume the external shear on the section to be V. For the dimensions shown on Fig. 30a, the general expression for Q, the statical moment of the area $abcd$ about the neutral axis, is

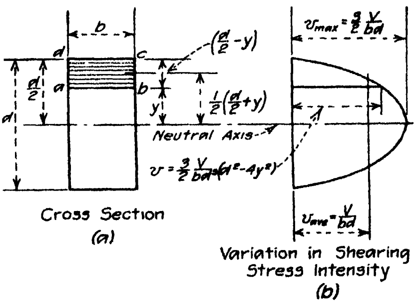


FIG. 30.

$$Q = b\left(\frac{d}{2} - y\right)\left[\frac{1}{2}\left(\frac{d}{2} + y\right)\right] = \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

It can be shown that the moment of inertia of a rectangle about the neutral axis is $I = \frac{bd^3}{12}$. Hence, from eq. (3)

$$v = \frac{\frac{b}{2}\left(\frac{d^2}{4} - y^2\right)}{b\left(\frac{bd^3}{12}\right)} V$$

from which

$$v = \frac{3}{2}\frac{V}{bd^3}(d^2 - 4y^2) \tag{4}$$

From analytical geometry it can be shown that this is the equation of a *parabola*. Figure 30b shows the curve as plotted, the origin being at point O.

At the top and bottom of the section, where $y = \frac{d}{2}$, we have $v = 0$; and at the neutral axis, where $y = 0$, we have

$$v = \frac{3}{2}\frac{V}{bd} \tag{5}$$

which is the maximum shearing stress intensity for the section. Note that the term bd in eq. (5) is the area of the section. Therefore, the shearing stress inten-

sity at the neutral axis, which is the maximum value, is $\frac{3}{2}$ of the average shearing stress intensity found by dividing the external shear by the area of the section.

I-Section.—The variation in shearing stress intensity will be studied for the typical *I*-section shown in Fig. 31*a*. In this section the rounded corners and sloping inner faces of the flanges have been replaced by parallel edges in order to simplify the discussion.

To determine the intensity of shearing stress on section *a-a* through the flange, substitute in eq. (3), Art. **51***b*, values of the statical moment, taken about the

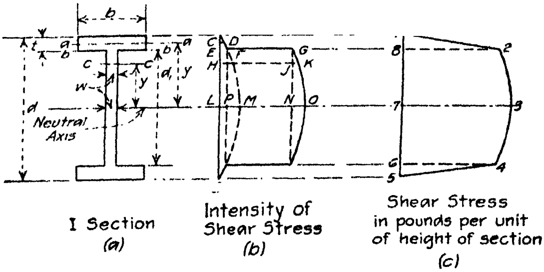


FIG. 31.

neutral axis, of the flange area above section *a-a*. From Fig. 31*a* the required statical moment is

$$Q = b\left(\frac{d}{2} - y\right)\left[y + \frac{1}{2}\left(\frac{d}{2} - y\right)\right] = \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

Then, assuming V and I as constants, we have from eq. (3),

$$v_a = \frac{V}{2I}\left(\frac{d^2}{4} - y^2\right) \tag{6}$$

At the lower edge of the flange, section *b-b* of Fig. 31*a*, where $y = \left(\frac{d}{2} - t\right)$, eq. (6) becomes

$$v_b = \frac{Vt}{2I}\left(d - t\right) \tag{7}$$

For a section *c-c* in the web, where the width of section is w, and the statical moment of the area of the beam section above *c-c* is

$$Q = \frac{bt}{2}(d - t) + \frac{w}{2}\left[\left(\frac{d}{2} - t\right)^2 - y^2\right]$$

the intensity of shearing stress at *c-c* is

$$v_b = \frac{V}{2I}\left\{\frac{bt}{w}\left(d - t\right) + \left[\left(\frac{d}{2} - t\right)^2 - y^2\right]\right\} \tag{8}$$

At section *b-b*, the under side of the flange, where $y = \left(\frac{d}{2} - t\right)$, eq. (8) becomes

$$v_b = \frac{V}{2I}\frac{bt}{w}(d - t) \tag{9}$$

Note that eq. (7) is taken an infinitesimal distance above section *b-b*, where the width of section is b, and that eq. (9) is taken an infinitesimal distance below section *b-b*, where the width of section is w, the web thickness. At the neutral axis, where $y = 0$, eq. (8) becomes

$$v_{N.A.} = \frac{V}{2I}\left[\frac{bt}{w}\left(d - t\right) + \left(\frac{d}{2} - t\right)^2\right] \tag{10}$$

Values given by eqs. (6) to (10), when plotted for special values of the several dimensions given on Fig. 31*a*, will form the shearing stress intensity diagram of Fig. 31*b*. Let *AB* represent a base line. The value of v_a from eq. (6) is represented by *CD*. Note that eq. (6) is the equation of a parabola. If the width of the section were constant, the resulting curve would be represented by the parabola *AMB*. The abscissa *EF* represents the value given by eq. (7) and *EG* represents the value given by eq. (9). Note the effect of the sudden change in the width of section at *b-b*. The abscissa *HK* represents the value given by eq. (8). It can be seen that the first part of the expression in brackets is exactly the same as a portion of eq. (9), while the last part of eq. (8) represents a parabola which is represented in Fig. 31*b* by the curve *GKO*. The curves *PFM* and *NGO* can be shown to be equal curves. At the neutral axis, the abscissa *LO* represents the value given by eq. (10). Figure 31*b* gives the complete curve, and represents the variation of shearing stress intensity across the whole section. Note that the stress intensity across the web is nearly uniform and greatly in excess of the shearing stress intensity for the flanges.

Figure 31*b* shows the distribution of shearing stress intensity. The distribution of the shear over the web may be studied by dividing the web and flanges into small vertical sections, determining the area of these sections, and multiplying each by the stress intensity shown in Fig. 31*b*. On plotting these values, the resulting curve will show the actual distribution of the shear across the web. Thus, at the neutral axis, consider a piece of the web whose height is unity. Since the thickness of the web is w, the area of this piece of web is w. On multiplying this area by the stress intensity given by eq. (10), and for other portions of the web area by eq. (8), we may plot the curve represented by 2-3-4 of Fig. 31*c*. For the flanges, a similar process gives the curves 1-2 and 4-5. The total area of the curve 1-2-3-4-5 is equal to the external shear on the section.

The curve of Fig. 31*c* shows that the amount of shear carried by the flanges is represented by the areas 1-2-8 and 4-5-6, while the shear carried by the web is represented by the area 2-4-6-8.

A general expression for the relative amount of shear carried by the flanges and the web may be determined by the process outlined above. Thus, at section *a-a* of the flange, the area of a strip of height dy is bdy. Equation (6) gives the stress intensity on this strip. Therefore, the stress on this strip is $v_a bdy = \frac{Vb}{2I}\left(\frac{d^2}{4} - y^2\right)dy$. The total shear stress on the top and bottom flange is then

$$V_f = 2\left[\frac{Vb}{2I} \quad \Sigma_{\left(\frac{d}{2} - t\right)}^{\frac{d}{2}} \left(\frac{d^2}{4} - y^2\right)dy\right] + \frac{Vbt^2}{6I}\left(3d - 2t\right) \tag{11}$$

where V_f = shear carried by the flanges.

To determine the amount of shear carried by the web, subtract the amount of shear carried by the flanges from the total. Thus, if V_w = shear carried by web, $V_w = V - V_f$, and

$$V_w = V\left[1 - \frac{bt^2}{6I}\left(3d - 2t\right)\right] \tag{12}$$

On substituting special values in eq. (12), the expression in brackets gives the percentage of the total shear carried by the web.

In practical designing it is generally assumed that the shearing stress intensity in the web is equal to the external shear divided by the web area, which is taken equal to the thickness of the web times the total depth of the beam. The results given in the illustrative problem at the end of this article for a section of the form given in Fig. 31 show that the maximum shearing stress intensity determined by the more exact method given in this article is about 16½ per cent greater than the value calculated on the above assumption.

It will now be shown that the maximum shearing stress intensity on the web is approximately equal to the external shear divided by the area of the web taken for the portion between the flanges. To derive this relation, approximate values of Q and I for the section will be used in eq. (3) of Art. **51***b*. As shown in Fig. 31*a*, let d_1 represent the depth of the web between flanges. Approximately, the moment of inertia of the section is $I = \frac{1}{12}wd_1^3 + 2bt\left(\frac{d_1}{2}\right)^2$. The approximation involved in this expression is that the distance from the gravity axis to the center of the flange is $\frac{d_1}{2}$ instead of its true value $\frac{1}{2}(d_1 + t)$. In the same manner, the statical moment of the area of the upper half of the section about the neutral axis is $Q = bt\frac{d_1}{2} + \frac{wd_1^2}{8}$. Substituting these values in eq. (3), noting that the width of the flange is w, we have

$$v_N = \frac{3}{wd_1}\left(\frac{2bt + \frac{wd_1}{2}}{6bt + wd_1}\right)V.$$

Making the further approximation that the expression in brackets is equal to ⅓, we have

$$v_N = \frac{V}{wd_1} \tag{13}$$

where v_N = shearing stress intensity at the neutral axis. Noting that wd_1 is the area of the web between flanges, we have a theoretical basis for the above assumption.

52. Limitations of the Ordinary Theory of Bending.—Figure 32*a* shows a simple beam with any set of applied loads. Consider a section *n-n* at which both moment and shear exist due to the applied loads. From Art. **50***b* the bending moment is resisted by bending stresses which act normal to the section, as shown in Fig. 32*b*, and Arts. **51***a* and **51***b* show that the external shear is resisted by shearing stresses acting parallel to the section and varying in amount from the edges to the center of the section, where they reach their maximum values. At any point in the section the existing stress is the resultant of the bending and shearing stresses. Figure 32*c* represents approximately the amount and direction of these internal stresses.

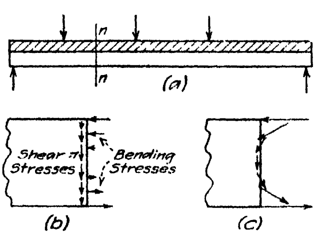


Fig. 32.

Since in most cases of bending, shear and moment exist at the same section and the nature of the internal stresses is as shown in Fig. 32*c*, it is evident that

the stress and deformation conditions in such sections are much more complicated than assumed in the preceding articles in the derivation of the flexure, and shearing stress formulas. St. Venant, a celebrated French mathematician, has shown that the assumptions made in deriving the flexure formulas of the preceding articles hold true only when the shearing force is constant. For the more exact theory developed by St. Venant, the reader is referred to his advanced works on the Theory of Elasticity.

For practically all cases of bending encountered in the design of engineering structures, it is sufficiently accurate, however, to assume independent action of the internal forces. The ordinary theory of bending is based upon this assumption. Since maximum moment occurs at the section for which the shear is zero, and also, since when the shear is large the bending moment is generally small, the ordinary theory of bending gives results in practice very close to those obtained by the exact theory. In these books, the ordinary theory of bending will be assumed to hold true.

53. Principal Stresses in the Web of a Beam.—When shear and moment fiber stresses exist on any section of a beam, as shown in Fig. 32*c*, the maximum fiber stress intensity will be a function of the resultant of the stress intensities due to bending and shear.

Let Fig. 33 represent any particle taken from the tension side of a beam, and let it be assumed that the stress conditions on this particle are as shown in Figs. 32*b* and *c*. These fiber stresses, which are shown in position on Fig. 33*a*, are the normal bending fiber stresses of intensity f acting on the vertical faces, and the shearing stresses of intensity v acting on the four faces and in the directions shown on the figure. Values of f and v to be used in Fig. 33*a* are given respectively by eq. (6) of Art. **50***b* and eq. (3) of Art. **51***b*.

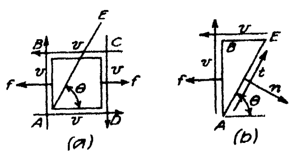


FIG. 33.

To determine the resultant stress intensity, consider any plane AE at an angle θ to the horizontal. Remove the portion of the particle to the left of plane AE, and represent all stress intensities, as shown in Fig. 33*b*. Let the internal stress conditions on plane AE be represented by a normal stress of intensity n and a tangential, or shear stress, of intensity t.

Resolving forces perpendicular to plane AE, we have

$$nAE - vAB\cos\theta - vBE\sin\theta - fAB\sin\theta = 0$$

Solving for n, noting that $\frac{AB}{AE} = \sin\theta$ and $\frac{BE}{AE}\cos\theta$, we have

$$n = 2v\sin\theta\cos\theta + f\sin^2\theta$$

Expressing $\sin\theta$ and $\cos\theta$ in terms of the double angle 2θ, we may write

$$n = v\sin 2\theta + \frac{f}{2}(1 - \cos 2\theta) \qquad \text{(A)}$$

Equation (A) gives the value of the normal stress on plane AE. By a similar process, it can be shown that the value of t, the tangential stress intensity on plane AE, is

$$t = v\cos 2\theta + \frac{f}{2}\sin 2\theta \qquad \text{(B)}$$

Equation (A) shows that the value of the normal stress intensity n depends upon the angle θ. By methods given in text books on the calculus, it can be shown that n is a maximum when θ has a value given by the equation

$$\tan 2\theta = -\frac{2v}{f} \tag{14}$$

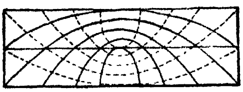
and that the maximum value of n is

$$n_{max} = \frac{f}{2} + \sqrt{\left(\frac{f}{2}\right)^2 + v^2} \tag{15}$$ [1]

On substituting values of $\sin 2\theta$ and $\cos 2\theta$, as given in the footnote, in eq. (B), it will be found that $t = 0$. That is, when the normal stress intensity n on plane AE of Fig. 33 reaches its maximum value, the tangential stress intensity t is equal to zero. Therefore the stress on plane AE is *entirely normal.* When the resultant stress intensity on any plane is normal to that plane, it is said to be the *principal stress intensity* for that plane. The plane on which a principal stress intensity occurs is said to be a *principal stress plane.* Thus n_{max} of eq. (15) is a principal stress intensity and AE of Fig. 33 is a principal stress plane when θ has the value given by eq. (14).

Equations (14) and (15) have been derived for a particle on the tension side of a beam. These equations may also be used where f is compressive by changing the sign of f to minus, and placing a minus sign in front of the radical in eq. (15). Since f under the radical is squared, the value of $\sqrt{\left(\frac{f}{2}\right)^2 + v^2}$ is the same for compression as for tension.

53a. Principal Stress Lines in the Web of a Rectangular Beam.—The amount and direction of the principal stresses vary for every section taken through a beam. Figure 34 shows the stress lines in a simple beam, supporting a uniform load. Note that at all sections on the neutral axis where shear exists, the principal stresses make angles of 45 deg. with the neutral axis.



——— *Lines of maximum compression*
------ *Lines of maximum tension*

Fig. 34.

These curves are interesting and prove especially instructive in the study of internal stresses in webs of beams composed of a material which is weak in tension but strong in compression—as for example, a concrete beam. The full

[1] *Derivations of Eqs.* (14) *and* (15).—To determine the value of θ given by eq. (14), place equal to zero the first derivative of n with respect to θ in eq. (A). Thus

$$\frac{dn}{d\theta} = 2v\cos 2\theta + f\sin 2\theta = 0$$

from which

$$\tan 2\theta = -\frac{2v}{f}$$

To derive eq. (15), we note from trigonometry that eq. (14) represents either a second or a fourth quadrant angle. On substituting values of 2θ from eq. (14) in the equation expressing the second derivative of n, eq. (A), with respect to θ, a negative value results. This can be shown, by methods given in text books on the calculus, to indicate a maximum value of n for second quadrant values of 2θ. From trigonometry, for second quadrant values of 2θ, we have

$$\sin 2\theta = +\frac{v}{\sqrt{\left(\frac{f}{2}\right)^2 + v^2}} \quad \text{and} \quad \cos 2\theta = -\frac{\frac{f}{2}}{\sqrt{\left(\frac{f}{2}\right)^2 + v^2}}$$

Substituting these values in eq (A),

$$n_{max} = \frac{f}{2} + \sqrt{\left(\frac{f}{2}\right)^2 + v^2}, \text{ as given by eq. (15).}$$

lines of Fig. 34 show the directions of principal tensile stress intensity, and the dotted lines show curves for principal compressive stress intensity. Note that these sets of curves cross at right angles. For beams in which the length is great compared to the depth, the curves of principal stress intensity are much flatter than for the conditions shown in Fig. 34.

53b. Principal Stresses in the Web of an I-Beam.—Where sudden changes occur in the thickness of the web of a beam, as in the case of an I-beam, it will often be found that the principal stress intensity, as given by eq. (15), is in excess of the extreme fiber stress in bending, as given by eq. (6) of Art. 50b. This is particularly true when heavy shear and moment exist on any section at the same time—as, for example, at the wall section in a cantilever beam. In such cases the intensity of stress at the junction of web and flanges requires careful consideration.

This matter was first called to the attention of engineers when it was noticed that I-beams under test loads showed signs of weakness at the point where the web joins the flanges. Later types of I-beams were rolled with larger fillets at the junction of flange and web. This change provides additional area at a weak point, thus reducing the principal stress intensity at these dangerous sections.

53c. Effect of Vertical Loads on the Principal Stresses in the Web.—In the preceding articles the effect of external vertical loads supported on the

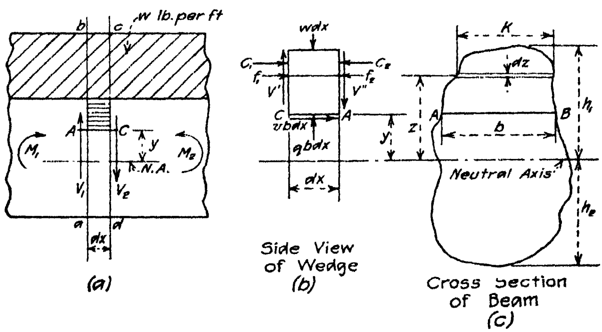


FIG. 35.

flanges of the beam has been neglected in the determination of the principa stresses in the web of the beam. Where heavy loads are supported by the top flange, the compressive principal stress intensity at the junction of the web and flange is considerably increased. If the loads are supported by the bottom flange, a corresponding increase takes place in the principal tensile stress intensity. An expression will now be derived for the intensity of stress at any point in the web of a beam due to vertical loads on the top flange.

Let Fig. 35a show a portion of a beam which supports a load of w lb. per unit of length. Consider two sections a-b and c-d at a distance dx apart, and let the moments and shears acting on these sections have the character shown by the arrows. Remove the portion of the section above the plane AC of Fig. 35a. Figure 35b represents this portion of the beam with all applied forces in position. These forces are a vertical downward load wdx; horizontal loads C_1 and C_2 which

represent the summation of total fiber stresses f_1 and f_2 above the plane AC; vertical forces V' and V'' which represent the total shearing stress on the vertical faces above the plane AC; a horizontal force on the plane AC representing the total horizontal shearing stress on that plane; and a vertical force $qbdx$ acting on the plane of AC, where q is the intensity of the vertical compressive stress in the web. The intensity of q will now be determined.

Placing a summation of vertical forces on Fig. 35b equal to zero, we have

$$+V' - V'' - wdx + qbdx = 0$$

from which

$$q = \frac{1}{bdx}\,[wdx - (V' - V'')]$$

The terms V' and V'' represent the total shear stresses above the plane AC, that is $V' = \Sigma_y^{h_1}\, v^z k dz$ for the dimensions shown in Fig. 35c. The value of v_z is given by eq. (3) of Art. **51**b in terms of V_1, the total external shear on section ab of Fig. 35a Let N represent the term involving the properties of the section. We may then write $V' = NV_1$ and $V'' = NV_2$, where V_2 = external shear on section cd of Fig. 35a. Noting that $V_1 - V_2$ = change in shear between sections ab and cd = wdx, the above expression for q may be written

$$q = \frac{1}{bdx}\,[wdq - N\,wdx]$$

or

$$q = \frac{w}{b}\,(1 - N) \tag{16}$$

The summation mentioned above leading to the value N is not readily accomplished for a general section of irregular outline. However, the value of N may be determined by placing moments about point A of Fig. 35b equal to zero. In making this moment summation, we may neglect the moments of forces wdx and $qbdx$, for they involve the term dx squared which is infinitely small when compared to the other terms. Also, the moments of C_1 and C_2 will be stated in terms of f_1 and f_2. We then have

$$+V'dx + \Sigma_y^{h} f_1 z(z-y)kdz - \Sigma_y^{h_1} f_2 z(z-y)\;kdz = 0$$

But

$$f_1 = \frac{M_1 z}{I},\; f_2 = \frac{M_2 z}{I},\;\text{and } V' = NV_1$$

Since $M_2 = M_1$, we may write

$$NV_1 = \frac{dM}{Idx}\left[\Sigma_y^{h_1} z^2 kdz - y\Sigma_y^{h_1} zkdz\right]$$

But $\dfrac{dM}{dx} = V_1$; $\Sigma_y^{h_1} z^2kdz$ = moment of inertia about neutral axis of area outside plane AB of Fig. 35c, which we will denote by I_y; and $\Sigma_y^{h_1} zkdz$ = statical moment about the neutral axis of the area outside plane AB, which will be denoted by Q_y. We then have

$$N = \frac{1}{I}\,(I_y - yQ_y)$$

On substituting this value of N in eq. (16), the general expression for stress intensity q is

$$q = \frac{w}{b}\left[1 - \frac{1}{I}(I_y - yQ_y)\right] \tag{17}$$

Equation (17) shows that the value of q varies across the section. At the top of the section, $I_y = 0$, $Q_y = 0$ and $q = \frac{w}{b}\cdot$ At the center of the section $I_y = \frac{1}{2} I$, $y = 0$ and $q = \frac{1}{2}\frac{w}{b}\cdot$ At the bottom of the section, $I_y = I$, $Q_y = 0$ and $q = 0$. Figure 36 gives the curve for q across the upper half of an I-beam web.

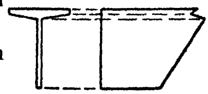


FIG. 36.

For a rectangle of width b and depth d, as shown in Fig. 37,

$$I_y = \frac{b}{24}\,(d^3 - 8y^3),\; Q_y = \frac{b}{2}\left(\frac{d^2}{4} - y^2\right),$$

and

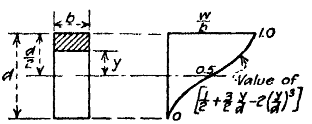
$$I = \frac{bd^3}{12}\cdot$$

Equation (17) then becomes

$$q = \frac{b}{w}\left[\frac{1}{2} + \frac{3}{2}\frac{y}{d} - 2\left(\frac{y}{d}\right)^3\right] \tag{18}$$

The curve shown in Fig. 37 represents the value of the term in brackets for various values of y.

When the load on the flange is concentrated instead of uniform, as assumed in the above analysis, the concentrated load may be reduced to a uniform load by assuming that it is uniformly distributed over a certain portion of the flange (see Arts. **51**c and **51**d, Sec. 2).



Curve showing variation in q across section

FIG. 37.

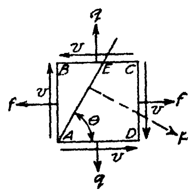


FIG. 38.

Figure 38 shows a particle on the tension side of the web of a beam which is acted upon by a vertical stress of intensity q in addition to the bending and shearing stresses considered in Art. **53.** The stress intensity q is determined from eq. (17). In Fig. 38 it has been assumed that q is a tensile stress.

Let p represent the intensity of principal stress on a plane AE at an angle θ with the horizontal. By methods similar to those employed in Art. **53,** it can be shown that

$$p_{max} = \frac{1}{2}(f + q) + \sqrt{\left(\frac{f - q}{2}\right)^2 + v^2} \tag{19}$$

and that the angle which the principal stress plane for p_{max} makes with the horizontal is given by the equation

$$\tan 2\theta = -\frac{2v}{f - q} \tag{20}$$

When p or q are compressive, negative values for these terms must be substituted in eqs. (19) and (20).

54. Plain Concrete Beams.—The first assumption in the common theory of flexure, as given in Art. **50**, may be applied directly to plain concrete and also to reinforced-concrete beams. Careful measurements seem to show some deviation from a plane, but in general this assumption seems to be warranted. From this fact it follows (as stated above) that deformations of the fibers are proportional to the distances of the fibers from the neutral axis. *OS* in Fig. 39 is the stress-deformation diagram for concrete in compression with the deformations represented vertically. The curve *OT* is the stress-deformation diagram for concrete in tension. For working loads the curves *OS* and *OT* do not vary materially from straight lines and the unit stresses in the fibers at any section of a plain concrete

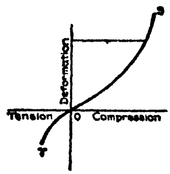


Fig. 39.

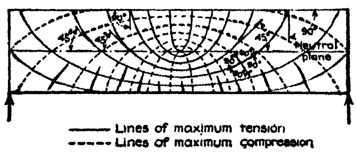


Fig. 40.

beam may thus be assumed to vary directly as the deformations and consequently as the distances of the fibers from the neutral axis. Hence, the common flexure formula for homogeneous beams applies when the loads are working loads. For ultimate loads, however, the formula does not strictly apply.

A plain concrete beam will fail by cracks opening up along the uneven lines which are shown in Fig. 40 on account of the low strength of concrete in tension. If concrete were only stronger in tension, then the plain concrete beam might be of some structural value. In order to offset this disadvantage of plain concrete, steel is used.

55. Purpose and Location of Steel Reinforcement in Concrete Beams.—Steel reinforcement should have the general directions shown in Fig. 41 in order

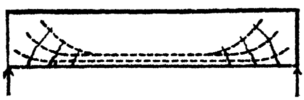


Fig. 41.

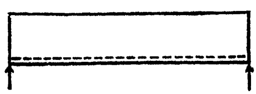


Fig. 42.

to take the tension in the beam and prevent the cracks starting along the lines indicated. Figure 42 is the simplest method of reinforcement and quite often used for light loads. In beams highly stressed, curved or inclined reinforcement is needed, in addition to the horizontal rods. The most common method is to use several bars for the horizontal reinforcement and then to bend up some of these at an angle of from 30 to 45 deg. as they approach the end of the beam and where they are not needed to resist bending stresses. The concrete is depended upon to take care of the compressive and pure shearing stresses, its resistance to such stresses being large.

56. Tensile Stress Lines in Reinforced-concrete Beams.—Lines of maximum tension in the concrete of reinforced-concrete beams are considerable inclined immediately above the line of the steel. The inclination of these lines is greater,

the greater the shear, and the less the horizontal tension. The inclination, therefore, increases toward the end of the beam. At points nearer the neutral plane, the horizontal tensile stresses become less and the inclined tension approaches the value of the shearing stress, while its inclinat on approaches 45 deg. Figure 43 is an attempt to represent roughly the general direction of the inclined tensile stresses in a simply supported beam uniformly loaded and with horizontal reinforcement.

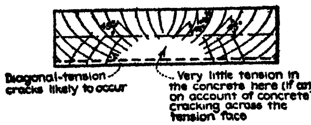


Fig. 43.

57. Flexure Formulas for Reinforced-concrete Beams.—A great many varieties of flexure formulas have been proposed from time to time to be used in the design of reinforced-concrete beams. As might be expected, many of the earlier formulas considered the concrete to carry its share of the tension which we know now cannot be done with safety. Only two classes of flexure formulas are at the present time in practical use. In each of these classes, tension in the concrete is neglected and a plane section before bending is assumed to be a plane after bending takes place.

The formulas almost universally used and made standard by the Joint Committee relate to working stresses and safe loads, and are based on the straight-line theory of stress distribution. The other formulas referred to above relate to ultimate strength and ultimate loads and the stress-deformation curve for concrete in compression is assumed to be a full parabola. Ultimate-load formulas are used to such a limited extent that they will not be considered here.

57*a*. Assumptions in Flexure Calculations.—The following assumptions are made in deriving the flexure formulas: (1) the adhesion of concrete to steel is perfect within the elastic limit of the steel; (2) no initial stresses are considered in either the concrete or the steel due to contraction or expansion; (3) the applied forces are parallel to each other and perpendicular to the neutral surface of the beam before bending; (4) sectional planes before bending remain plane surfaces after bending within the elastic limit of the steel; (5) no tension exists in the concrete; (6) modulus of elasticity of concrete is constant.

57*b*. Flexure Formulas for Working Loads.—Straight-line Theory.—The unit stress in the steel is within the elastic limit, and the unit tresses in the concrete at the given section of the beam are considered to vary as the ordinates to a straight line (see Fig. 44). Tension in the concrete is neglected. The formulas follow[1] (see *Notation, Appendix A*):

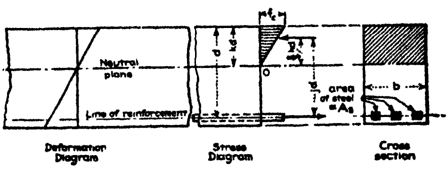


Fig. 44.

[1] The formulas may be derived as follows:

Total compressive resistance = total tensile resistance, or

$$\tfrac{1}{2} f_c k b d = A_s f_s \qquad (a)$$

From the assumption that deformations vary as the distances of the fibers from the natural axis and assuming stress proportional to deformation

$$\frac{f_c}{E_c k d} = \frac{f_s}{E_s d(1 - k)}$$

which reduces to

$$k = \sqrt{2pn + (pn)^2} - pn = \frac{1}{1 + \frac{f_s}{nf_c}} \tag{1}$$

$$j = 1 - \tfrac{1}{3}k \tag{2}$$

$$p = \frac{A_s}{bd} = \frac{\tfrac{1}{2}}{\frac{f_s}{f_c}\left(\frac{f_s}{nf_c} + 1\right)} = \frac{f_ck}{2f_s} \tag{3}$$

$$M_c = \tfrac{1}{2}f_ckj(bd^2), \text{ or } bd^2 = \frac{2M}{f_ckj}, \text{ or } f_c = \frac{2M}{kjbd^2} \tag{4}$$

$$M_s = pf_sj(bd^2), \text{ or } bd^2 = \frac{M}{pf_sj}, \text{ or } f_s = \frac{M}{A_sjd} \tag{5}$$

$$f_c = \frac{2f_sp}{k} \text{ or } \frac{f_sk}{n(1-k)} \tag{6}$$

The above formulas show that for a given ratio of $\frac{f_s}{f_c}$ p and k remain the same for all sizes of beams The formula for M_c gives the resisting moment when the maximum allowable value of f_c is introduced as the limiting factor and the formula for M_s gives the resisting moment when the maximum allowable value of f_s is the limiting factor. The lesser of these two resisting moments, when proper working values are assigned to f_c and f_s, is the safe resisting moment of the beam in question.

Unlike steel beams, reinforced-concrete beams require a preliminary formula to be solved before the formula for resisting moment may be employed. Solving this preliminary formula locates the position of the neutral axis which is in the same position only for beams of a given percentage of steel reinforcement.

The method of procedure in flexure formulas is to determine the vertical section of the beam where the moment is a maximum and apply the formulas at

---

$$f_s = f_cn\frac{1-k}{k}, \text{ or } f_c = \frac{f_sk}{n(1-k)}, \text{ or } \frac{1}{1 + \frac{f_s}{nf_c}} \tag{b}$$

The total resisting moment of the beam is the sum of the moments of the total compressive stresses and of the total tensile stresses about the neutral axis, or

$$\begin{aligned} M &= \tfrac{2}{3}kd(\tfrac{1}{2}f_ckbd) + d(1-k)A_sf_s \\ &= \tfrac{1}{3}f_ck^2bd^2 + A_sf_sd(1-k) \end{aligned} \tag{c}$$

Eliminating k between eq. (a) and (b), the following formula for steel ratio results:

$$p = \frac{\tfrac{1}{2}}{\frac{f_s}{f_c}\left(\frac{f_s}{nf_c} + 1\right)}$$

Introducing the value of f_s from eq. (b) into eq. (a), we have

$$\tfrac{1}{2}k^2bd - A_sn(1-k) = 0$$

or

$$\tfrac{1}{2}k^2b - pbn(1-k) = 0$$

from which

$$k = \sqrt{2pn + (pn)^2} - pn$$

Substituting the value of A_sf_s from (a) into (c), we get

$$M_c = \tfrac{1}{2}f_ck(1 - \tfrac{1}{3}k)bd^2$$

or

$$M_c = \tfrac{1}{2}f_ckjbd^2$$

Substituting the value of f_c from (a) into (c), and remembering that $A_s = pbd$

$$M_s = pf_sjbd^2$$

Equation (a) may be solved to give

$$f_c = \frac{2f_sp}{k}, \text{ or } p = \frac{f_ck}{2f_s}$$

that section. Either formula for p, containing the value of f_c and f_s, determines the amount of steel reinforcement which is needed to cause the beam to be of equal strength in tension and compression. The formulas for resisting moment determine the bending moment which a beam will safely withstand (for an existing structure) or the size of the beam needed to resist a given bending moment (for a proposed structure).

If a beam is over-reinforced, its resisting moment depends on M_c, and if under-reinforced on M_s.

If it is desired to find the fiber stresses in concrete and steel of a given beam, the formulas $f_s = \frac{M}{A_s jd}$ and $f_c = \frac{2M}{kjbd^2}$ $\left(\text{or } f_c = \frac{2f_s p}{k}\right)$ should be used, where M is the external bending moment in each case. For a given external M, either $bd_2 = \frac{2M}{f_c kj}$ or $bd_2 = \frac{M}{pf_s j}$ may be used to determine cross-section, when the p used is obtained from the formula $p = \frac{\frac{1}{2}}{\frac{f_s}{f_c}\left(\frac{f_s}{nf_c} + 1\right)}$, or from $p = \frac{f_c k}{2f_s}$, in which $k = \frac{1}{1 + \frac{f_s}{nf_c}}$

Illustrative Problem.—What will be the resisting moment (M) for a beam whose breadth (b) is 8 in. with a distance from the center of the reinforcement to the compression surface (d) of 12 in., the area of steel section being 0.96 sq. in.? Assume $n = 15$; $f_c = 650$ lb. per sq. in.; and $f_s = 16{,}000$ lb. per sq. in.

$$p = \frac{A_s}{bd} = \frac{0.96}{(8)(12)} = 0.01$$

From (1)

$$k = \sqrt{(2)(0.01)(15) + (0.01)^2(15)^2} - (0.01)(15) = 0.418$$

$$j = 0.861$$

From (4)

$$M_c = \tfrac{1}{2}(650)(0.418)(0.861)(8)(12)^2 = 134{,}700 \text{ in.-lb.}$$

From (5)

$$M_s = (0.01)(16{,}000)(0.861)(8)(12)^2 = 158{,}700 \text{ in.-lb.}$$

M_c is the lesser of the two resisting moments and hence controls in the design.

Illustrative Problem.—Assume the beam of the preceding problem to be 14 in. deep and subjected to a bending moment of 130,000 in.-lb. Compute the maximum unit stresses in the steel and concrete.

$$p = \frac{A_s}{bd} = \frac{0.96}{(8)(14)} = 0.0086$$

From (1)

$$k = \sqrt{(2)(0.0086)(15) + (0.0086)^2(15)^2} - 0.0086)(15) = 0.395$$

$$j = 0.868$$

From (4)

$$130{,}000 = \left(\frac{f_c}{2}\right)(0.395)(0.868)(8)(14)^2$$

$$f_c = 480 \text{ lb. per sq. in.}$$

From (5)

$$130{,}000 = (0.0086)(f_s)(0.868)(8)(14)^2$$

$$f_s = 11{,}100 \text{ lb. per sq. in.}$$

Illustrative Problem.—A beam is to be designed to withstand a bending moment of 300,000 in.-lb. and to have equal strength in tension and compression. A concrete will be

used with $E_c = 2{,}000{,}000$ and $f_c = 600$ lb. per sq. in. The pull in the steel is to be limited to 14,000 lb. per sq. in. Its modulus of elasticity E_s is 30,000,000.

$$n = \frac{E_s}{E_c} = 15 \qquad \frac{f_s}{f_c} = \frac{70}{3}$$

From (1) and (2)

$$k = \frac{1}{1 + \frac{14{,}000}{(15)(600)}} = 0.391 \text{ and } j = 0.870$$

From (3)

$$p = \frac{(600)(0.391)}{(21)(14{,}000)} = 0.0084$$

Either (4) or (5) may now be used in determining b and d since the amount of steel to be employed will cause simultaneous maximum working stresses.

From (5)

$$bd^2 = \frac{300{,}000}{(0.0084)(14{,}000)(0.870)} = 2{,}930$$

Many different values of b and d will satisfy the last equation. If b is taken as 10 in., then

$$d^2 = \frac{2{,}930}{10} = 293, \text{ or } d = 17\tfrac{1}{4} \text{ in.}$$

Finally

$$A_s = (0.0084)(10)(17.25) = 1.45 \text{ sq. in.}$$

If $1\frac{3}{4}$ in. is allowed between the tension surface of the concrete and the center of the steel, the entire depth of the beam should be 19 in.

58. Shearing Stresses in Reinforced-concrete Beams.—In Fig. 45 is shown a small portion of a concrete beam, so short that no appreciable portion of the load on the beam acts directly upon it. The opposing total compressive forces are denoted by C' and C; and the tension in the steel on each face by T' and T. The tension in the concrete may be neglected. Let V be the total shear on this small portion of the beam. From conditions of equilibrium, $C' = T'$ and $C = T$. The total horizontal shearing stress upon a horizontal section immediately above the steel is $T' = T$, and if b denotes the breadth of the beam and V the unit shear (horizontal or vertical) at any point between the neutral axis and the steel, then

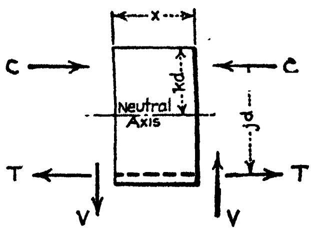


Fig. 45.

$$v = \frac{T' - T}{bx} \tag{1}$$

The various couples acting upon the element produce equilibrium; hence

$$Vx = (T' - T)jd$$

or

$$(T' - T) = \frac{Vx}{jd}$$

Substituting this value in eq. (1) there results

$$v = \frac{V}{bjd} \tag{2}$$

which is the value of shear intensity at any point between the neutral axis and the steel.

The value of j for working loads varies within narrow limits and v will change but slightly if the different values of j are inserted in eq. (2). The average value of j for beams in ordinary construction is $\frac{7}{8}$. Using this value, eq. (2) reduces to

$$v = \frac{8}{7} \cdot \frac{V}{bd} \qquad (3)$$

Shearing stress is the same at all points between the neutral axis and the steel, and above the neutral axis it follows the parabolic law. Figure 46 represents the distribution of shearing stress on a vertical cross-section assuming no tension in the concrete.

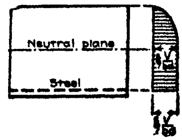


FIG. 46.

The longitudinal tension in the concrete near the end of beam modifies the distribution of the shear, increasing the shearing stress somewhat at the neutral axis and decreasing it at the level of the reinforcement. Equation (2), however, gives results which are sufficiently accurate and are derived for beams having the horizontal bars straight throughout. When any web reinforcement is used, the distribution and the amount of the shearing stresses at the end of a simply supported beam are materially different from the foregoing. The analysis of the stresses becomes more complex and a determination of their value impracticable. Even here, however, the above formula serves a useful purpose. It is found that shear is the chief factor in the failure of a beam by diagonal tension and either eq. (2) or eq. (3) may be used in design if properly controlled by the results of experiments.

Failure by the actual shearing of the concrete in a beam is not a likely occurrence under any conditions as the shearing strength of concrete is at least one-half the crushing strength.

59. Methods of Strengthening Reinforced-concrete Beams Against Failure in Diagonal Tension.—The intensity of the diagonal tensile stress at any point in a beam depends upon the shear and horizontal tension in the concrete, with shear as the chief factor. The percentage of horizontal reinforcement must also be considered, since the amount of steel employed affects the horizontal deformation and consequently the tension in the concrete. Thus beams may be strengthened against failure in diagonal tension by keeping the horizontal tension small through the use of considerable horizontal steel at points of heavy shear, by avoiding heavy shearing stresses, and by providing some type of web reinforcement. A low unit working stress in whatever type of web reinforcement is employed is also much to be preferred.

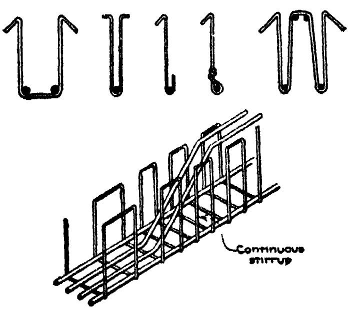


FIG. 47.

The most unfavorable part of a beam as regards diagonal tension is at points of excessive shear combined with considerable bending moment. A sufficient number of reinforcing rods should be extended horizontally to the ends of the beam to provide for bending with low unit stresses in the steel. In small beams, vertical stirrups looped about the horizontal rods may be employed throughout for web reinforcement but in large beams under heavy shearing stresses, both

stirrups and bent rods should be used. The stirrups in large beams should be securely fastened to the longitudinal rods in such a way as to prevent slipping of bar past the stirrup. Inclined web members may also be used in place of vertical stirrups if securely attached to the horizontal rods. Vertical stirrups may be made in various forms, as indicated in Fig. 47.

60. Bond Stress.—The tension in the horizontal steel near the lower surface of a reinforced-concrete beam is a maximum near the center of beam and decreases each way toward the end. The difference in the tension between any two points is transmitted to the concrete by the bond between the steel and the concrete.

A formula for bond may be derived for beams in which the reinforcement is horizontal or straight throughout. The total shearing stress per linear inch between the steel and the concrete, considering a length of beam equal to x, is

$$\frac{T' - T}{x}$$

From Fig. 45

$$Vx = (T' - T)jd$$

or

$$\frac{T' - T}{x} = \frac{V}{jd} \quad \text{(bond stress per linear inch)}$$

and the bond stress per square inch of the surface of the steel bars is $\frac{V}{jd}$ divided by the sum in inches of the circumference of the bars at the given vertical cross section. If u = unit bond stress, and Σo the total circumference of all bars in a beam at the given section, then

$$u = \frac{V}{\Sigma ojd}$$

The above formula shows that theoretically the bond stress is a simple function of the shear and varies with the shear. Thus, shear diagrams may be used to represent the variation of bond stress along a beam. When using the above formula, the average value of $j = \frac{7}{8}$ may be taken.

61. Web Reinforcement.—Inclined web reinforcement may be separate members firmly connected with the horizontal reinforcement to prevent slipping, or some of the horizontal bars may be bent up near the ends of the beam where they are not needed to resist bending. The vertical reinforcement may be used separately or in combination with inclined reinforcement, depending upon the preference of the designer and upon the amount of diagonal tension to be provided for. Vertical stirrups should be looped around the horizontal bars and in important beams should also be firmly secured to these bars by wiring or otherwise. Stirrups should usually be looped or hooked at the top in order to prevent slipping due to insufficient bond.

The proportioning of web reinforcement cannot be done with any degree of exactness since very little experimental work has been performed along this line. However, rough determinations of what is required may be obtained on rational grounds. The only information from tests is the value of the maximum shearing stress which measures diagonal tension failure—(1) for beams with horizontal bars only, and (2) for beams having an effective system of web reinforcement. Also, tests on beams, with and without web reinforcement, show that when

reinforcement is provided for diagonal tension, the concrete may be assumed to carry its full value of the shear and the steel the remainder.

Consider now Fig. 48, in which V represents the average total shear over the portion s of the beam. Let v' represent average unit horizontal shear on any plane below the neutral axis, A_v the total stress in the stirrup, and f_v the tensile stress in the stirrup. Then (see Art. **58**)

$$v' = \frac{V}{bjd}$$

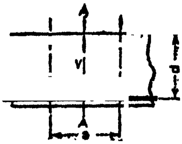


Fig. 48.

The total shear over any such horizontal plane is $v'bs$; whence

$$v'bs = \frac{Vs}{jd}$$

The function of stirrups, either vertical or inclined, is to resist by their tensile strength that portion of the above shearing stress which is not carried by the concrete.

Assume a vertical stirrup to be placed at the section A-A, and to oppose the shear over the portion of the beam. The total stress in the stirrup is $A_v f_v$ (in a U-shaped stirrup, A_v is the sum of the areas of the two legs), and it is produced by that part of the total shear over the horizontal plane bs not taken by the concrete. Let V' represent the shear carried by the web reinforcement. Then

$$A_v f_v = \frac{V's}{jd} \qquad (1)$$

Solving

$$A_v = \frac{V's}{f_v jd} \text{ (vertical stirrups)} \qquad (2)$$

or

$$s = \frac{A_v f_v jd}{V'} \qquad (3)$$

For inclined members and bent-up bars, the lines on a beam representing the direction in which the diagonal tensile cracks are likely to occur, are crossed more times per unit of length for a given horizontal spacing than would be the case if vertical stirrups were employed; that is, a given amount of inclined steel is much more effective in taking diagonal tension than the same amount of vertical steel. If α represents the angle between the web bars and longitudinal bars, then

$$A_v = \frac{V's \sin \alpha}{f_v jd} \qquad (4)$$

or

$$s = \frac{A_v F_v jd}{V' \sin \alpha} \qquad (5)$$

DEFLECTION OF BEAMS

By W. S. Kinne and Chas. A. Ellis[1]

62. Methods of Computing Deflection.—The structural engineer frequently desires to determine the deflection of a beam or girder at one or more points in its length. This in itself makes a study of deflections desirable. However, a more important use for the theory involved is its application to the analysis of statically indeterminate structures.

[1] The Area-Moment Method of Art. **63** contributed by Chas. A. Ellis.

There are several methods by which deflections caused by bending moments may be determined. The oldest and most widely known method, which was used by the mathematician Euler as early as 1744, is known as the *Elastic Curve Method*, and also as the *Double Integration Method.* In this method the change in slope of the tangent to the elastic curve is expressed as an ordinary differential equation of the second order in terms of the bending moment at any point. This equation must be integrated, the constants of integration determined, and the complete equation of the elastic curve derived before the deflection at a given point may be determined. The method is long and greatly involved, except for the simplest conditions of loading.

A simpler but less known method, which is called the *Area-Moment Method,* was derived independently by Prof. C. E. Greene of the University of Michigan, and Prof. Otto Mohr of Dresden, Germany. The methods derived by Profs. Greene and Mohr differ in certain respects, although both express the deflection at any point as a function of the moment due to a loading which is proportioned to the bending moment diagram for any given set of applied loads.

The Area-Moment Method, as derived by Prof. Greene, and as treated in this chapter, establishes a relation between the intercepts on a given axis of the tangents at adjacent points on the elastic curve and the moment about the given axis of the moment diagram area for the portion of the beam between the points at which the tangents are drawn.

The Area-Moment Method as derived by Prof. Mohr is based on the observed similarity between the deflection and slope diagrams for the given beam and the moment and shear diagrams for a similar, or properly chosen, beam due to loading which is proportional to the moment diagram for the loading causing the deflection. As first presented, Prof. Mohr's method was applicable only to simple beams. It was later extended by Prof. Mueller-Breslau so that it was applicable to any type of beam. Prof. Mohr's method, as presented in this chapter, is called the *Method of Elastic Weights.* This name is applied to the method due to the fact that in deriving the fundamental principles, use is made of an arbitrary weight which is a function of the elastic deformation of the beam elements.

Another useful method for the determination of deflections, which is known as the *Unit-Load Method,* was derived by Prof. Fraenkel. This method is based on the necessary condition that for elastic equilibrium the internal work due to fiber stresses and the external work done by the loads during deflection must be equal.

The Area-Moment and the Unit-Load methods are particularly useful when the deflection of a certain point is desired, without reference to the deflection at any other point. In this respect, these methods have an advantage over the Elastic Curve method, for as stated above, the complete equation of the elastic curve must be derived in this latter method before the deflection, or slope of the elastic curve, at any point may be determined. However, if so desired, the general equation of the elastic curve may also be determined by means of the Area-Moment and Unit-Load methods.

Prof. Greene's Area-Moment Method and the Method of Elastic Weights are treated in this chapter. The Elastic Curve and Unit-Load Methods are treated in Appendix *C.*

63. Area-Moment Method (Prof. Greene's Method)[1]

63a. Method in General.—Let A and B, Fig. 49, represent any two points on the neutral axis of a beam, which is bent by any arrangement of loading. Through A and B draw the tangnets AD and BC intersecting at C, and the normals AI and BI intersecting at I. Then $\angle AIB = \angle BCD = \phi$. Let $QPRS$ represent the bending moment diagram for the portion of the beam between A and B. Let $EFHG$ represent an element of the beam between two right sections EG and FH (drawn to a larger scale in Fig. 50) which were parallel

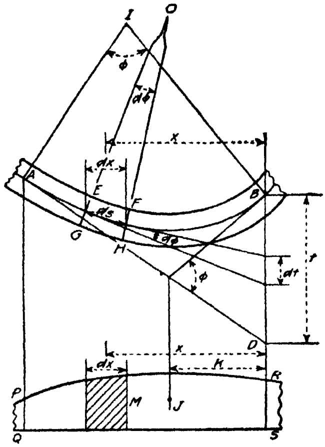


FIG. 49.

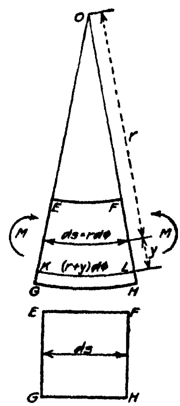


FIG. 50.

and a distance ds apart before the element was bent by the bending moment M. Let r, Fig. 50, represent the radius of curvature of the neutral axis for this element. The fiber at the neutral axis remains unchanged in length, while the fiber KL at a distance y below the neutral plane has been increased in length from $ds = rd\phi$ to $(r + y)d\phi$. Hence the total deformation in the length ds is $yd\phi$ and the unit deformation is $\frac{yd\phi}{ds}$. Let f represent the unit stress on the fiber KL; and let E represent the modulus of elasticity. Then

$$E = \frac{f}{\frac{yd\phi}{ds}} \text{ or } f = \frac{Eyd\phi}{ds}$$

Let I = moment of inertia of the cross section about the neutral plane. Then

$$f = \frac{My}{I}$$

whence

$$\frac{Eyd\phi}{ds} = \frac{My}{I}$$

or

$$d\phi = \frac{Mds}{EI}$$

and

$$\phi = \int_A^B d\phi = \int^B \frac{Mds}{EI}$$

[1] Contributed by Chas. A. Ellis.

If the beam in its natural state is straight (not arched) and is properly designed, the curvature will be so slight that ds may be replaced by dx, allowing the integration to be made horizontally between A and B instead of along the path of the elastic curve.
Then

$$\phi = \int_A^B \frac{Mdx}{EI}$$

If the beam is homogeneous and has a uniform cross section, E and I are constants, and the equation may be written thus:

$$= \frac{1}{EI} \int_A^B Mdx \tag{1}$$

The expression Mdx represents the area of the cross-hatched element in the bending moment diagram. Hence the integral expression $\int_A^B Mdx$ is the area of the M-diagram between the ordinates RS and PQ; and if this area is divided by EI, the quotient is the angle ϕ. If M is expressed in in.-lb., the area Mdx is expressed in in.^{2}-lb. If E is expressed in lb. per sq. in. and I in in.^{4}, then EI is also expressed in in.^{2}-lb., and the angle ϕ is a ratio. In any practical beam ϕ is comparatively very small; hence, when the tangent CB, Fig. 49, is horizontal, the ratio ϕ may be taken as the slope of the tangent AD. Likewise when AD is horizontal, the ratio ϕ may be taken as the slope of the tangent CB.

From this analysis the first principle may be deduced, namely: *If tangents are drawn through any two points on the elastic curve of a homogeneous beam of uniform cross-section, the angle which one tangent makes with the other tangent equals the area of the M-diagram between the two points, divided by EI.*

Now imagine that the unstrained position of the beam was in the direction AD, and that the beam was subsequently bent so that the point D moved to B, the point A remaining stationary. This movement is caused by the bending of all the elements from A to B. The bending of the element $EFGH$ causes the point, in its travel from D to B, to move a distance $dt = xd\phi$. Since the curvature is comparatively small, the path of the point moving from D to B deviates but slightly from the straight line DB. Hence

$$t = \int_A^B dt = \int_A^B xd\phi = \frac{1}{EI} \int_A^B Mxdx \tag{2}$$

The distance $DB = t$ is called the tangential deviation; since it represents the distance through which the point B has been displaced by the curvature of the beam, when AD is assumed as the original position.

In eq. (1) and (2), I is the *gross* moment of inertia of the cross-section. No deductions are made for holes, as is the case when the strength of a beam is being computed.

The expression $Mxdx$ represents the moment of the elemental area Mdx about the ordinate through B. Hence the integral expression $\int_A^B Mxdx$ represents the moment of the area $QPRS$ about the ordinate through B, and is called the *area-moment* of $QPRS$ about B. The area-moment is expressed in in.^{3}-lb., when M is expressed in in.-lb. Since EI is expressed in in.^{2}-lb., the tangential deviation t is expressed in inches. The second principle may now be stated: *If the tangent to the elastic curve is drawn through any point A, the tangential deviation at any other*

point B may be obtained by finding the area of the M-diagram between ordinates through A and B, and dividing by EI the moment of this area about the ordinate through B.

Let J represent the centroid of the area $QPRS$, and let k be the distance from J to the ordinate through B. Then

$$t = \frac{\text{area } QPRS}{EI} k$$

Since, from the first principle

$$\frac{\text{area } QPRS}{EI} = \phi$$

then

$$t = k\phi$$

Hence the tangents to the elastic curve at any two points A and B intersect on the ordinate through the centroid of the M-diagram included between the ordinates through A and B.

63b. Application of the Method.—*Beam with Single Concentrated Load (Determination of Deflection Under the Load).*—The beam in Fig. 51 is a 2 × 1-in. piece of wood laid flatwise. $I = \frac{1}{6}$ in.^{4} $E = 1{,}500{,}000$ lb. per sq. in. Hence EI

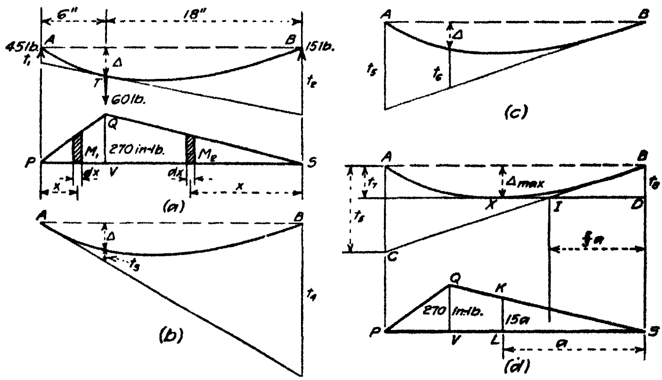


Fig. 51.

= 250,000 in.^{2}-lb. The M-diagram is PQS. The deflection Δ under the load will be determined in several ways, by drawing the tangent to the elastic curve through different points as shown in Figs. 51*a*, 51*b* and 51*c*. Considerable time and labor may be saved by exercising good judgment in choosing the most advantageous point in the elastic curve through which the tangent is to be drawn.

In Fig. 51*a*, the tangent to the elastic curve ATB is drawn through T. The deflection Δ is readily found after the tangential deviations t_1 and t_2 have been computed. From the second principle,

$$t_1 = \frac{1}{EI}\int_0^6 M_1 x dx$$

where M_1 is the bending moment at any distance x from the ordinate on which the tangential deviation is required. Hence, $M_1 = 45x$, and

$$t_1 = \frac{1}{EI}\int_0^6 45x^2 dx = \frac{3{,}240}{EI}$$

The origin for t_2 is at B; hence $M_2 = 15x$, and

$$t_2 = \frac{1}{EI}\int_0^{18} M_2 x\,dx = \frac{29{,}160}{EI}$$

From similar triangles in Fig. 51a

$$\Delta = t_1 + \frac{6}{24}(t_2 - t_1) = \frac{9{,}720}{EI} = \frac{9{,}720}{250{,}000} = 0.039 \text{ in.}$$

When the M-diagram can conveniently be divided into portions whose areas and centroids are easily found, a semigraphic or geometric solution can readily be made. The area of the M-diagram to be considered in each case is included between two ordinates. One ordinate passes through the point of tangency, on the other ordinate the tangential deviation is found; the moment of this area is taken about the latter ordinate.

The expression $\int_0^6 M_1 x\,dx$ represents the area-moment of PQV about P. Hence

$$t = \frac{1}{EI}(270)(3)(4) = \frac{3{,}240}{EI}$$

Likewise, the expression $\int_0^{18} M_2 x\,dx$ represents the area-moment of SQV about S. Hence

$$t_2 = \frac{1}{EI}(270)(9)(12) = \frac{29{,}160}{EI}$$

In Fig. 51b the tangent is drawn through A. The area-moment for t_4 is PQS about S; and for t_3, the area-moment is PQV about QV. The geometrical solution is as follows:

$$t_4 = \frac{1}{EI}\begin{bmatrix} (270)(9) & (12) = 29{,}160 \\ (270)(3) & (18+2) = 16{,}200 \end{bmatrix}^{+} \quad \frac{45{,}360}{EI}$$

$$t_3 = \frac{1}{EI}(270)(3)(2) = \frac{1{,}620}{EI}$$

From similar triangles in Fig. 51b,

$$\Delta + t_3 = \frac{t_4}{4} = \frac{11{,}340}{EI}$$

and

$$\Delta = \frac{11{,}340 - 1{,}620}{EI} = \frac{9{,}720}{EI} \text{ (as before)}$$

In the algebraic solution, the origin for t_4 is at S. $M = 15x$ for values of x between 0 and 18, and $M = 15x - 60(x - 18) = 1{,}080 - 45x$, for values of x between 18 and 24. Hence

$$t_4 = \frac{1}{EI}\int_0^{24} Mx\,dx = \frac{1}{EI}\int_0^{18} 15x^2\,dx + \frac{1}{EI}\int_{18}^{24}(1{,}080x - 45x^2)\,dx$$
$$= \frac{29{,}160 + 16{,}200}{EI} = \frac{45{,}360}{EI}$$

The origin for t_3 is at V. Hence, $M = 45(6 - x)$, and

$$t_3 = \frac{45}{EI}\int_0^6 (6x - x^2)\,dx = \frac{1{,}620}{EI}$$

The geometric solution is considerably shorter when M is not a continuous function of x as in the case of t_4.

In Fig. 51c the tangent is drawn through B. Hence

$$t_5 = \frac{1}{EI}\begin{bmatrix}(270)(3) & (4) = 3{,}240 \\ (270)(9) & + \ (6+6) = 29{,}160\end{bmatrix} \qquad \frac{32{,}400}{EI}$$

$$t_6 = \frac{1}{EI}(270)(9)(6) = \frac{14{,}580}{EI}$$

$$\Delta + t_6 = \frac{3}{4}t_5 = \frac{24{,}300}{EI}$$

$$\Delta = \frac{24{,}300 - 14{,}580}{EI} = \frac{9{,}720}{EI} \text{ (as before)}$$

Beam with Single Concentrated Load (Determination of Maximum Deflection). Let X, Fig. 51d, represent the point of maximum deflection. Since the tangent through X is horizontal, $\Delta_{max} = t_7 = t_8$. Let KL be the ordinate in the M-diagram at the point of maximum deflection, and let $LS = a$. Then $KL = 15a$. Let ϕ represent the angle which the tangent through B makes with the horizontal tangent through X, then

$$\angle ABC = \angle BID = \phi$$

$$24\phi = t_5 = \text{moment area } PQS \text{ about } P$$

Hence

$$\phi = \frac{t_5}{24} = \frac{1{,}350}{EI}$$

Also

$$\phi = \frac{\text{area } KLS}{EI} = \frac{7.5\,a^2}{EI}$$

Hence

$$7.5\,a^2 = 1{,}350$$

and

$$a = 13.42$$

Since the centroid of the triangular area KLS is on the ordinate through I,

$$ID = \frac{2}{3}a$$

$$t_8 = \frac{2}{3}a\phi = \left(\frac{2a}{3}\right)\left(\frac{1{,}350}{EI}\right) = \frac{900a}{EI} = \frac{12{,}078}{EI} = 0.048 \text{ in.}$$

The value of t_8 may also be determined as follows:

$$t_8 = \begin{pmatrix}\text{area-moment of} \\ KLS \text{ about } S\end{pmatrix} \div EI = \frac{5a^3}{EI} = \frac{12{,}078}{EI}$$

Since $EI = 250{,}000$, we have

$$t_8 = \frac{12{,}078}{250{,}000} = 0.048 \text{ in. (as before)}$$

The distance a might also be found by equating the values of t_7 and t_8 without any reference to the tangent through B.

Beam with Single Concentrated Load (Derivation of Equation of Elastic Curve). The general expression will be developed for the deflection of a simple beam l inches long when supporting a single concentrated load of P pounds, at any dis-

tance kl from the left support, (Figs. 52 and 53). The deflection Δ is found at T, a distance cl from the left support,

When c < k

In Fig. 52 the tangent CD is drawn through T.

$$EIt_1 = \text{area moment of } PKL \text{ about } P = c(1-k)Pl\left(\frac{cl}{2}\right)\left(\frac{2}{3}cl\right)$$

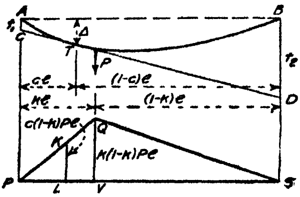


Fig. 52.

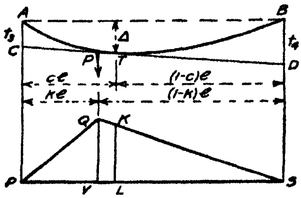


Fig. 53.

EIt_2 equals the area-moment of $KQSL$ about S which is the area-moment of PQS about S, minus the area-moment of PKL about S.

$$EIt_2 = k(1-k)Pl\left(\frac{l}{2}\right)\frac{2}{3}\left(l - \frac{kl}{2}\right) - c(1-k)Pl\left(\frac{cl}{2}\right)\left(l - \frac{2}{3}cl\right)$$

Simplifying these expressions, we have

$$t_1 = \frac{Pl^3}{6EI}(1-k)2c^3$$

and

$$t_2 = \frac{Pl^3}{6EI}(1-k)(2k - k^2 - 3c^2 + 2c^3)$$

By similar triangles,

$$\Delta = t_1 + c(t_2 - t_1)$$

Substituting values of t_1 and t_2, we have

$$\Delta = \frac{Pl^3c}{6EI}[k(2-k) - c^2](1-k) \qquad (3)$$

When c > k

In Fig. 53 the tangent CD is drawn through T as before.

$$t_3 = \frac{Pl^3}{6EI}k(3c^2 - 2c^3 - k^2)$$

$$t_4 = \frac{Pl^3}{6EI}2k(1-c)^3$$

$$\Delta = t_3 + c(t_4 - t_3)$$

$$\Delta = \frac{Pl^3k}{6EI}[c(2-c) - k^2](1-c) \qquad (4)$$

The deflection at the load may be obtained from either eq. (3) or (4). Since $c = k$ for this condition, either equation reduces to

$$\Delta = \frac{Pl^3}{3EI}k^2(1-k)^2$$

The maximum deflection occurs in the longer segment of Fig. 53 where c is greater than k, and at the point where the tangent through T is horizontal; hence

the value of c for Δ_{max} may be found by equating the expressions for t_3 and t_4, whence

$$c = 1 - \sqrt{\frac{(1 - k^2)}{3}}$$

Since the limits of k are 0 and 0.5, all values of c for maximum deflection will fall between 0.4227 and 0.5. Hence the point of maximum deflection for a single load is between the load and the center of the span, and always relatively near the center. The most eccentric loading which a simple beam of uniform cross-section and span l can experience, occurs when a single load is adjacent to one of the supports, and k is on the point of becoming zero. Under this condition the point of maximum deflection cannot be at a distance greater than $0.0773l$ from the center of the span. Any second load applied to the beam must necessarily throw the point of maximum deflection nearer the center. Hence the point of maximum deflection of a simple beam of uniform cross-section, loaded in any manner, will be near the center and not more than 0.0773 of its length from the center.

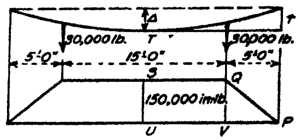


Fig. 54.

A 20-in. 65.4-lb. I-beam supports two loads of 30,000 lb. each, Fig. 54. Since the loads are symmetrically placed, the elastic curve and M-diagram are symmetrical about the center. The tangent to the elastic curve at the center, drawn through T, is horizontal and $\Delta = t$; $E =$ 29,000,000 lb. per sq. in., $I = 1169.5$ in.^{4}

Hence

$$EI = 33{,}915{,}500{,}000 \text{ in.}^2\text{-lb.}$$

Area-moment of PQV about P is

$$150{,}000(2.5)(3.33) = 1{,}250{,}000$$

Area-moment of $SUVQ$ about P is

$$150{,}000(7.5)(8.75) = 9{,}843{,}750$$

Area-moment of $PQSU$ about $P = 11{,}093{,}750$ ft.^{3}-lb.

$$\Delta = t = \frac{11{,}093{,}750\ (1{,}728)}{33{,}915{,}500{,}000} = 0.565 \text{ in.}$$

When the length of a beam is expressed in feet, and the loads are expressed in pounds, the area-moment will be expressed in ft.^{3}-lb. and the factor 1,728 is introduced if EI is expressed in in.^{2}-lb.

EI may be expressed in ft.^{2}-lb. by dividing by 144, whence

$$EI = 235{,}521 \text{ ft.}^2\text{-lb.}$$

Then

$$\Delta = t = \frac{11{,}093{,}750 \text{ ft.}^3\text{-lb.}}{235{,}521 \text{ ft.}^2\text{-lb.}} = 0.0471 \text{ ft.} = 0.565 \text{ in.}$$

Beam with Uniform Load.—The beam in Fig. 55 supports a uniform load and the M-diagram PQS is a parabola. The maximum deflection is at the center of the span. The tangent to the elastic curve at C is horizontal, and $\Delta = t$. $EIt =$ the area-moment of SQV about S. The area SQV is two-thirds the area

of the rectangle $QRSV$, and the centroid of the area SQV is five-eighths of VS, hence

$$\Delta = t = \frac{1}{EI}\,(19{,}200)(8)\left(\frac{2}{3}\right)(5)(1{,}728) = \frac{884{,}736{,}000}{EI}$$

Equation of Elastic Curve, Beam with Uniform Load.—In Fig. 56 the span is

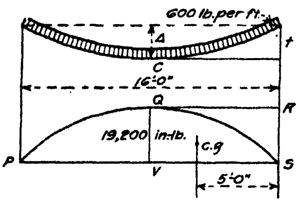


FIG. 55.

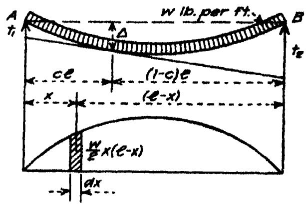


FIG. 56.

l in., and the uniform load is w lb. per in. The deflection at any distance cl from A will be foumd. At any distance x from either support, $M = \frac{w}{2}x(l - x)$,

$$t_1 = \frac{1}{EI}\int_0^{cl} Mx\,dx = \frac{wl^4}{24}\,(4c^3 - 3c^4)$$

$$t_2 = \frac{1}{EI}\int_0^{(1-c)l} Mx\,dx = \frac{wl^4}{24}\,(1 - 6c^2 + 8c^3 - 3c^4)$$

$$\Delta = t_1 + c(t_2 - t_1)$$

$$\Delta = \frac{wl^4}{24EI}\,(c - 2c^3 + c^4) = \frac{wl^4}{24EI}\,c(1 - c)(1 + c - c^2)$$

Let W = the total uniform load, when $W = wl$. Then

$$\Delta = \frac{Wl^3}{24EI}\,(c - 2c^3 + c^4) = \frac{Wl^3}{24EI}c(1 - c)(1 + c - c^2) \qquad (6)$$

The formula for deflection at any point may be determined from eq. (6) by substituting the proper value of c. Thus, at the quarter point, where $c = \frac{1}{4}$, we have

$$\Delta = \frac{19}{2{,}048}\,\frac{Wl^3}{EI}$$

At the beam center, where $c = \frac{1}{2}$, the deflection is a maximum, and from eq. (6),

$$\Delta = \frac{5}{384}\,\frac{Wl^3}{EI}$$

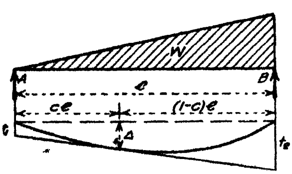


FIG. 57.

Beam with Load of Uniformly Varying Intensity.—The beam in Fig. 57 supports a load of uniformly varying intensity. The total load is W lb., and the length of the beam is l in. The bending moment at any distance x from A is

$$M_1 = \frac{W}{3l^2}\,(l^2x - x^3)$$

The bending moment at any distance x from B is

$$M_2 = \frac{W}{3l^2}(2l^2x - 3lx^2 + x^3)$$

$$t_1 = \frac{1}{EI}\int_0^{cl} M_1 x\,dx = \frac{Wl^3}{180EI}(20c^3 - 12c^5)$$

$$t_2 = \frac{1}{EI}\int_0^{(1-c)l} M_2 x\,dx = \frac{Wl^3}{180EI}(7 - 30c^2 + 20c^3 + 15c^4 - 12c^5)$$

$$\Delta = t_1 + c(t_2 - t_1) = \frac{Wl^3}{180EI}(7c - 10c^3 + 3c^5)$$

The value of c for Δ_{max} may be found by equating t_1 and t_2, whence

$$15c^4 - 30c^2 = -7$$

$$c = 0.519$$

$$\Delta_{max} = \frac{Wl^3}{180EI}(2.348) = 0.0131\frac{Wl^3}{EI}$$

The intensity of the load in Fig. 58 increases uniformly from each support to the center of the span. The total load is W lb., and the length of the beam is l in. The bending moment at any distance x between the end and center is

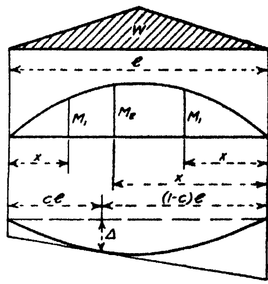


Fig. 58.

$$M_1 = \frac{W}{6l^2}(3l^2x - 4x^3)$$

The bending moment at any distance x, when x is greater than ½ l, is

$$M_2 = \frac{W}{6l^2}(-l^3 + 9l^2x - 12lx^2 + 4x^3)$$

$$t_1 = \frac{1}{EI}\int^{cl} M_1 x\,dx = \frac{Wl^3}{60EI}(10c^3 - 8c^5)$$

$$t_2 = \frac{1}{EI}\int_0^{\frac{l}{2}} M_1 x\,dx = \frac{1}{EI}\int_{\frac{l}{2}}^{(1-c)l} M_2 x\,dx$$

$$= \frac{Wl^3}{60EI}\left(\frac{25}{8} - 15c^2 + 10c^3 + 10c^4 - 8c^5\right)$$

$$\Delta = t_1 + c(t_2 - t_1) = \frac{Wl^3}{480EI}(25c - 40c^3 + 16c^5)$$

in which c may have any value between 0 and ½. The value of c for Δ_{max} may be found by equating t_1 and t_2, whence,

$$10c^4 - 15c^2 = -\frac{25}{8}$$

$$c = \frac{1}{2}$$

$$\Delta_{max} = \frac{Wl^3}{60EI}$$

Beam with Several Concentrated Loads.—A 24-in. 79.9-lb. I-beam supports three loads (Fig. 59). The linear dimensions of the beam are expressed in feet and the units in which E and I are usually expressed will be changed accordingly.

$I = 2{,}087$ in.^{4} $E = 29{,}000{,}000$ lb. per sq. in. Hence $EI = 60{,}523{,}000{,}000$ in.^{2}-lb. or 420,300,000 ft.^{2}-lb. The tangent is drawn through C at the center of the span. The deflection due to the weight of the beam will be considered later.

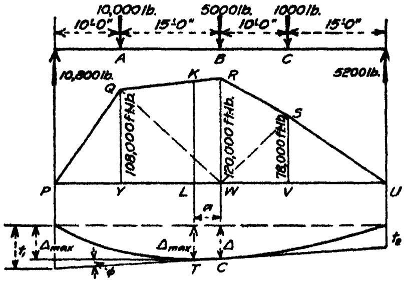


FIG. 59.

Area-moment about P

Area PQY	108,000 (5) (6.67)	=	3,600,000
Area YQW	108,000 (7.5) (15)	=	12,150,000
Area QRW	120,000 (7.5) (20)	=	18,000,000
			33,750,000 ft.^{3}-lb.

$$t_1 = \frac{33{,}750{,}000}{420{,}300{,}000} = 0.0803 \text{ ft.}$$

Area-moment about U

Area USV	78,000 (7.5) (10)	=	5,850,000
Area SVW	78,000 (5) (18.33)	=	7,150,000
Area SRW	120,000 (5) (21.67)	=	13,000,000
			26,000,000 ft.^{3}-lb.

$$t_2 = \frac{26{,}000{,}000}{420{,}400{,}000} = 0.0618 \text{ ft.}$$

$$\Delta = \frac{t_1 + t_2}{2} = 0.071 \text{ ft.} = 0.85 \text{ in.}$$

The maximum deflection caused by the three loads is at T, where the tangent is horizontal, and the ordinate in the M-diagram is KL. Let $LW = a$ and let ϕ be the angle made by the two tangents; then ϕ represents the slope of the tangent through C. The beam is 50 ft. long; hence

$$\phi = \frac{t_1 - t_2}{50} = \frac{0.0185}{50} = 0.00037$$

also

$$\phi = \frac{\text{area } KRWL}{EI}$$

Thus

$$\text{area } KRWL = \phi EI = 155{,}000 \text{ ft.}^2\text{-lb.}$$

and

$$a = 1.3 \text{ ft.}$$

The centroid of the area $KRWL$ is approximately 24.35 ft. from P, and the area-moment of $KRWL$ about P is

$$155{,}000\ (24.35) = 3{,}774{,}000 \text{ ft.}^3\text{-lb.}$$

$$\Delta_{max} = t_1 - \frac{3{,}774{,}000}{EI} = \frac{29{,}976{,}000}{EI}$$

$$\Delta_{max} = \frac{29{,}976{,}000}{420{,}300{,}000} = 0.0713 \text{ ft.} = 0.86 \text{ in.}$$

Although the loads are eccentric, it is clear that there is practically no difference between the deflection at the center and the maximum deflection.

The deflection at the center may be found from eq. (3), p. 77. The coefficients F[1] are given in Table I, p. 79. $c = 0.5$; $k = 0.2$ for the load at A; 0.5, for the load at B; and 0.3, for the load at C. Hence

$$\frac{10{,}000(50)^3}{6}(0.071) = 14{,}792{,}000$$

$$\frac{5{,}000(50)^3}{6}(0.125) = 13{,}021{,}000$$

$$\frac{1{,}000(50)^3}{6}(0.099) = 2{,}062{,}000$$

$$29{,}875{,}000 \text{ ft.}^3\text{-lb.}$$

$$\Delta = \frac{29{,}875{,}000}{420{,}300{,}000} = 0.071 = 0.85 \text{ in.}$$

The deflection at the center, due to the weight of the beam, may be found from eq. (7), p. 80. The coefficient J[1] when $c = 0.5$ is 0.3125. $W = 50\ (79.9) = 4{,}000$.

$$\Delta = \frac{4{,}000(50)^3}{24(420{,}300{,}000)}(0.3125) = 0.0155 \text{ ft.} = 0.19 \text{ in.}$$

The total deflection at the center is $0.85 + 0.19 = 1.04$ in., which in this case may be assumed without appreciable error as the maximum deflection. A deflection of $\frac{1}{360}$ of the span is not considered excessive.

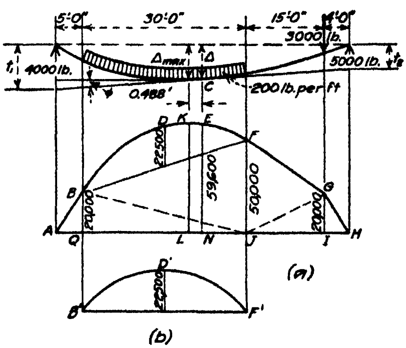


Fig. 60.

Beam with Partial Uniform Load. In Fig. 60a, the tangent is drawn through C at the center of the span. Assume that EI is expressed in ft.^{2}-lb. The M-diagram under the uniform load cannot be accurately divided into triangles, and an integration is necessary if an accurate solution is desired. A sufficiently accurate solution for all practical purposes may be obtained by the geometric process of dividing the area $QBDFJ$ by vertical ordinates into strips, so narrow that their areas may be considered trapezoidal. The accurate method by integration is given below. Let M_1 represent the bending moment under the

[1] See Art. 65.

uniform load at any distance x from the left support; and M_2 the bending moment under the uniform load at any distance x from the right support. Then

$$M_1 = -100x^2 + 5{,}000x - 2{,}500$$

and

$$M_2 = -100x^2 + 5{,}800x - 24{,}100$$

Area-moment about A

$$\begin{aligned}
\text{Area } ABQ \quad 20{,}000\ (2.5)\ (3.33) &= 166{,}667 \\
\text{Area } BENQ \quad \int_5^{27} M_1 x\,dx &= \underline{18{,}446{,}266} \\
& 18{,}612{,}933 \text{ ft.}^3\text{-lb.} \\
t_1 &= \frac{18{,}612{,}933}{EI}\text{ ft.}
\end{aligned}$$

Area-moment about H

$$\begin{aligned}
\text{Area } GHI \quad 20{,}000\ (2)\ (2.67) &= \phantom{1{,}}106{,}667 \\
\text{Area } GIJ \quad 20{,}000\ (7.5)\ (9) &= 1{,}350{,}000 \\
\text{Area } FGJ \quad 50{,}000\ (7.5)\ (14) &= 5{,}250{,}000 \\
\text{Area } FENJ \quad \int_{19}^{27} M_2 x\,dx &= \underline{10{,}332{,}600} \\
& 17{,}039{,}267 \text{ ft.}^3\text{-lb.}
\end{aligned}$$

$$t_2 = \frac{17{,}039{,}267}{EI}\text{ ft.}$$

The deflection at the center is

$$\Delta = \frac{t_1 + t_2}{2} = \frac{17{,}826{,}100}{EI}\text{ ft.}$$

The slope of the tangent is

$$\phi = \frac{t_1 - t_2}{54} = \frac{29{,}142}{EI}$$

Let KL represent the ordinate in the M-diagram at the point of maximum deflection, then

$$\phi = \frac{\text{area } KENL}{EI}$$

Therefore

$$\text{area } KENL = 29{,}142$$

Whence

$$LN = 0.488 \text{ ft.}$$

The area-moment of $KENL$ about A is

$$29{,}142\ (26.756) = 779{,}723$$

$$\Delta_{max} = t_1 - \frac{779{,}723}{EI} = \frac{17{,}833{,}210}{EI}$$

The area-moment of $KENL$ about H is

$$29{,}142\ (27.244) = 793{,}945$$

$$\Delta_{max} = t_2 + \frac{793{,}945}{EI} = \frac{17{,}833{,}212}{EI}$$

When the tangent is drawn to the elastic curve at the right end of the uniform load, the tangential deviations t_3 and t_4 at the left and right supports respectively may be determined by the geometric process; for if a straight line be drawn from

B to F, the area of the M-diagram BDF has all the properties of the M-diagram $B'D'F'$ for a beam 30 ft. long when uniformly loaded with 200 lb. per ft. (Fig. 60b).

Area-moment about A

Area ABQ	20,000 (2.5) (3.33)	=	166,667
Area BQJ	20,000 (15) (15)	=	4,500,000
Area FBJ	50,000 (15) (25)	=	18,750,000
Area BDF	22,500 (30) (⅔) (20)	=	9,000,000
			32,416,667 ft.^{3}-lb

Area-moment about H

Area GHI	20,000 (2)(2.67)	=	106,667
Area GIJ	20,000 (.75) (9)	=	1,350,000
Area FGJ	50,000 (7.5) (14)	=	5,250,000
			6,706,667 ft.^{3}-lb.

The slope of the tangent is

$$\phi = \frac{t_3 - t_4}{54} = \frac{476{,}315}{EI} = \frac{\text{area } KFJL}{EI}$$

The area $EFJN$, when considered as four trapezoidal areas, each 2 ft. wide, is 446,400. Hence, the approximate area of $KENL$ is

$$476{,}315 - 446{,}400 = 29{,}915$$

from which we find that the ordinate KL is located about 0.5 ft. to the left of the center of the beam as before.

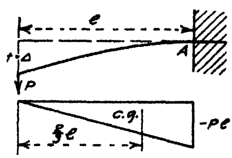


Fig. 61.

Cantilever Beam with Single Concentrated Load.—The beam in Fig. 61 supports a single load P at the free end. It is fixed in the wall at A in such a way that the tangent to the elastic curve at A remains horizontal. Hence the deflection at the free end is

$$\Delta = t = \frac{1}{EI}\left(-Pl\right)\left(\frac{l}{2}\right)\left(\frac{2}{3}l\right) = -\frac{Pl^3}{3EI}$$

The negative sign indicates that the elastic curve deviates below the tangent.

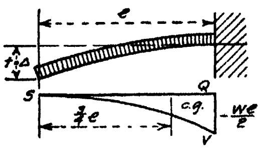


Fig. 62.

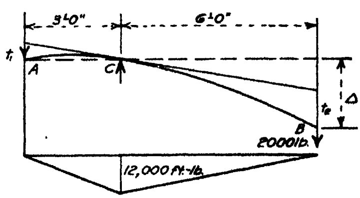


Fig. 63.

Cantilever Beam with Uniform Load.—The beam in Fig. 62 supports the load W uniformly distributed. The M-diagram is SQV. The curve SV is a parabola with the vertex at S. Hence the deflection at the free end is

$$\Delta = t = \frac{1}{EI}\left(-\frac{Wl}{2}\right)\left(\frac{l}{3}\right)\left(\frac{3}{4}l\right) = -\frac{Wl^3}{8EI}$$

Cantilever Beam with Overhanging Ends.—A 7-in. 15.3-lb. I-beam, Fig. 63, supports a load of 2,000 lb. EI = 1,050,000,000 in².-lb. The tangent is drawn through C.

$$t_1 = \frac{-12{,}000(1.5)(2)(1{,}728)}{1{,}050{,}000{,}000} = -0.0592 \text{ in.}$$

$$t_2 = \frac{-12{,}000(3)(4)(1{,}728)}{1{,}050{,}000{,}000} = -0.2368 \text{ in.}$$

$$\Delta = 2t_1 + t_2 = -\ 0.3552 \text{ in.}$$

The deflection at B may also be found by drawing the tangent through either A or B.

A cantilever beam is shown in Fig. 64. In finding t_1 positive and negative areas are encountered in the M-diagram. These may be treated in one of two ways. The point of zero bending moment at I may be determined, and the areas

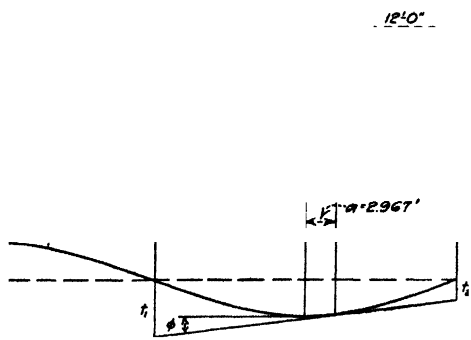


FIG. 64.

PIW and IQV treated separately, or the area WIV may be included with both positive and negative areas as follows:

Area-moment about P

Area PVW	$-$ 30,000 (3) (2)	=	$-$180,000
Area WQV	+ 36,000 (3) (4)	=	+432,000
Area QVU	+ 36,000 (6) (10)	=	+2,160,000
Area QRU	+ 48,000 (6) (14)	=	+4,032,000
			+6,444,000 ft³.-lb.

If EI is expressed in ft.²-lb., then

$$t_1 = \frac{6{,}444{,}000}{EI} \text{ ft.}$$

Area-moment about S

Area SRU 48,000 (6) (8) = 2,304,000 ft.³-lb.

$$t_2 = \frac{2{,}304{,}000}{EI} \text{ ft.}$$

$$\phi = \frac{t_1 - t_2}{30} = \frac{138{,}000}{EI}$$

Let KL represent the ordinate in the M-diagram at the point of maximum deflection. Then

$$\text{Area } KRUL = 138{,}000$$
$$a = 2.967 \text{ ft.}$$
$$KL = 45{,}033$$

The maximum deflection may now be found as in previous cases.

Beams with Cross-section not Constant.—The moment of inertia of beams having uniform cross-section is constant, and for this reason I appears outside the integral sign in eqs. (1) and (2), p. 46. When the cross-section is not uniform, the moment of inertia varies, and eqs. (1) and (2) become

$$\phi = \frac{1}{E}\int_A^B \frac{Mdx}{I} \tag{8}$$

$$t = \frac{1}{E}\int_A^B \frac{Mxdx}{I} \tag{9}$$

In order to perform the integration, I as well as M must be expressed as a function of x. This is relatively a simple matter when the beam has a rectangu-

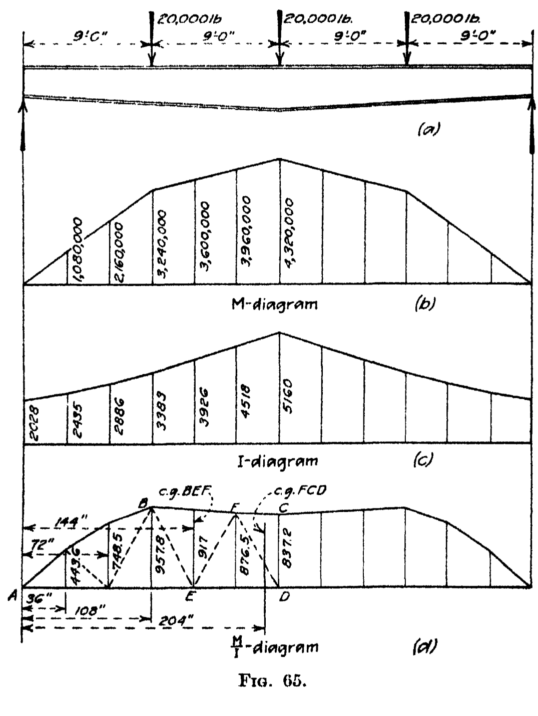


Fig. 65.

lar cross-section varying uniformly in breadth or depth; but this method often results in long and cumbersome expressions when applied to structural steel sections. In all such instances the geometric method is preferable.

The beam in Fig. 65*a* is a plate girder. The 3/8-in. web plate is 24 in. wide at the ends, and the width increases uniformly to 36 in. at the center. Each flange is composed of 2 angles, 5 × 3½ × 3/8, with the 3½-in. leg against the web. The distance back to back of angles is the width of the web plus one-half inch. Ordinates in the M-diagram, Fig. 65*b*, are given in inch pounds every 3 ft. An I-diagram is shown in Fig. 65*c*. The ordinates represent the moment of inertia in inches^4 at 3-ft. intervals. Each ordinate in the M-diagram has been divided by the corresponding ordinate in the I-diagram, and the quotient recorded in the $\frac{M}{I}$ diagram, Fig. 65 *d*. The ordinates in this diagram are expressed in lb.-in.^{3} Since the girder and the loads are symmetrical, the $\frac{M}{I}$ diagram is symmetrical about the vertical ordinate through the center of the span; the maximum deflection is at the center and the tangent to the elastic curve (not drawn) at the center is horizontal; hence the tangential deviation t at the left support equals the area-moment $ABCD$ about A divided by E.

Area-moment about A

443.6 (36) (36)	=	575,000
748.5 (36) (72)	=	1,940,000
957.8 (36) (108)	=	3,724,000
917 (36) (144)	=	4,754,000
876.5 (36) (180)	=	5,680,000
837.2 (18) (204)	=	3,074,000
		19,747,000

In calculating these area-moments, the $\frac{M}{I}$ diagram was broken up into triangles as shown in Fig. 65*d*. The lever arms to the centers of gravity of the several triangles are shown in position.

Then

$$\Delta_{max} = t = \frac{19{,}747{,}000}{29{,}000{,}000} = 0.68 \text{ in.}$$

When the ordinates in the $\frac{M}{I}$ diagram are computed only for the ordinates at B and C, and AB and BC considered as straight lines, the computations are as follows:

Area-moment about A

957.8 (54) (72)	=	3,724,000
957.8 (54) (144)	=	7,448,000
837.2 (54) (180)	=	8,138,000
		19,310,000

$$\Delta_{max} = \frac{19{,}310{,}000}{29{,}000{,}000} = 0.67 \text{ in.}$$

Thus, it is clear that if the ordinates in the $\frac{M}{I}$ diagram were relatively close

together, say at every foot or closer, or even if I were expressed as an exact function of x in eq. (9) and the integration performed, the results in either case would not differ materially with those obtained above.

The plate girder in Fig. 66a consists of a 24 × ⅜-web plate, 4 angles 5 × 3½ × ⅜, and 2 cover plates 12 × ⅜ × 24 ft. 0 in. symmetrical about the center

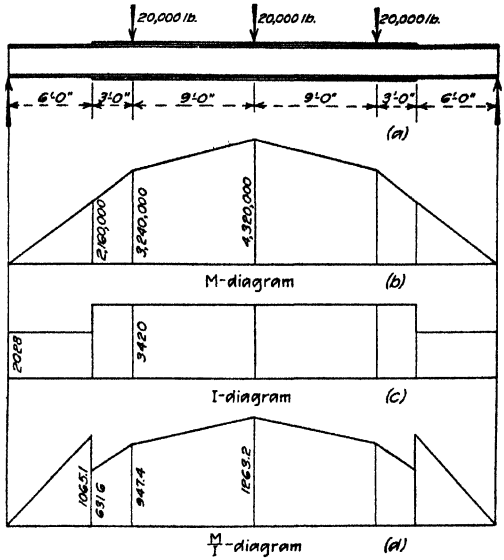


FIG. 66.

line. The M-diagram is shown in Fig. 66b; the I-diagram, in Fig. 66c; and the $\frac{M}{I}$ diagram in Fig. 66d.

Area-moment about A

$$
\begin{aligned}
1{,}065.1\ (36)\ \ (48) &= \ \ 1{,}840{,}500\\
631.6\ (18)\ \ (84) &= \ \ \ \ 955{,}000\\
947.4\ (18)\ \ (96) &= \ \ 1{,}637{,}100\\
947.4\ (54)\ (144) &= \ \ 7{,}367{,}000\\
1{,}263.2\ (54)\ (180) &= 12{,}278{,}300
\end{aligned}
$$

The deflection at the center is

$$\Delta = \frac{24{,}077{,}900}{29{,}000{,}000} = 0.83 \text{ in.}$$

The solution by integration may be obtained as follows: Let M_1 represent the bending moment for values of x between 0 and 9; and M_2, the bending moment for values of x between 9 and 18. Then

$$M_1 = 30{,}000x$$
$$M_2 = 10{,}000x + 180{,}000$$

Then for values of x between 0 and 6, $EI_1 = 408,417,000$ ft.²-lb. and for values of x between 6 and 18, $EI_2 = 688,750,000$ ft.²-lb. The deflection at the center expressed in feet is

$$\Delta = t = \frac{1}{EI_1}\int_0^6 M_1 x dx + \frac{1}{EI_2}\int_6^9 M_1 x dx + \frac{1}{EI_2}\int_9^{18} M_2 x dx$$

The solution by integration is much more simple in this problem than in the preceding one, for in this problem I is constant between certain limits of x and is therefore not a function of x.

64. Method of Elastic Weights.

64a. Derivation of General Formulas.—As stated in Art. **62,** the Method of Elastic Weights is based on the observed similarity between the deflection and slope diagrams for a beam due to a set of applied loads, and the bending moment and shear diagrams due to a loading which is a function of the elastic distortion of the beam elements. Due to its nature, as explained below, this loading is called an *elastic weight.* The beam on which this elastic weight loading is applied is known as a *conjugate beam.* It must be selected so that it meets certain initial conditions imposed by the character of the true deflection and slope diagrams. In the discussion which follows, the general principles will be given on which this method is based and a few simple typical cases will be followed through in detail. For a more complete discussion of this subject, the reader is referred to an article by Prof. H. M. Westergaard on "Deflection of Beams by the Conjugate Beam Method."[1]

Let ACB of Fig. 67*a* represent a simple beam which supports any set of applied loads. This beam will hereafter be referred to as the *given beam.* Assume for the present that the beam element at C, a distance a from the left end of the given beam, is elastic and that all other elements are non-elastic. Figure 67*b* shows the elastic element at C. Assuming that the flexural stress and strain on this element are subject to the conditions stated in Art. **50,** p. 21, it can readily be seen that $\delta = \frac{f}{E}$, where δ = deformation of extreme fiber, f = stress intensity on that fiber due to bending, and E = modulus of elasticity of the material composing the beam element. The angular rotation of the face of the distorted beam element is then $d\phi = \frac{\delta}{c} = \frac{f}{Ec}$. From Art. **50***b*, p. 23, $f = \frac{Mc}{I}$, and hence

$$d\phi = \frac{M}{EI} \tag{1}$$

The effect of the distortion of the elastic element at C on the deflection of the beam is shown (greatly exaggerated) in Fig. 67*c*. All points from A to C rotate about point A through an angle α and all points from C to B rotate about B through an angle β, taking the positions shown in Fig. 67*c*. To determine the values of the angles α and β, note from Fig. 67*b* that the angle between AC produced and CB of Fig. 67*c* is equal to $d\phi$ as given by eq. (1). Since the angles are all very small, we may write

$$\alpha = \frac{BB_1}{l}$$

[1] See *Journal of the Western Society of Engineers,* vol. 26, No. 11, Nov., 1921.

But

$$BB_1 = (l - a)\, d\phi$$

Therefore

$$\alpha = \frac{l-a}{l} d\phi = \left(\frac{l-a}{l}\right) \frac{M}{EI} \tag{2}$$

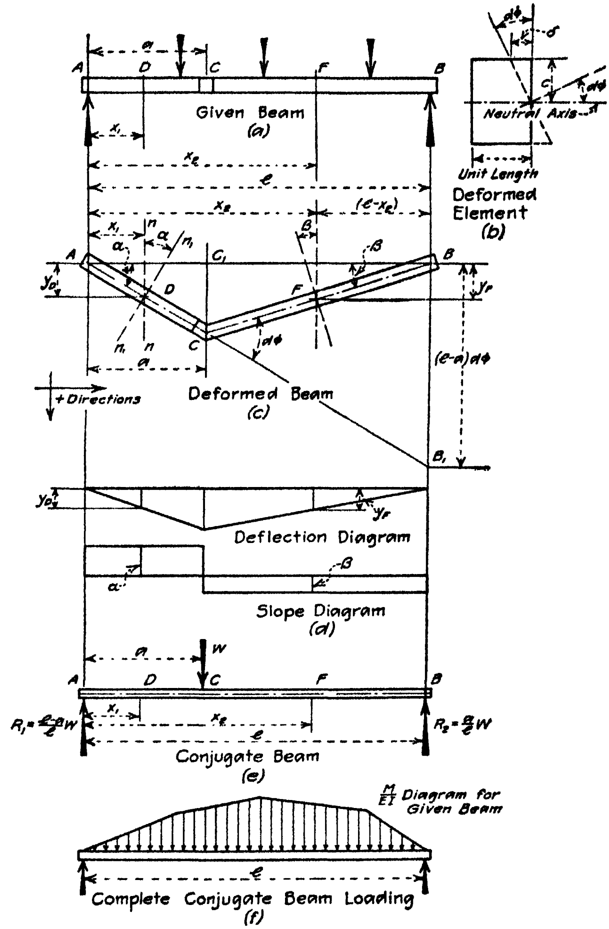


Fig. 67.

Also

$$\beta = \frac{CC_1}{l - a}$$

But

$$CC_1 = a\alpha = \frac{a(l-a)}{l} d\phi$$

Therefore

$$\beta = \frac{a}{l}\, d\phi = \frac{a}{l}\frac{M}{EI} \tag{3}$$

The deflection of point D on the neutral axis at a distance x_1 from the left end of the beam is

$$y_D = x_1\alpha = \frac{x_1}{l}(l - a)\frac{M}{EI} \tag{4}$$

and the deflection of point F on the neutral axis at a distance x_2 from the left end of the beam is

$$y_F = (l - x_2)\beta = \frac{a}{l}(l - x_2)\frac{M}{EI} \tag{5}$$

Equations (4) and (5) give the deflection of points on either side of the distorted element in terms of $\frac{M}{EI}$, the elastic deformation of that element. The deflections given by eqs. (4) and (5), when plotted, give a deflection diagram which is represented by the triangle of Fig. 67*d*.

The slope of the tangent to the elastic curve at any point due to the elastic deformation of the element at C of Fig. 67*a* is equal to the angular rotation for the portion of the beam containing the point in question. Thus in Fig. 67*c*, a vertical section n-n through D of the undeformed beam is rotated through an angle α to n_1-n_1 after the deformation has taken place. Since the tangent to the elastic curve is perpendicular to n-n, it also rotates through the angle α. In the same manner it can be shown that the tangent at F rotates through an angle β. We may then write, substituting for α and β the values given by eqs. (2) and (3).

For point D

$$\frac{dy}{dx} = \alpha = +\left(\frac{l - a}{l}\right)\frac{M}{EI} \tag{6}$$

For point F

$$\frac{dy}{dx} = \beta = -\frac{a}{l}\frac{M}{EI} \tag{7}$$

The signs given to α and β are determined by the direction of rotation shown in Fig. 67*c*. Denoting clockwise rotation as positive, α is a positive rotation and β is negative. The slope given by eqs. (6) and (7) is plotted in the slope diagram of Fig. 67*d*.

As stated above, a conjugate beam is to be selected and a loading condition determined for that beam such that its bending moment diagram will represent the deflection of the given beam and its shear diagram will represent the slope diagram of the given beam. On examining the diagrams given in Art. **46**, p. 15, we note that the moment and shear diagrams for a simple beam with a single concentrated load are of the same form as the deflection and shear diagrams of Fig. 67*d*.

Consider a simple beam of span l, Fig. 67*e* carrying a single load W at a distance a from the left end. The values of moment and shear at point D are

$$M_D = R_1x = \frac{x_1}{l}(l - a)W \tag{8}$$

$$V_D = R_1 = +\frac{l - a}{l}W \tag{9}$$

At point F, the values are

$$M_F = R_2(l - x_2) = \frac{a}{l}(l - x_2)W \tag{10}$$

$$V_F = -R_2 = -\frac{a}{l}W \tag{11}$$

On comparing the right hand members of eqs. (4) and (8); eqs. (5) and (10); eqs. (6) and (9); and eqs. (7) and (11), we note that they differ only in that eqs. (4) to (7) have a term M/EI where eqs. (8) to (11) have a term W. Therefore, if W of eqs. (8) to (11) be replaced by M/EI for the distorted element at C, the resulting moments and shears are exactly the same as the deflections and slopes given by eqs. (4) to (7). Note that to secure agreement in signs the load $\frac{M}{EI}$ must act downward.

This observed similarity between the two sets of equations suggests the type of conjugate beam and the character of the loading to be used on the conjugate beam in order to determine the deflection and slope at any point on a given simple beam due to the distortion of a single beam element. Thus, on a beam of the same span as the given beam and supported in the same manner, apply at the position of the distorted element, a load M/EI, which is equal to the elastic distortion of the given beam element due to the applied loads on the given beam. Calculate the moment and shear at a point on the conjugate beam at a position corresponding to the location on the given beam of the point whose deflection and slope is required. This moment and shear are respectively equal to the desired deflection and slope. Positive moment indicates downward deflection. Positive shear indicates clockwise rotation, and negative shear indicates counter-clockwise rotation.

The above analysis is general and holds true for each and every element of the given beam. Hence if all elements are considered to be elastic, each element must be loaded with its M/EI value. The resulting load on the conjugate beam is then the M/EI diagram for the given beam, as shown in Fig. 67f. Moments and shears calculated at any point on the conjugate beam loaded as shown in Fig. 67f, will give the deflection and slope at a corresponding point in the given beam due to the applied loads.

Since M/EI is a function of the elastic distortion of a beam element, it has been called the *elastic weight* of that element, and the method for the determination of deflections and slopes is called the *Method of Elastic Weights*. When E and I are constant for the entire beam, the conjugate beam loading may be taken as the moment diagram for the given beam. After the moments and shears, representing deflection and slope, have been calculated, they must be divided by the constant EI.

Cantilever beam deflections may also be determined by the method of elastic weights, suitably modified to meet the new conditions. Figure 68a shows a cantilever beam which is fixed at B and free at A and supporting any set of applied loads. Assume as before that an element at C at a distance a from the left end is elastic and that all others are rigid. For the conditions shown in Fig. 68b, the deflection of any point D with respect to an origin at O, the free end of the undeformed beam is

$$y_D = (a - x)d\phi = (a - x)\frac{M}{EI} \tag{12}$$

A positive value in eq. (12) indicates downward deflection. The deflection diagram plotted from eq. (12) is shown in Fig. 68c.

The slope of the tangent to the elastic curve at point D is equal to $d\phi$. According to the assumed direction of rotation, $d\phi$ is a negative, or counter-clockwise angle. Therefore

$$\frac{dy}{dx} = d\phi = -\frac{M}{EI} \tag{13}$$

The slope diagram plotted from eq. (13) is given in Fig. 68c.

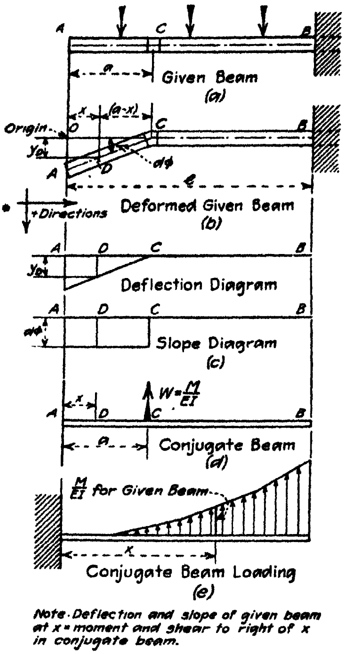


FIG. 68.

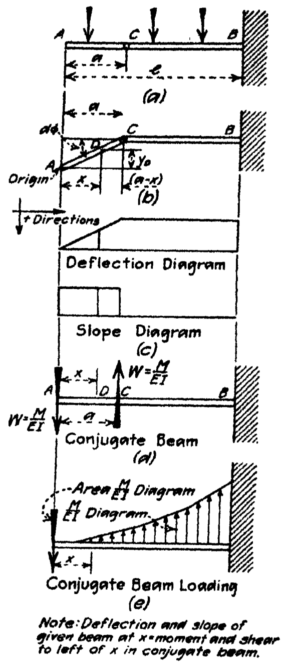


FIG. 69.

A conjugate beam must now be selected subject to the conditions that the moment and shear diagrams due to an elastic weight M/EI will be similar to the deflection and slope diagrams of Fig. 68c. A cantilever beam fixed at A and free at B, as shown in Fig. 68d, and loaded with an upward force M/EI will answer the given conditions. This load must act upward, for, according to the direction of rotation, the deflection is upward or positive. Hence the conjugate beam moment must also be positive. Positive moment (compression in the top fibers) will occur for a load directed as shown in Fig. 68d. This conjugate beam of Fig. 68d applies when only element C is elastic. Similar conditions hold for all other elements, when they are also considered as elastic. The complete loading for the conjugate beam is then as shown in Fig. 68e, being the M/EI diagram for the applied loads on the given beam of Fig. 68a.

Figure 69a shows the conditions for the beam of Fig. 68a when the origin is taken at the deflected position of the free end of the beam, as shown in Fig. 69b. The deflection and slope diagrams for a single elastic element at C are shown in Fig. 69c. A conjugate beam fixed at the right end, free at the left end, and loaded with a couple $\frac{M}{EI}\cdot a$, as shown in Fig. 69d, will give moment and shear diagrams similar to the deflection and slope diagrams of Fig. 69c. When all elements are considered as elastic, the conjugate beam loading will be as shown in Fig. 69e. The M/EI values for each element will form the M/EI diagram shown in Fig. 69e, and the concentrated load at the free end will be equal in magnitude to the area of the M/EI diagram.

Algebraic or graphical methods may be used in the solution of problems in the deflection of beams by the method of elastic weights. If the equation for the M/EI diagram can be expressed as a continuous function of x, an equation for the deflection at any point may be derived. This equation is exactly the same as the one derived by the elastic curve method of Appendix C. When the equation for the M/EI diagram cannot be expressed as a function of x, it is possible to plot the moment curve from values calculated at several points. By the use of semi-graphical methods, similar to those used in the area-moment method in Art. 63b, the desired deflection of any point may be determined.

Graphical methods for the determination of the deflection of beams by the method of elastic weights are based on properties of the equilibrium polygon. The equilibrium polygon may be used to determine the moment of forces about a given point, and it may be so drawn that it represents the moment diagram for a given set of forces. To apply these principles to the determination of the deflection of beams, the M/EI diagram for the given loading may be divided into small areas. At the center of gravity of each of these areas the corresponding M/EI is applied as a force. An equilibrium polygon drawn for these forces represents the moment diagram for these forces, and is therefore a graphical representation of the deflection of the beam as shown by the elastic curve for the beam.

64b. Application of the Method.—The method of elastic weights offers a very convenient method for the determination of the deflection and slope of simple and cantilever beams of variable cross-section supporting complicated loading systems. In the following articles the method will be applied to the solution of typical problems.

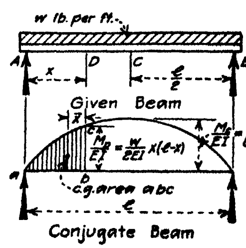


Fig. 70.

Simple Beam with Uniform Load (Moment of Inertia Constant).—As stated in the preceding article the deflection at any point due to the given uniform load is equal to the moment at that point due to a loading represented by the M/EI diagram for the given loading. The bending moment diagram for a beam with a uniform load is a parabola. In Fig. 70 this bending moment is shown as a load applied to the beam. For the conditions shown

$$y_D = R_1x - (\text{area } abc)\bar{x}$$

Now R_1 = one-half the area of the moment diagram $= \left(\frac{1}{2}\right)(l)\left(\frac{wl^2}{8}\right)\left(\frac{2}{3}\right) = \frac{1}{24}wl^3$, $\bar{x}$ = distance to center of gravity of area $abc = \frac{x}{2}\frac{(2l - x)}{(3l - 2x)}$ and area $abc = \frac{wx^2}{12}(3l - 2x)$. Then

$$y_D = \frac{wx}{24EI}(l^3 - 2lx^2 + x^3)$$

The slope of the tangent to the elastic curve has been shown to be equal to the shear in the conjugate beam at the required point. Hence at point D,

$$\frac{dy}{dx} = \text{Shear} = R_1 - \text{area } abc.$$

Substituting the values given above

$$\frac{dy}{dx} = \frac{w}{24EI}(l^3 - 6lx^2 + 4x^3)$$

Illustrative Problem.—A simple beam 16 ft. long supports a uniform load of 600 lb. per ft. Determine the maximum deflection of this beam. Assume that the moment of inertia of the beam is 100 in.^{4}, and that the material is steel for which E = 30,000,000 lb. per sq. in.

The conjugate beam loaded with the bending moment diagram for the given beam is shown in Fig. 71. The ordinate $cb = \frac{1}{8}wl^2 = (\frac{1}{8})(600)(16)^2(12) = 230{,}400$ in.-lb. and other ordinates follow the parabolic law. For the conditions shown it is evident that the maximum moment in the conjugate beam, and therefore the maximum deflection in the given beam will occur at the beam center. Therefore

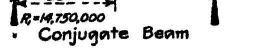


FIG. 71.

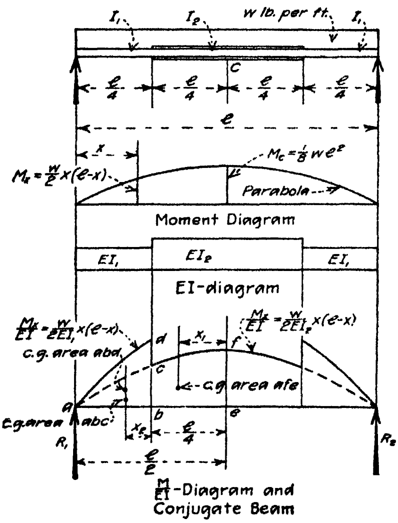


FIG. 72.

$$\text{Max. deflection} = \frac{1}{EI}\text{(Moment about beam center).}$$

For the loading shown R_1 = one-half the area of moment diagram = $(\frac{1}{2})(\frac{2}{3})(192)(230{,}400)$ = 14,750,000. Therefore,

$$\text{Max. deflection} = \frac{(14{,}750{,}000)(96 - 36)}{(30{,}000{,}000)(100)} = 0.295 \text{ in.}$$

Simple Beam with Uniform Load (Moment of Inertia Not Constant).—Figure 72 shows a simple beam carrying a uniform load. The moment of inertia of this beam is not-constant, being I_1 for the end quarters and I_2 for the center half. The moment diagram, EI diagram, and the M/EI diagram are shown. Let it

be required to determine the general expression for center deflection of the given beam. As shown in Art. 64a, the required deflection is equal to the moment at the center of the conjugate beam.

For the conditions shown in Fig. 72, the moment at the center of the conjugate beam is

$$R_1\frac{l}{2} - \left[(\text{area } abd)\left(x_2 + \frac{l}{4}\right) + (\text{area } afe)x_1 - (\text{area } abc)\left(x_2 + \frac{l}{4}\right)\right]$$

It can readily be shown that the several areas and lever arms have the following values, which were obtained by substituting in the general equations for area abc and $\bar{x}$ given on p. 577.

$$\text{area } abc = \frac{5}{384}\frac{wl^3}{EI_2} \qquad \text{area } abd = \frac{5}{384}\frac{wl^3}{EI_1}$$

$$\text{area } afe = \frac{wl^3}{24EI_2}$$

$$x_1 = \frac{3}{16}l \qquad x_2 = \frac{7}{80}l \qquad x_2 + \frac{l}{4} = \frac{27}{80}l$$

Now R_1 = one-half the area of the M/EI diagram = area abd + area afe − area ack. Substituting values given above

$$R_1 = \frac{5}{384}\frac{wl^3}{EI_1} + \frac{11}{384}\frac{wl^3}{EI_2}$$

The complete expression for y_C, the center deflection, is then

$$y_C = \frac{wl^4}{384E}\left(\frac{335}{80I_2} + \frac{65}{80I_1}\right)$$

However, we may write

$$\frac{335}{80I_2} = \frac{(400 - 65)}{80I_2} = \frac{5}{I_2} - \frac{65}{80I_2}$$

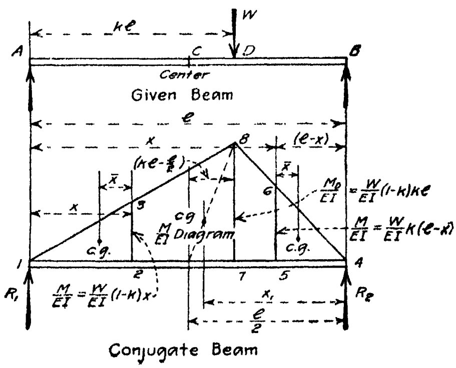


FIG. 73.

Substituting this value in the above expression, we have finally

$$y_C = \frac{wl^4}{384E}\left[\frac{5}{I_2} + \frac{13}{16}\left(\frac{I_2 - I_1}{I_1I_2}\right)\right]$$

Simple Beam with a Single Concentrated Load.—A simple beam with a single concentrated load W at a distance kl from the left end of a beam is shown in Fig. 73. The conjugate beam is shown with the M/EI loading in position. For

the conditions shown, $R_1 =$ (area $1 - 8 - 4$) $\frac{x_1}{l}$ and $R_2 =$ area $(1 - 8 - 4)$ $\frac{(l - x_1)}{l}$. In these equations, area $(1 - 8 - 4) = \frac{1}{2}\left[\frac{W}{EI}(1 - k)kl\right]l = \frac{1}{2}\frac{W}{EI}$ $(1 - k)kl^2$, and x_1 = distance from right end of beam to the center of gravity of the M/EI diagram. From Fig. 73 it can readily be seen that

$$x_1 = \frac{l}{2} - \frac{1}{3}\left(kl - \frac{l}{2}\right) = \frac{l}{3}(2 - k).$$

With these values we readily derive,

$$R_1 = \frac{W}{6EI}(1 - k)(2 - k)kl^2$$

and

$$R_2 = \frac{W}{6EI}(1 - k^2)kl^2.$$

The equation of the elastic curve for the portion of the beam from A to D is given by the general expression for moments in the conjugate beam for a section $2-3$ at a distance x from the left end. Thus

$$y = R_1x - (\text{area } 1 - 2 - 3)\,\bar{x}$$

Now area $1-2-3 = \frac{1}{2}\left[\frac{W}{EI}(1 - k)\,x\right]x = \frac{1}{2}\frac{W}{EI}(1 - k)x^2$, and $\bar{x}$ = distance to center of gravity of area $1-2-3 = \frac{x}{3}$. Then, with R_1 as given above, we have

$$y = \frac{W(1 - k)x}{6EI}\left[(2 - k)kl^2 - x^2\right]$$

In the same manner, the equation of the elastic curve for the portion of the given beam from D to B is equal to the moment at section $5-6$ of the conjugate beam, from which

$$y = R_2(l - x) - (\text{area } 4-5-6)\bar{x}$$

Substituting values in this expression and reducing, we have finally

$$y = \frac{Wk}{6EI}(l - x)[x(2l - x) - k^2l^2]$$

The slope of the tangent to the elastic curve has been shown to be equal to the conjugate beam shear at the point where the slope is required. Hence, at the left end of the given beam,

$$\frac{dy}{dx} = R_1 = \frac{W}{6EI}$$

At the right end of the given beam,

$$\frac{dy}{dx} = -R_2 = -\frac{W}{6EI}(1 - k^2)kl^2$$

The maximum deflection in the given beam will occur at the point of maximum moment in the conjugate beam. As shown in Art. **48**, the moment is a maximum at the point of zero shear. For the conditions shown in Fig. 73, zero shear will occur when the

$$\text{Area } 1-2-3 = R_1 = \frac{W}{6EI}(1 - k)(2 - k)kl^2$$

If x_0 denotes the distance from the left end of the conjugate beam to the point of zero shear

$$\text{Area } 1-2-3 = \frac{W}{2EI}(1-k)x_0^2$$

Equating these expressions and solving for x_0, we have

$$x_0 = l\left[\frac{k}{3}(2-k)\right]^{1/2}$$

The maximum deflection of the given beam is equal to the moment at the zero shear point. Thus

$$y_{max} = R_1x_0 - (\text{area } 1-2-3)\frac{x_0}{3}$$

Substituting values as given above, we have finally

$$y_{max} = \frac{Wl^3}{3EI}\left[\frac{k}{3}(2-k)\right]^{3/2}(1-k)$$

Illustrative Problem.—A 2 × 1-in. piece of wood laid flatwise spans a 24-in. opening. The beam carries a 60-lb. load at a distance of 18 in. from the left end of the beam. Determine the deflection under the load and the maximum deflection of the beam. Assume E = 1,500,000 lb. per sq. in.

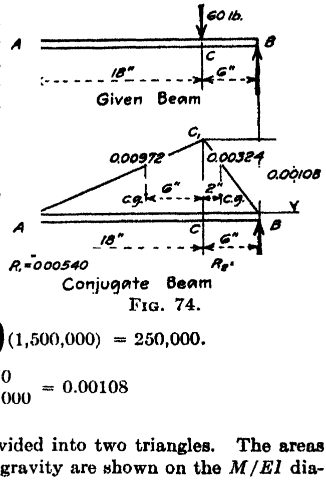


FIG. 74.

Figure 74 shows the given beam and the conjugate beam with its loading diagram. For the given beam, the maximum moment occurs under the load and the moment is

$$M = \frac{(60)(6)}{24}(18) = 270 \text{ in.-lb.}$$

The moment of inertia is

$$\frac{1}{12}bd^3 = \left(\frac{1}{12}\right)(2)(1)^3 = \frac{1}{6}, \text{and } EI = \left(\frac{1}{6}\right)(1{,}500{,}000) = 250{,}000.$$

$$\text{Therefore maximum } \frac{M}{EI} = \frac{270}{250{,}000} = 0.00108$$

the value shown on the conjugate beam.

To calculate R_1 and R_2, the M/EI diagram is divided into two triangles. The areas of these triangles and the location of their centers of gravity are shown on the M/EI diagram. Values of R_1 and R_2 as calculated are shown on Fig. 74.

The deflection under the load is given by a moment equation about point C, from which

$$y_C = (0.00540)(18) - (0.00972)(6) = 0.0389 \text{ in.}$$

Maximum deflection occurs where the shear in the conjugate beam is zero. Let x_0 be the distance from the left end of the beam to the zero shear point. It can readily be shown that the area of the triangle AC_1C from A to a point distant x_0 from the left end is $\frac{0.00108x_0^2}{36}$. This must be equal to R_1. Therefore $\frac{0.00108x_0^2}{36} = 0.0054.$

Solving for x_0, we have x_0 = 13.42 in. The moment about a point 13.42 in. from the left end of the beam gives the required maximum deflection. Therefore,

$$y_{max} = (0.00540)(13.42) - \left(\frac{0.00108}{18}\right)(13.42)\left(\frac{13.42}{2}\right)\left(\frac{13.42}{3}\right) = 0.0482 \text{ in.}$$

Cantilever Beams.—Figure 75 shows a cantilever beam carrying a uniform load. The equation of the elastic curve and the maximum deflection will be determined with respect to an origin at point O. The conjugate beam and the loading for this case is of the type shown in Fig. 68e. This calls for a cantilever beam fixed

at the left end and carrying an upward loading represented by the M/EI diagram for the given beam. Figure 75 shows the conjugate beam with the loading in position.

The equation of the elastic curve is given by moments to the right of point C of the conjugate beam. Thus

$$y_C = \text{Area } (CBED)\ (x_1 - x)$$

Substituting and reducing, we have

$$y_C = \frac{w}{24EI}(x^4 - 4l^3x + 3l^4)$$

The slope of the tangent to the elastic curve is equal to the shear due to forces to the right of C; that is

$$\text{Slope} = \frac{dy}{dx} = \text{Area } CBED = -\frac{w}{6EI}(l^3 - x^3)$$

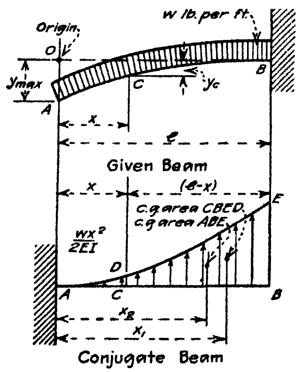


FIG. 75.

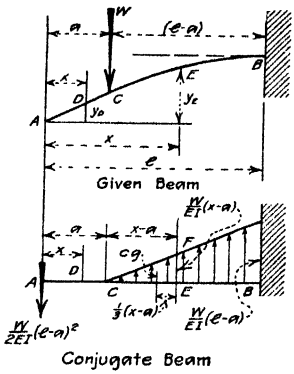


FIG. 76.

The maximum deflection, which occurs at point A of the given beam, is equal to the moment to the right of point A of the conjugate beam. Thus

$$y_{max} = \text{Moment of area } ABE \text{ about point } A = (\text{Area } ABE)(x_2) = +\frac{wl^4}{8EI}$$

The slope of the tangent to the elastic curve at point A of the given beam is equal to the shear for forces to the right of point A of the conjugate beam. Thus

$$\text{Slope} = \frac{dy}{dx} = -\text{Area } ABE = -\frac{wl^3}{6EI}$$

Figure 76 shows a cantilever beam carrying a single concentrated load at a distance a from the free end. The equation of the elastic curve and the maximum deflection at the free end will be determined with respect to an origin at point A of Fig. 76, the free end of the beam. Figure 69e shows the type of conjugate beam and M/EI loading for this case, and Fig. 76 shows the actual conjugate beam with the M/EI values in position. The load at point A is equal to the area of the M/EI diagram =

$$\frac{1}{2}\left[\frac{W}{EI}(l-a)\right](l-a) = \frac{1}{2}\frac{W}{EI}(l-a)^2$$

The equation of the elastic curve for the portion of the given beam from A to C is given by a moment equation for M/EI values to the left of point D of the conjugate beam. Thus

$$y_D = -\frac{Wx}{2EI}(l-a)^2$$

The slope of the tangent to the elastic curve at D is equal to the shear to the left of D, thus

$$\frac{dy}{dx} = -\frac{W}{2EI}(l-a)^2$$

Note that the moment and shear have negative values.

For the portion of the given beam from C to B, the equation of the elastic curve is given by a moment equation for M/EI values to the left of any point, as E. Since the M/EI diagram is a triangle, whose center of gravity is readily located, the moment equation is readily written out. Thus

$$\begin{aligned} y_E &= -(M/EI \text{ load at } A)x + (\text{area triangle } CEF)\ (\tfrac{1}{3}\ CE) \\ &= -\frac{W}{2EI}(l-a)^2x + \left\{\frac{1}{2}\left[\frac{W}{EI}\ (x-a)\ \right](x-a)\right\}\frac{(x-a)}{3} \end{aligned}$$

from which

$$y_E = \frac{W}{6EI}[x^3 - 3ax^2 - 3lx(l-2a) - a^3]$$

The slope of the tangent to the elastic curve at E is equal to the shear for M/EI values to the left of that point. Thus

$$\begin{aligned} \frac{dy}{dx} &= -(M/EI \text{ load at } A) + (\text{area triangle } CEF) \\ &= -\frac{W}{2EI}(l-a)^2 + \frac{1}{2}\left[\frac{w}{EI}(x-a)\right](x-a) \end{aligned}$$

from which

$$\frac{dy}{dx} = \frac{W}{2EI}[x^2 - 2ax - l(l-2a)]$$

The above values are the same as those given on p. 588.

The maximum deflection occurs at point A and the value of this deflection is given by moments about point B of the conjugate beam. Thus

$$y_{max} = -\frac{W}{2EI}(l-a)^2l + \frac{W}{6EI}(l-a)^3$$

from which

$$y_{max} = -\frac{W}{6EI}(l-a)^2(2l+a)$$

Illustrative Problem.—A 24-in. cantilever beam made up of a 2 × 1-in. piece of wood laid flatwise supports a 20-lb. load at a distance of 18 in. from the free end of the beam. Determine the maximum deflection of the beam. Assume E = 1,500,000 lb. per sq. in.

In this case $I = \frac{bd^3}{12} = \frac{2(1)^3}{12} = \frac{1}{6}$ in.^{4}, l = 24 in., and a = 18 in. Substituting in the above equation for y_{max}, we have

$$y_{max} = -\frac{20}{6(1{,}500{,}000)(\frac{1}{6})}(24-18)^2[2(24)+18] = 0.307 \text{ in.}$$

64c. Graphical Methods for the Determination of the Deflection of Beams.—As stated on p. 67 of Art. **64a**, a graphical determination of the

deflection of a beam may be made by dividing the M/EI diagram into small areas, each of which is replaced by a force proportional to the area in question. These forces are applied at the center of gravity of each area. An equilibrium polygon drawn for these forces represents their moment diagram. Hence, as stated in Art. 64*a*, this moment diagram represents the deflection diagram for the given beam.

Graphical methods for the determination of the deflection of beams are particularly useful when the loading is complicated or when the moment of inertia of the beam is not constant. An algebraic solution for such cases is long and tedious. However, the results obtained by graphical methods are in general not as precise as those given by the algebraic solutions of the preceding articles, even when great care is taken in constructing the graphical diagrams. Therefore, graphical methods are recommended for use when a reasonably precise result is desired in a short time. Whenever possible use algebraic methods.

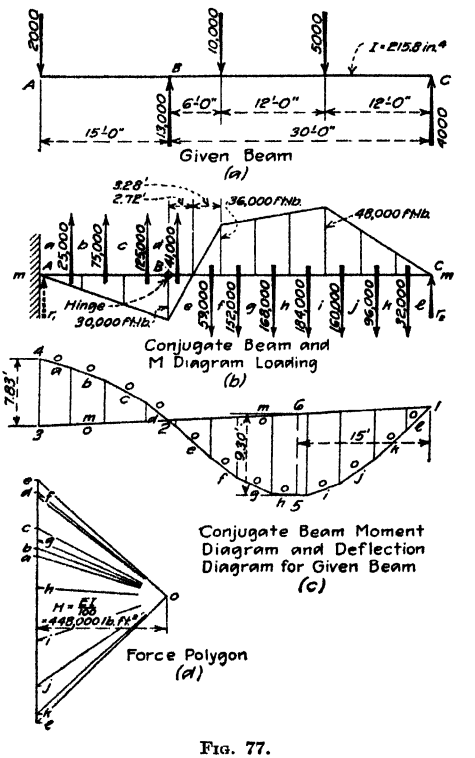


Fig. 77.

To illustrate the application of graphical methods, graphical solutions will be given for the problems shown in Figs. 64 and 65. An algebraic solution of these problems is given in Art. 63*b*, p. 58.

Figure 77*a* shows a beam with an overhanging end supporting concentrated loads (see Fig. 64 for an algebraic solution). This problem was chosen in order to explain the graphical method for beams with overhanging ends. The moment diagram for this beam is shown in Fig. 77*b*. This moment diagram is applied as a load to the conjugate beam *ABC*.

The form of the conjugate beam is determined by the following conditions: (*a*) The conjugate beam for the span *BC* of the given beam is a simple beam of the same span; (*b*) the conjugate beam for the cantilever *AB* of the given beam is a cantilever of the same span fixed at *A* and free at end *B*; (*c*) the deflections at points *B* and *C* of the given beam are zero as these points form rigid supports; and (*d*) in the given beam the tangent to the elastic curve at *B* for the cantilever *AB* must have the same slope as the tangent for the span *BC*.

To satisfy the above conditions, the conjugate beam of Fig. 77b must consist of a cantilever AB fixed at A and supporting by a hinged connection at B a simple beam BC, which is also freely supported at C. Thus the conjugate beam moments at B and C will be zero, indicating zero deflection at these points in the given beam. The conjugate beam shears on either side of the hinge at B will be equal, indicating equal slopes for the tangents to the elastic curve in the given beam on either side of point B. All of the above conditions are therefore satisfied.

To construct the conjugate beam moment diagram, divide the M-diagram of Fig. 77b into small areas and apply at the center of gravity of each area a force equal to that area. These forces are shown in amount and direction on the figure. The greater the number of subdivisions the greater will be the accuracy of the construction. Next construct the force diagram of Fig. 77d, plotting the forces in order beginning at the left end of the conjugate beam. The pole distance H may be chosen at will. A convenient value is some multiple of EI. Thus if we make $H = \frac{EI}{n}$, the ordinates to the resulting equilibrium polygon will be n times the actual deflection.

Suppose the deflection for the case under consideration is desired in foot units. For the given beam $I = 215.8$ in.^{4} and $E = 30{,}000{,}000$ lb. per sq. in. Assume that the ordinates to the deflection diagram are to be represented at 100 times their true value. Then

$$H = \frac{EI}{100} = \frac{(30{,}000{,}000)(144)(215.8)}{(100)(12)^4} = 448{,}000 \text{ lb.-ft.}^2$$

Figure 77c shows the equilibrium polygon constructed from the force polygon described above. The construction is started at any convenient point, as 1, and the complete equilibrium polygon is shown by 1-5-2-4. Since the deflections of points B and C of the given beam are zero, the conjugate beam moments for these points must be zero. To represent this condition in the equilibrium polygon extend verticals from points B and C of the given beam to intersections at 1 and 2 of Fig. 77c. Through these points draw the line 1-2-3, which is the closing line for the equilibrium polygon and the base line for the conjugate beam moment diagram Finally, the deflection of any point in the given beam is measured by the intercept on a vertical through that point between the equilibrium polygon and the base line 1-2-3. Ordinates below the base line indicate downward deflections. The scale for deflections is the same as the distance scale for the given beam. Remember that these distances represent 100 times the true deflection.

The deflection of A, the free end of the cantilever AB of the given beam is shown by the ordinate 1-4 of Fig. 77c. By scale this distance is 7.83 ft. as shown. The true deflection of A is therefore $7.83/100 = 0.0783$ ft. To determine the maximum deflection locate by trial the maximum ordinate to the equilibrium polygon. This ordinate is found to be 5-6, and the distance as scaled is 9.30 ft. The corresponding deflection is $9.30/100 = 0.0930$ ft.

Figure 78 shows a beam of variable moment of inertia carrying concentrated loads (see also Fig. 65, p. 59). Two methods will be given for the graphical determination of the deflection of the given beam. In each case the conjugate beam is a simple beam of the same span as the given beam.

First Method.—The M/EI diagram for the conjugate beam is divided into small areas as shown on the left side of Fig 78b. At the center of gravity of each

area, a load is applied, which is equal to the area in question. The information for the determination of these areas was taken directly from Fig. 65*d*. The resulting forces are shown in amount and in direction in Fig. 78*b*. Figure 78*d* shows the force diagram drawn for these forces. The pole distance H has been taken as $E/100$, or 290,000. Therefore, the equilibrium polygon shown on the left half of Fig. 78*c* represents 100 times the true deflection. By scale the maximum ordinate of this diagram is 67.0 in. Hence maximum deflection = 0.67 in.

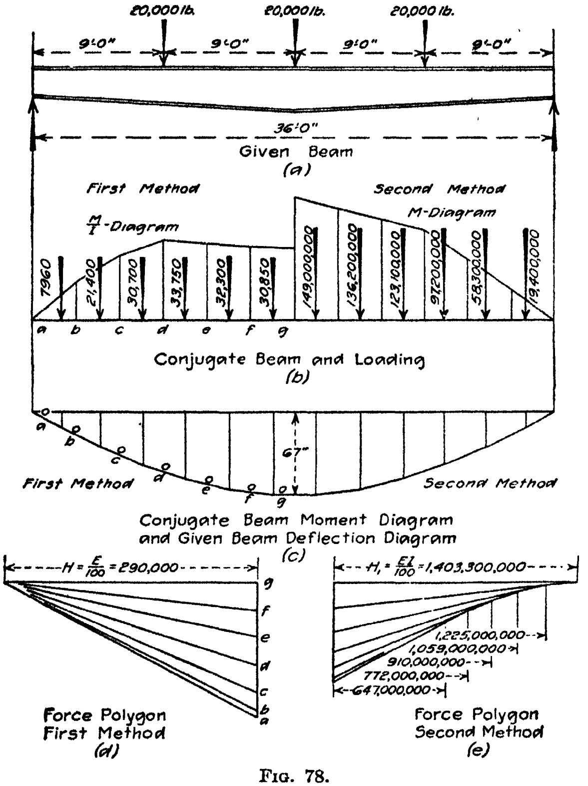


FIG. 78.

Second Method.—The M-diagram shown on the right half of Fig. 78*b* is divided into small areas each of which is replaced by a force applied at the center of gravity of the area. Information for the determination of these forces was taken from Fig. 65*b*. Figure 78*e* shows the force polygon plotted from these forces. The pole distance used for each force is the average of the values of $EI/100$ taken from Fig. 65*c* for the corresponding moment area. Thus for the first force

$$H = \frac{(29{,}000{,}000)(4{,}518 + 5{,}160)(\frac{1}{2})}{100} = 1{,}403{,}300{,}000 \text{ lb.-in.}^2$$

Other pole distances were determined in the same manner. The right half of Fig. 78c shows the equilibrium polygon drawn for these forces and pole distances. Deflections given by this equilibrium polygon represent 100 times the true deflection.

65. Deflection Coefficients.[1]—The detail work involved in the solution of problems in the deflection of beams is greatly reduced by means of tables or diagrams. A few such tables and diagrams will be given for standard beams.

Simple Beam with a Single Concentrated Load.—Figure 79 shows a simple beam with a load P at a distance kl from the left support, where k is a fraction

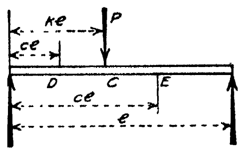


Fig. 79.

less than unity. From the preceding articles, the deflection at a point distance cl from the left support, where c is a fraction less than unity, is given by the following formulas:

For point D, where $c < k$

$$y = \frac{Pl^3}{6EI}\left\{ c[k(2 - k) - c^2](1 - k) \right\} \tag{1}$$

For point E, where $c > k$

$$y = \frac{Pl^3}{6EI}[c(2 - c) - k^2](1 - c) \tag{2}$$

In these equations y is the deflection in inches at any distance cl from the left support, E is the modulus of elasticity in pounds per square inch, and I is the moment of inertia of the constant cross-section of the beam about the neutral axis, measured in inches.[4]

Let F represent the expression in the parenthesis of eq. (1) when c is less than k, and the expression in the parenthesis of eq. (2) when c is greater than k. Then, in general,

$$y = \frac{Pl^3}{6EI}F \tag{3}$$

The values of F for various values of c and k are given in Table 1, or may be found from Fig. 80. It has also been shown on p. 51 that when the deflection is a maximum, the relation between c and k is given by the equation

$$1 \quad \sqrt{\frac{1}{3}}(1 - k^2) \tag{4}$$

Values of c from eq. (4) are given in Table 1 and Fig. 80.

[1] By C. A. Ellis.

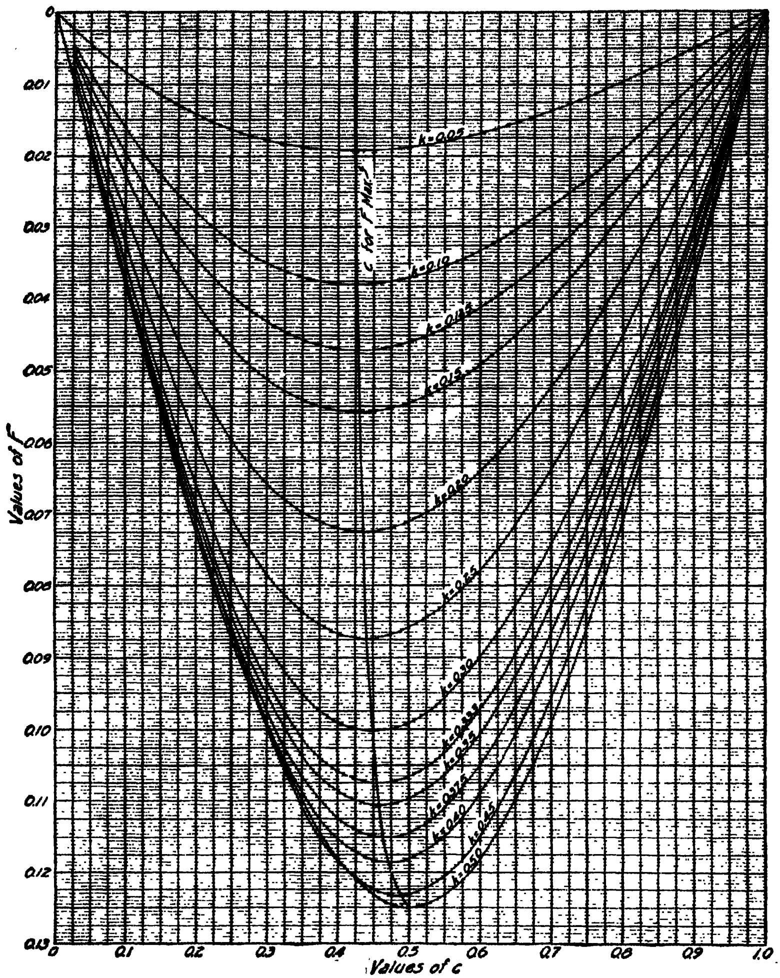

Values of F
Values of c
c for F Max.
k=0.05
k=0.10
k=0.125
k=0.15
k=0.20
k=0.25
k=0.30
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.10
0.11
0.12
0.13
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

FIG. 80.

TABLE 1.—VALUES OF F

$c \backslash k$	0.05	0.10	0.125	0.15	0.20	0.25	0.30	0.333	0.35	0.375	0.40	0.45	0.50
0.05	0.0045	0.0084	0.0101	0.0117	0.0143	0.0163	0.0178	0.0184	0.0187	0.0190	0.0191	0.0191	0.0187
0.10	0.0084	0.0162	0.0196	0.0227	0.0280	0.0321	0.0350	0.0364	0.0369	0.0375	0.0378	0.0378	0.0370
0.15	0.0117	0.0227	0.0278	0.0325	0.0405	0.0467	0.0512	0.0533	0.0541	0.0550	0.0556	0.0557	0.0546
0.20	0.0143	0.0280	0.0344	0.0405	0.0512	0.0596	0.0658	0.0687	0.0699	0.0712	0.0720	0.0723	0.0710
0.25	0.0163	0.0320	0.0396	0.0467	0.0596	0.0703	0.0783	0.0822	0.0837	0.0855	0.0866	0.0873	0.0859
0.30	0.0178	0.0350	0.0433	0.0512	0.0658	0.0783	0.0882	0.0931	0.0951	0.0974	0.0990	0.0999	0.0990
0.35	0.0187	0.0369	0.0457	0.0541	0.0699	0.0837	0.0951	0.1011	0.1035	0.1065	0.1087	0.1107	0.1098
0.40	0.0191	0.0378	0.0468	0.0556	0.0720	0.0866	0.0990	0.1058	0.1087	0.1124	0.1152	0.1183	0.1180
0.45	0.0191	0.0378	0.0469	0.0557	0.0723	0.0873	0.1002	0.1075	0.1107	0.1149	0.1183	0.1225	0.1232
0.50	0.0187	0.0370	0.0459	0.0546	0.0710	0.0859	0.0990	0.1065	0.1098	0.1143	0.1180	0.1232	0.1250
0.55	0.0179	0.0354	0.0440	0.0523	0.0682	0.0827	0.0955	0.1030	0.1063	0.1108	0.1148	0.1205	0.1232
0.60	0.0168	0.0332	0.0412	0.0491	0.0640	0.0778	0.0900	0.0972	0.1005	0.1049	0.1088	0.1148	0.1180
0.65	0.0153	0.0304	0.0377	0.0449	0.0586	0.0713	0.0827	0.0894	0.0925	0.0967	0.1005	0.1063	0.1098
0.70	0.0136	0.0270	0.0335	0.0399	0.0522	0.0636	0.0738	0.0799	0.0827	0.0866	0.0900	0.0955	0.0990
0.75	0.0117	0.0232	0.0288	0.0343	0.0449	0.0547	0.0636	0.0689	0.0713	0.0747	0.0778	0.0827	0.0859
0.80	0.0096	0.0190	0.0236	0.0281	0.0368	0.0449	0.0522	0.0566	0.0586	0.0615	0.0640	0.0682	0.0710
0.85	0.0073	0.0145	0.0180	0.0215	0.0281	0.0343	0.0390	0.0433	0.0449	0.0471	0.0491	0.0523	0.0546
0.90	0.0049	0.0098	0.0122	0.0145	0.0190	0.0232	0.0270	0.0293	0.0304	0.0319	0.0332	0.0354	0.0370
0.95	0.0025	0.0019	0.0061	0.0073	0.0096	0.0117	0.0136	0.0148	0.0153	0.0161	0.0168	0.0179	0.0187
For maximum deflection													
c	0.4234	0.4255	0.4272	0.4292	0.4343	0.4409	0.4492	0.4557	0.4592	0.4648	0.4708	0.4850	0.5000
F	0.0192	0.0379	0.0470	0.0558	0.0724	0.0873	0.1002	0.1075	0.1107	0.1150	0.1185	0.1234	0.1250

Illustrative Problem.—A 20-in. 65.4-lb. I-beam supports two loads of 30,000 lb. each symmetrically placed about the center line of the beam. The span of the beam is 25 ft. and the distance between loads 15 ft. Determine the deflection at the beam center using deflection coefficients. $E = 29{,}000{,}000$ lb. per sq. in. and $I = 1{,}169.5$ in.^{4}

For the given conditions $k = 0.2$ and $c = 0.5$, for each load, since the loads are symmetrically placed on the beam. From Table 1, or Fig. 80, for $k = 0.2$ and $c = 0.5$, we find $F = 0.071$ for each load. Then

$$y = \frac{2Pl^3}{6EI}(0.071) = \frac{60{,}000(25 \times 12)^3 0.071}{(6)(33{,}915{,}500{,}000)} = 0.565 \text{ in.}$$

Illustrative Problem.—A 15-in. 37.3-lb. I-beam, 10 ft. long, carries a load of 10,000 lb. at a point 3 ft. from one end and a load of 5,000 lb. at a point 2 ft. from the other end. Determine the deflection at the center using deflection coefficients $E = 29{,}000{,}000$ lb. per sq. in. and $I = 405.5$ in.^{4}

With $k = 0.3$ and $c = 0.5$ for the 10,000-lb. load we find from Table 1 or Fig. 80, that $F = 0.991$. With $k = 0.2$ and $c = 0.5$ for the 5,000-lb. load we find from Table 1, or Fig. 80, that $F = 0.710$. The total deflection at the center equals the sum of the deflections caused by the two loads considered independently. Then, substituting in eq. (3) of the preceding article, we have

$$y = \frac{l^3}{6EI}[10{,}000(0.991) + 5{,}000(0.710)]$$

$$= \frac{(120)^3(13{,}460)}{6(29{,}000{,}000)(405.5)} = 0.33 \text{ in.}$$

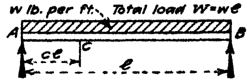


FIG. 81.

Simple Beam with a Uniform Load.—Figure 81 shows a simple beam of span l supporting a uniform load of w lb. per ft. The deflection at any point distance

cl from the end of the beam, where c is a fraction less than unity, is as follows:

$$y = \frac{wl^4}{24EI}[c(1 - c)(1 + c - c^2)] \tag{5}$$

which may also be written, in terms of the total load $W = wl$, in the form

$$y = \frac{Wl^3}{24EI}[c(1 - c)(1 + c - c^2)] \tag{6}$$

Let J represent the expression in brackets. Equations (5) and (6) may then be written

$$y = \frac{wl^4}{24EI}J = \frac{Wl^3}{24EI}J \tag{7}$$

Values of J are given in Table 2 and in Fig. 82.

TABLE 2.—VALUES OF J

c	J	c	J
0.05	0.0498	0.30	0.2541
0.10	0.0981	0.35	0.2793
0.15	0.1438	0.40	0.2976
0.20	0.1856	0.45	0.3088
0.25	0.2227	0.50	0.3125

Illustrative Problem.—A simple beam 12 ft. long, composed of a 10-in. 25.4-lb. I-beam carries a uniformly distributed load of 800 lb. per ft. Find the deflection at the center and at the quarter-point using deflection coefficients. E = 29,000,000 lb. per sq. in. and I = 122.1 in.[4]

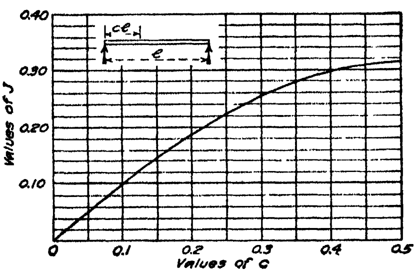


FIG. 82.

From Table 2 or Fig. 82, when c = 0.5, J = 0.3125 and when c = 0.25, J = 0.2227. Substituting these values of J in eq. (7) of the preceding article we find that at the center

$$y = \frac{12(200)(144)^3(0.3125)}{29,000,000(122.1)} = 0.633 \text{ in.}$$

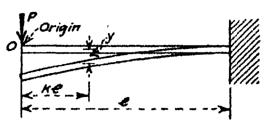


FIG. 83.

and at the quarter-point

$$y = \frac{12(200)(144)^3(0.2227)}{29,000,000(122.1)} = 0.451 \text{ in.}$$

Cantilever Beam with a Single Concentrated Load.—Figure 83 shows a cantilever beam with a single concentrated load at the free end. The deflection at any point distance kl from the free end with respect to an origin at O, Fig. 83, is given by the expression

$$y = \frac{Pl^3}{6EI}[(1 - k)^2(2 + k)] \tag{8}$$

Denoting the term in brackets by L, eq. (8) may be written

$$y = \frac{Pl^3}{6EI} L \tag{9}$$

Values of L are given in Table 3 and Fig. 84.

TABLE 3.—VALUES OF L

k	L	k	L
0.0	2.0	0.6	0.416
0.1	1.701	0.7	0.243
0.2	1.408	0.8	0.112
0.3	1.127	0.9	0.029
0.4	0.864	1.0	0.000
0.5	0.625		

Illustrative Problem.—A cantilever beam 8 ft. long, composed of a 10-in. 25.4-lb. I-beam carries a concentrated load of 4,000 lb. at its free end. Find the deflection at the free end and at a point 4 ft. from the free end using deflection coefficients. $E = 29{,}000{,}000$ lb. per sq. in. and $I = 122.1$ in.[4]

From Table 3 or Fig. 84, when $k = 0.5$, $L = 0.625$ and when $k = 0.0$, $L = 2.0$. Substituting these values of L in eq. (9) we find that at a point 4 ft. from the free end

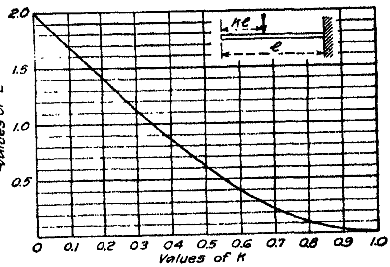


FIG. 84.

$$y = \frac{4{,}000(96)^3(0.625)}{6(29{,}000{,}000)(122.1)} = 0.104 \text{ in.}$$

and at the free end

$$y = \frac{4{,}000(96)^3(2.000)}{6(29{,}000{,}000)(122.1)} = 0.333 \text{ in.}$$

66. Deflection in Terms of Fiber Stress.—In the solution of problems in certain statically indeterminate structures, it is convenient to have an expression for beam deflection in terms of the extreme fiber stress on the section due to bending. To illustrate, consider the case of a simple beam of span l carrying a uniform load of w lb. per ft. As stated on p. 52, the maximum deflection for the given conditions is

$$y_{max} = \frac{5}{384}\frac{wl^4}{EI}$$

To express this deflection formula in terms of fiber stress, we may proceed as follows: From the general formula for fiber stress, we have $f = \frac{Mc}{I}$. Solving for M, and letting d = depth of section = $2c$, we have, $M = \frac{2fI}{d}$. For the given loading conditions, we have, $M = \frac{wl^2}{8}$. Equating these values of M, and solving for

w, we find $w = \frac{16fI}{dl^2}$. On substituting this value of w in the above deflection formula, we derive

$$y_{max} = \frac{5}{24}\frac{fl^2}{Ed} \tag{1}$$

Equation (1) expresses the deflection as a function of fiber stress instead of the loading. Note that the deflection is proportional to the fiber stress.

Similar values may also be derived for other loading conditions. Such values are given as a part of Table 6, p. 95.

Illustrative Problem.—A 10-in. 25.4-lb. steel I-beam, 20 ft. long, supports a uniform load. If the maximum fiber stress for the given loading conditions is 15,000 lb. per sq. in., determine the maximum deflection in inches.

For the given conditions the several terms of eq. (1) have the following values: f = 15,000 lb. per sq. in., l = 20 ft. = 240 in., E = 30,000,000 lb. per sq. in., and d = 10 in. From eq. (1)

$$y_{max} = \frac{(5)(15{,}000)(240)^2}{(24)(30{,}000{,}000)(10)} = 0.60 \text{ in.}$$

67. Limiting Deflection of Beams.—In designing beams, it is the usual practice to fix the allowable fiber stresses in bending and determine the beam section subject to the given conditions. However, there are certain types of construction in which it is desirable to place certain limits on the deflection under the applied loads.

In practice this limiting deflection is obtained by specifying that the depth of a beam shall not be less than a certain fractional part of its span length. To illustrate the methods employed in determining these practical rules, consider the case of a simple beam uniformly loaded. Suppose the allowable fiber stress in bending to be f lb. per sq. in. and suppose the deflection is limited to $\frac{1}{300}$ part of the span, a usual limit placed on ordinary construction.

The desired relation may be determined by equating the allowable deflection, $\frac{l}{300}$, to the maximum deflection in terms of fiber stress, as given by eq. (1), p. 82. Thus

$$\frac{l}{300} = \frac{5}{24}\frac{fl^2}{Ed}$$

Solving this expression for $\frac{l}{d}$, the desired ratio, we have

$$\frac{l}{d} = 0.016\frac{E}{f} \tag{2}$$

Suppose the beam is of steel for which E = 30,000,000 lb. per sq. in. and allowable f = 16,000 lb. per sq. in. From eq. (2)

$$\frac{l}{d} = \frac{(0.016)(30{,}000{,}000)}{(16{,}000)} = 30$$

That is, the depth of the beam shall not be less then $\frac{1}{30}$ of the span. When this condition has been satisfied, the deflection will not exceed the limiting value specified. For plastered ceilings, it is usual to specify a limiting deflection of $\frac{1}{360}$ of the span.

Table 4 gives general formulas and limiting spans for steel and wooden beams for various standard loadings. These values are based on the formulas given in Table 6 on p. 95.

TABLE 4

Loading condition	Limiting deflection $= \frac{\text{Span}}{300}$			Limiting deflection $= \frac{\text{Span}}{360}$		
	General formula	Limiting ratio span to depth: Steel beams $E = 30{,}000{,}000$, $f = 16{,}000$	Limiting ratio span to depth: Wooden beams $E = 750{,}000$, $f = 1{,}000$	General formula	Limiting ratio span to depth: Steel beams $E = 30{,}000{,}000$, $f = 16{,}000$	Limiting ratio span to depth: Wooden beams $E = 750{,}000$, $f = 1{,}000$
w lb per ft.	$\frac{l}{d} = 0.0167\frac{E}{f}$	$\frac{l}{d} = 30.0$	$\frac{l}{d} = 12.0$	$\frac{l}{d} = 0.0133\frac{E}{f}$	$\frac{l}{d} = 25.0$	$\frac{l}{d} = 10.0$
W	$\frac{l}{d} = 0.020\frac{E}{f}$	$\frac{l}{d} = 37.5$	$\frac{l}{d} = 15.0$	$\frac{l}{d} = 0.0167\frac{E}{f}$	$\frac{l}{d} = 31.25$	$\frac{l}{d} = 12.5$
	Limiting deflection $= \frac{\text{Span}}{150}$			Limiting deflection $= \frac{\text{Span}}{180}$		
w lb. per ft.	$\frac{l}{d} = 0.01333\frac{E}{f}$	$\frac{l}{d} = 25$	$\frac{l}{d} = 10$	$\frac{l}{d} = 0.0111\frac{E}{f}$	$\frac{l}{d} = 20.83$	$\frac{l}{d} = 8.33$
W	$\frac{l}{d} = 0.0100\frac{E}{f}$	$\frac{l}{d} = 18.75$	$\frac{l}{d} = 7.50$	$\frac{l}{d} = 0.00833\frac{E}{f}$	$\frac{l}{d} = 15.66$	$\frac{l}{d} = 6.25$

68. Maxwell's Theorem of Reciprocal Displacements.[1]—Maxwell's theorem of reciprocal displacements establishes a mutual relation between the deflections at any two points in a structure. This theorem, when considered in connection with the deflection of beams, may be stated as follows: If the load P at A, Fig. 85, causes a deflection Δ_2 at B, and the load P at B, Fig. 86, causes a deflection Δ_3 at A, then, according to Maxwell's theorem, $\Delta_2 = \Delta_3$. Let Figs.

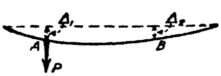


FIG. 85.

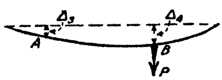


FIG. 86.

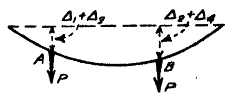


FIG. 87.

85, 86 and 87 represent the deflections of a beam when the loads are applied gradually. When A (Fig. 85) has received its full load, the work done is $\frac{1}{2}P\Delta_1$. With a full load P at A, let another load P be gradually added at B. The deflections as shown in Fig. 87 will result. The point A with the full load P. moves through the additional distance Δ_3; and the point B moves through the additional distance Δ_4 as the load P is gradually applied at B. Hence the total work done is

$$\tfrac{1}{2}P\Delta_1 + P\Delta_3 + \tfrac{1}{2}P\Delta_4$$

If B is loaded first and then A is loaded, the total work done is

$$\tfrac{1}{2}P\Delta_4 + P\Delta_2 + \tfrac{1}{2}P\Delta_1$$

The total amount of work done in each case is the same, hence

$$\Delta_2 = \Delta_3$$

Maxwell's law may be verified by Table 1 on p. 79. When A and B are on the same side of the center, the values of k and c for Fig. 85 become interchanged for Fig. 86. For example, $F = 0.0658$ when $k = 0.2$ and $c = 0.3$; likewise $F =$

[1] Contributed by C. A. ELLIS.

0.0658 when $k = 0.3$ and $c = 0.2$. When A and B are on opposite sides of the center, the application is made as follows: In Fig. 85 let $k = 0.3$ and $c = 0.8$; then in Fig. 86, $k = 0.2$ and $c = 0.7$; whence $F = 0.0522$ in each case. Maxwell's law renders excellent service in the solution of statically indeterminate structures.

69. Approximate Method for Determination of Deflection of Beams.—Professor J. B. Kommers[1] has devised and published[2] a very useful and readily applied approximate method for the determination of the deflection of beams. This method, which is based on Maxwell's theorem of reciprocal deflections is intended for use in cases where combinations of concentrated and uniform loading would lead to a long and tedious solution by ordinary methods. The article mentioned discusses simple, cantilever, and restrained beams of uniform cross-section.

Suppose a simple beam of span l carries a concentrated load W. When this load is at the span center, the maximum deflection occurs at a point under the load. When the load is at the right support, the maximum deflection occurs at a point $0.0773l$ to the right of the beam center (see Art. **63***b*, p. 51). Therefore, no matter where a load may be placed on the beam the point of maximum deflection can not be more remote from the beam center than $0.0773l$. It is therefore evident that the mid-span deflection and the maximum deflection for any load, regardless of the position of that on the span, are very nearly equal. The method under discussion assumes that these deflections are equal.

Figure 88 shows a simple beam supporting two concentrated loads W and P. According to Maxwell's theorem of the preceding article, the deflection produced at the beam center O by the load W at A is equal to the deflection at A produced by the load W at O. The same statement holds true for load P at B. Therefore, to determine the deflection of O for the loads shown in Fig. 88, place load W at O and calculate the deflection at A; place load P at O and calculate the deflection at B. The sum of these two deflections will give very closely the total deflection produced at O by the two loads when in the position shown in Fig. 88.

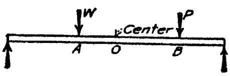


Fig. 88.

The deflection at a point distance x from the nearest support due to a central load W is given by the expression

$$\text{Defl.} = \frac{Wx}{48EI}(4x^2 - 3l^2)$$

This equation may be used for the determination of the deflections required in the above discussion. However, the calculations may readily and rapidly be made by means of Table 1, p. 79, or Fig. 80, p. 78, and eq. (3), p. 77, which is

$$\text{Defl.} = \frac{Pl^3}{6EI}F$$

where P is any load, l = span, and F = a coefficient which is a function of k and c where c = a fraction which expresses the distance from one support to the point of deflection in terms of the span length, and k = a corresponding fraction expressing the distance from a support to the load point.

When the applied loads consist of a uniform load covering a part of the span, the load may be broken up into short sections which may be replaced by the total load on that section considered as applied at its center of gravity.

[1] Associate Professor of Mechanics, The University of Wisconsin.

[2] *Eng. News-Record*, Jan. 2, 1919, p. 44.

Illustrative Problem.—Determine the maximum deflection for the conditions shown in Fig. 89, using the approximate method.

Assuming the maximum deflection to occur at the beam center, $c = \frac{1}{2}$. For the load at A, $k = \frac{5}{15} = \frac{1}{3}$ and for the load at B, $k = \left(\frac{15-3}{15}\right) = 0.8$. From Fig. 80, p. 78, or Table 1, p. 79, for the load at A, ($c = \frac{1}{2}$, $k = \frac{1}{3}$), we find $F = 0.107$ and for the load at B, ($c = \frac{1}{2}$, $k = 0.8$) we find $F = 0.071$. From eq. (3), p. 78, we have

$$PF = (7{,}000)(0.107) + (8{,}000)(0.071) = 1{,}310$$

$$\text{Defl. at } O = \frac{PFl^3}{6EI} = \frac{(1{,}310)(15)^3(1{,}728)}{(6)(30{,}000{,}000)(215.8)} = 0.1971 \text{ in.}$$

As a check, the correct maximum deflection was calculated by the methods given in the preceding articles from which it was found that Defl. = 0.1970 in.

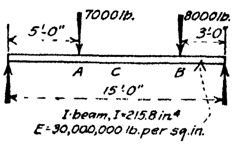


Fig. 89.

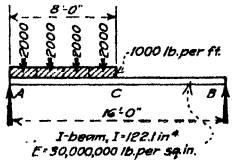


Fig. 90.

Illustrative Problem.—Determine the maximum deflection for the conditions shown in Fig. 90, using the approximate method. Assume the uniform load to be divided into four sections.

The values of k for the several loads, taken in order from the left end of the span, are as follows: 0.0625; 0.1875; 0.3125; and 0.4375. With $c = \frac{1}{2}$ and the several values of k given above, the term PF of eq. (3) becomes

$$PF = 2{,}000(0.0230 + 0.0680 + 0.1025 + 0.1225) = 632$$

$$\text{Defl. at } C = \frac{(632)(16)^3(1{,}728)}{(6)(30{,}000{,}000)(122.1)} = 0.203 \text{ in.}$$

By the exact methods, the true deflection is 0.2029 in. and the point of maximum deflection is 7.36 ft. from the left end of the beam. It may be of interest to note that the time required for the approximate solution was less than five minutes while the exact solution required nearly half an hour.

For cantilever beams the maximum deflection always occurs at the free end. Hence no approximation is involved in the application of the method given above

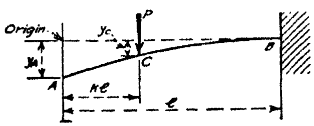


Fig. 91.

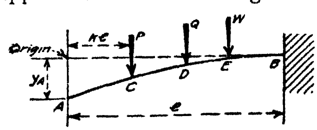


Fig. 92.

to cantilever beams. Figure 91 shows a cantilever beam with a load P at a distance kl from the free end. According to Maxwell's theorem, the deflection at the free end A due to a load at C is equal to the deflection at C due to a load at A. Therefore eq. (9), p. 81, may be used to determine the deflection of point A for a load at C, or for the deflection of point C for a load at A. In one case kl represents the distance to the load point and in the other case it represents the distance to the deflection point. Figure 84, p. 81, and Table 3, p. 81, may then be used in either case.

To apply the method given above to the determination of the deflection of point A of Fig. 92 due to the loading shown, consider each load in turn as applied

at the free end A, and determine the deflection at the true position of that load by means of Table 3, p. 81, or Fig. 84, p. 81. The sum of these deflections is the required total deflection for point A.

When the cantilever beam is covered by a partial uniform load, consider the uniform load as broken into short sections. Apply the total load on each section and solve as for concentrated load systems.

Illustrative Problem.—Calculate the deflection of the free end of the cantilever beam of Fig. 93 for the loads shown.

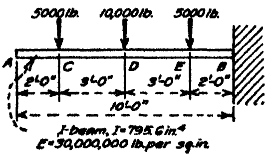


FIG. 93.

The values of k for the several loads, taken in order from left to right are 0.2, 0.5, and 0.8. From Table 3 or Fig. 84, the value of PL of eq. (9), p. 81, is as follows:

Load at C	= (1.408)(5,000) =	7,040
Load at D	= (0.625)(10,000) =	6,250
Load at E	= (0.112)(6,000) =	672
	ΣPL =	13,962

Then from eq. (9),

$$y_A = \Sigma PL \frac{l^3}{6EI} 1{,}728$$

$$= \frac{(13{,}962)(1{,}728)(10)^3}{(6)(30{,}000{,}000)(795.6)} = 0.167 \text{ in.}$$

70. Deflection Due to Shearing Stresses.—The deflection of beams due to shearing stresses is generally small and may be neglected in most cases. However, in short beams, the deflection due to shear may be so large that it cannot safely be neglected. To assist in deciding whether shearing deflections need be considered, a method will be given for the determination of these deflections, and comparisons will be made with the deflection due to bending stresses.

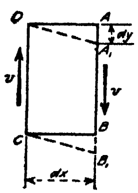


FIG. 94.

Figure 94 represents a beam element $ABCD$ acted upon by shearing stresses of intensity v applied to the faces AB and CD. Due to the action of these stresses, the element is deformed to A_1B_1CD. The modulus of elasticity for shear is equal to the unit shearing stress divided by the unit shearing strain. In Fig. 94, dy represents the shearing strain and hence $\frac{dy}{dx}$ represents the unit shearing strain. If v = intensity of shearing stress and E_s = modulus of elasticity for shear, we have

$$E_s = \frac{\text{unit shearing stress}}{\text{unit shearing strain}} = \frac{v}{\frac{dy}{dx}}$$

from which

$$\frac{dy}{dx} = \frac{v}{E_s}$$

If V = shear on section due to external forces, and A = area of cross-section, $v = \frac{V}{A}$, and the above equation becomes

$$\frac{dy}{dx} = \frac{V}{AE_s} \tag{1}$$

Equation (1) is a general expression for the differential equation of the elastic curve for shearing deflections when the shearing stress intensity is uniform over the section. Equation (1) is applicable to beam sections for which it is reasonable to assume uniform distribution of shearing stress, such as plate girder or I-beam webs.

To determine the shearing deflection when the shearing stress intensity is variable the external work due to shear must be placed equal to the internal work due to the shearing fiber stresses. Figure 95*a* shows a beam element whose length along the beam axis is dx. If V represents the shear on a face of this element due to external forces, and dy represents the vertical deformation of the element due to shear, the average external work done during the deflection of the element is $\frac{1}{2}Vdy$. The internal work done by the fibers of the element due to shearing distortion is equal to the elastic resilience due to shear, which is $\frac{1}{2}\frac{v^2}{E_s}$ per unit of volume, where v = shearing stress intensity on any area as 1-2, Fig. 95*b*. From eq. (3), p. 26, $v = \frac{V}{bI}\Sigma_z^{h} bzdz$, where V = external shear, I = moment of inertia of section, and the other terms have the values shown on Fig. 95*b*. Substituting this value of v in the above equation, noting that the volume of any fiber such as 1-2, Fig. 95*b*, is $bdzdx$, we have, for all fibers in the beam cross-section

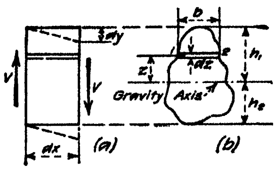


Fig. 95.

$$\text{Average internal work} = \frac{1}{2E_s}\Sigma_{h_2}^{h_1}\frac{V^2}{b^2I^2}(\Sigma_z^{h_1} bzdz)^2\, bdzdx$$

Equating the expressions for internal and external work, and solving for dy/dx, we have

$$\frac{dy}{dx} = \frac{V}{E_s}\int_{h_2}^{h_1}\frac{1}{bI^2}\left(\int_z^{h_1} bzdz\right)^2 dz \tag{2}$$

In eq. (2) the integral expression depends for its value on the form of the beam section. It will hereafter be called the Section Constant and it will be denoted by N, that is

$$N = \text{Section constant} = \int_{h_2}^{h_1}\frac{1}{bI^2}\left(\int_z^{h_1} bzdz\right)^2 dz \tag{3}$$

Equation (2) may then be written

$$\frac{dy}{dx} = \frac{VN}{E_s} \tag{4}$$

Equation (4) is a general expression for the differential equation of the elastic curve for shearing deflection when the shearing stress intensity is assumed as variable.

The above analysis is based on the assumption that the effective shearing area of the beam is constant over the entire length of the beam. When the area of cross section is variable, the formulas become more complicated.[1]

The value of the section constant N of eq. (3) depends upon the form of the section. As an example of the application of eq. (3) to a specific case, consider the

[1] Methods for the determination of shearing deflection for beams of variable cross-section are given in an article by Prof. S. E. Slocum which appeared in the *Journal of the Franklin Institute*, April, 1911.

rectangular section of Fig. 96. For the dimensions shown on Fig. 96, $I = \frac{bd^3}{12}$, $h_1 = h_2 = \frac{d}{2}$ and $b = b$. Equation (3) then becomes

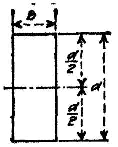


Fig. 96.

$$N = \frac{144}{b^2d^6}\int_{-\frac{d}{2}}^{+\frac{d}{2}} \frac{1}{b}\left(\int_z^{\frac{d}{2}} bz\,dz\right)^2 dz$$

Integrating the expression in parenthesis,

$$N = \frac{144}{b^2d^6}\int_{-\frac{d}{2}}^{+\frac{d}{2}} b\left(\frac{d^2}{8} - \frac{z^2}{2}\right)^2 dz$$

Integrating again, we have finally

$$N = \frac{1.2}{bd} = \frac{1.2}{A}$$

where $A = bd$ = area of section. By a similar process it can be shown that, for a circle,

$$N = \frac{10}{9A}$$

70a. Application of General Formulas to the Determination of Shearing Deflection.—When the deflection at a certain point is desired without reference to that at any other point, the desired deflection may be obtained by writing eqs. (1) and (4) in the form of a definite integral, thus:

From eq. (1)

$$y = \frac{1}{AE_s}\int_0^x V\,dx$$

From eq. (4)

$$y = \frac{N}{E_s}\int_0^x V\,dx \qquad (5)$$

where y = deflection of any point at a distance x from a convenient origin, as shown in Fig. 97. It is evident that the deflection at C of Fig. 97 is the sum of all such values as dy of Fig. 94 from A to C of Fig. 97. Hence the limits of integration for eqs. (5) are x to 0, as indicated. But $\int_0^x V\,dx$ = area shear diagram from x to 0 = moment at point x. Hence, when the shear is uniform across the section,

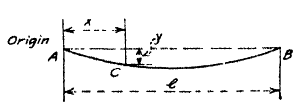


Fig. 97.

$$y = \frac{1}{AE_s}\text{(area shear diagram from end of beam to deflection point)} = \frac{M_x}{AE_s} \qquad (6)$$

and when the shear is variable across the section,

$$y = \frac{N}{E_s}\text{(area shear diagram from end of beam to deflection point)} = \frac{NM_x}{E_s} \qquad (7)$$

where M_x = moment due to applied loads at the point whose deflection is desired. A few typical problems will now be worked out in detail to illustrate the application of the method described above.

Simple Beam with Uniform Load.—As shown above, the shearing deflection at any point is proportional to the moment at that point due to the applied loading. For the conditions shown in Fig. 98 the moment at any point distance x from the left end of the beam is $M_x = \frac{w}{2}x(l - x)$. Hence the general equation for shearing deflection at that point, is given by eqs. (6) and (7),

$$\left.\begin{aligned} y &= \frac{wx}{2AE_s}(l - x) \\ \text{and} \qquad y &= \frac{wxN}{2E_s}(l - x) \end{aligned}\right\} \tag{8}$$

Since the shearing deflection is proportional to M_x, evidently the deflection is a maximum when M_x is a maximum. Now M_x is a maximum at the beam center and is equal to $\frac{1}{8}wl^2$. Hence

$$\left.\begin{aligned} y_{max} &= \frac{wl^2}{8AE_s} \text{ for uniform shearing stress} \\ \text{and} \qquad y_{max} &= \frac{wNl^2}{8E_s} \text{ for variable shearing stress} \end{aligned}\right\} \tag{9}$$

Illustrative Problem.—Assume that the beam of Fig. 98 is a 3 × 12-in. wooden member and that the span is 12 ft. and the uniform load is 300 lb. per foot. Determine the deflection in inches at the center of the beam due to shear, assuming variable shearing stress across the beam section.

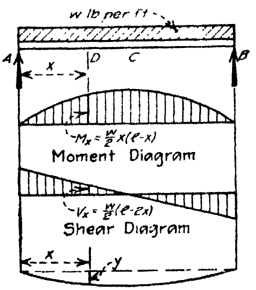


Fig. 98.

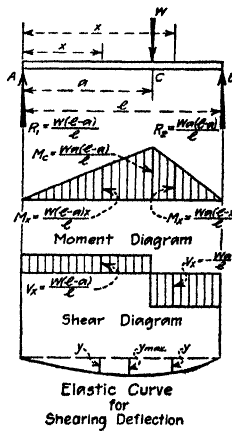


Fig. 99.

The desired deflection is given by eq. (9) with $l = 12$, $w = 300$, $N = \frac{1.2}{A} = \frac{2}{(3)}\frac{1}{(12)} = 0.0333$, and $E_s = \frac{1}{4}E = (\frac{1}{4})(1{,}500{,}000) = 375{,}000$. Then

$$y_{max} = \frac{(300)(0.0333)(144)(12)}{(8)(375{,}000)} = 0.00576 \text{ in.}$$

Simple Beam with a Single Concentrated Load.—Figure 99 shows the moment, shear, and deflection diagrams for the given conditions. In the pairs of equations given below, the deflection for uniform shear is given first, and the deflection for variable shear follows:

At a point distance x from the left end on the portion AC of the given beam, $M_x = \frac{Wx}{l}(l - a)$. Hence

$$\left.\begin{aligned} y &= \frac{Wx}{AE_sl}(l-a) \\ \text{and} \qquad y &= \frac{WNx}{E_sl}(l-a) \end{aligned}\right\} \tag{10}$$

On the portion CB, where $M_x = \frac{Wa}{l}(l - x)$,

$$\left.\begin{aligned} y &= \frac{Wa}{AE_sl}(l-x) \\ \text{and} \qquad y &= \frac{WNa}{E_sl}(l-x) \end{aligned}\right\} \tag{11}$$

Since x is of the first power in eqs. (10) and (11), the elastic curves are straight lines. Note that the shear is constant for each of the two portions of the beam. Hence, when the shear is constant, and therefore also, the slope is constant, the elastic curve is a straight line for shearing deflection.

The maximum shearing deflection occurs where M_x has its maximum value. This occurs at the load point, and

$$M_{max} = \frac{Wa(l-a)}{l}$$

Hence

$$\left.\begin{aligned} y_{max} &= \frac{Wa(l-a)}{AE_sl} \\ \text{and} \qquad y_{max} &= \frac{WNa(l-a)}{E_sl} \end{aligned}\right\} \tag{12}$$

At some point in the beam, the shearing deflection will be greater than at any other point. This is evidently the point of greatest moment in the beam. In Art. 48, p. 17, this point has been shown to be at the beam center, where, for the conditions shown in Fig. 99, $M = \frac{Wl}{4}$. Hence, the greatest shearing deflection is

$$\left.\begin{aligned} y &= \frac{Wl}{4AE_s} \\ \text{and} \qquad y &= \frac{WNl}{4E_s} \end{aligned}\right\} \tag{13}$$

Illustrative Problem.—Assume that the beam of Fig. 99 is a 3 × 12-in. wooden member 12 ft. long, and that the applied load is a single concentrated load of 2,400 lb. placed 8 ft. from the left end of the beam. Determine the deflection in inches at the load point, assuming variable shearing stress across the beam section.

The desired deflection is given by eq. (12) with $l = 12$, $W = 2{,}400$, $N = \frac{1.2}{A} = \frac{1.2}{(3)(12)} = 0.0333$, and $E_s = \frac{1}{4}E = (\frac{1}{4})(1{,}500{,}000) = 375{,}000$. Then

$$y_{max} = \frac{(2{,}400)(8)(4)(12)}{(36)(375{,}000)(12)} = 0.00569 \text{ in.}$$

Cantilever Beams.—Methods similar to those explained for simple beams may also be used for the determination of the shearing deflection for cantilever beams. Figure 100*a* shows a cantilever beam fixed at the right end and supporting a uniform load. A deformed element is shown in Fig. 100*b* and the shear diagram for the applied loads is shown in Fig. 100*d*. The form of the elastic curve of the deflected beam is shown in Fig. 100*e*.

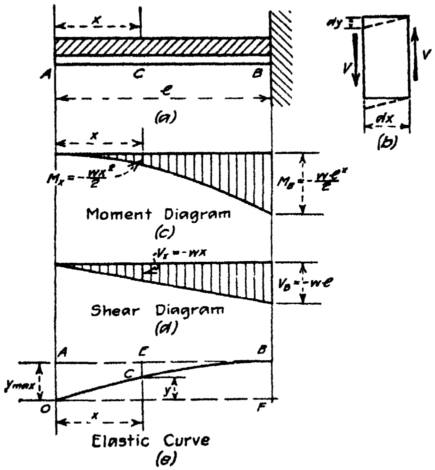


FIG. 100.

If the deflection is measured from an origin at point O of Fig. 100*e*, the deflection of any point C is given by the distance y, which is the sum of the dy values from O to C. We then have, assuming uniform shearing stress over the section, $y = \frac{1}{AE_s}\int_0^x V dx$. But $\int_0^x V dx = M_x$ which is the moment shown in the moment diagram of Fig. 100*d* at a distance x from the free end of the beam. Hence, as before

$$y = \frac{M_x}{AE_s} \text{ for uniform shear}$$

and (14)

$$y = \frac{M_x N}{E_s} \text{ for variable shear}$$

When the deflection is measured from an origin at point A of Fig. 100*e*, the deflection of point C is represented by the distance EC, which is the sum of the dy values from the fixed end B to point C. We then have, for uniform shearing stress on the section, $y = \frac{1}{AE_s}\int_x^l V dx$. This integral represents the shear diagram area (Fig. 100*d*) from C to B, and does not represent the moment at C, as in the former case. However, the above integral may be written, $y = \frac{1}{AE_s}\left[\int_0^l V dx - \int_0^x V dx\right]$. Now $\int_0^l V dx$ = total shear diagram area $= M_B$, and $\int_0^x$

Vdx = moment at $C = M_x$. Hence for an origin at A, we may write

$$y = \frac{1}{AE_s}[M_B - M_x]$$

where M_B = maximum moment for cantilever beam and M_x = moment at point where the deflection is desired. Therefore, the general equations become

$$\left.\begin{aligned} y &= \frac{1}{AE_s}[M_B - M_x] \text{ for uniform shear} \\ y &= \frac{N}{E_s}[M_B - M_x] \text{ for variable shear} \end{aligned}\right\} \tag{15}$$

These values are general and hold for any type of loading.

The maximum deflection, y_{max} of Fig. 100*c*, occurs at the free end of the beam. For an origin at O it is measured by FB, and for an origin at A it is measured by AO. In both cases, eqs. (14) and (15) reduce to the common form,

$$\left.\begin{aligned} y_{max} &= \frac{M_{max}}{AE_s} \text{ for uniform shear} \\ \text{and} \quad y_{max} &= \frac{NM_{max}}{E_s} \text{ for variable shear} \end{aligned}\right\} \tag{16}$$

For the loading conditions shown in Fig. 100*a*, $M_{max} = \frac{1}{2} wl^2$, and we have

$$\left.\begin{aligned} y_{max} &= \frac{wl^2}{2AE_s} \text{ for uniform shear} \\ \text{and} \quad y_{max} &= \frac{wl^2N}{2E_s} \text{ for variable shear} \end{aligned}\right\} \tag{17}$$

71. Comparative Values of Bending and Shearing Deflection.—As shown in the preceding articles, the greatest deflection due to bending and to shearing stresses for symmetrical loadings occurs at the center of the beam. It will be of interest to compare these maximum values for standard common loading conditions.

To illustrate the method employed in making this comparison, consider the case of a simple beam with a uniform load. From the preceding articles, the maximum deflection due to bending is $D_B = \frac{5}{384}\frac{wl^4}{EI}$ and the maximum deflection due to shear, assuming the shearing stress to be uniformly distributed, is $D_S = \frac{wl^2}{8AE_s}$. Noting that $I = Ar^2$, where A = area of section and r = its radius of gyration, the ratio of shearing to bending deflection may be written

$$\frac{D_S}{D_B} = 9.6\frac{E}{E_s}\left(\frac{r}{l}\right)^2 \tag{1}$$

This ratio is given in terms of E and E_s, the bending and shearing moduli of elasticity of the material composing the beam, and the radius of gyration of the beam section and the span length.

Tests show that for steel beams, $E_s = 0.4E$, and for wooden beams, $E_s = 0.25E$, approximately. From an examination of the properties of rolled I-beams and channels given in the rolling mill handbooks it will be found that $r = 0.4d$, approximately, for all sections, where d = depth of section. For plate girders,

considering only the flange areas, it will be found that $r = 0.5d$. Wooden beams are generally of rectangular form. For such sections it is shown in Art. **9**, p. 581, that $r = \frac{d}{\sqrt{12}}$. Substituting these general values in eq. (1), we have

For plate girders

$$\frac{D_S}{D_B} = 6.0\left(\frac{d}{l}\right)^2 \tag{2}$$

For I-beams and channels

$$\frac{D_S}{D_B} = 3.84\left(\frac{d}{l}\right)^2 \tag{3}$$

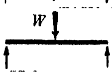
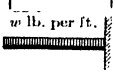
For rectangular sections, the shearing stresses are generally not uniformly distributed over the section, as shown in Art. **51c**, p. 27. Under such conditions the multiplier 1.2 calculated as explained in Art. **70**, p. 88, should be introduced. Substituting above values of E, E_s and r in eq. (1), we have

For rectangular wooden beams

$$\frac{D_S}{D_B} = 3.84\left(\frac{d}{l}\right)^2 \tag{4}$$

Equations (2) to (4) express the relative shearing and bending deflection in terms of the depth of section and length of beam. Table 5 gives corresponding values for simple and cantilever beams for standard loadings.

TABLE 5.—RELATIVE DEFLECTION DUE TO BENDING AND SHEAR

Loading condition	Bending deflection D_B	Shearing deflection D_S	D_S/D_B					
			Steel beams				Wooden beams (rectangular)	
			Plate girders		I-beams and channels		General formula	Limiting span $D_S = D_B$
			General formula	Limiting span $sD = D_B$	General formula	Limiting span $sD = D_B$		
	$\frac{5}{384}\frac{wl^4}{EI}$	$\frac{wl^2}{8AE_S}$	$6.0\left(\frac{d}{l}\right)^2$	$l = 2.45d$	$3.84\left(\frac{d}{l}\right)^2$	$l = 1.96d$	$3.84\left(\frac{d}{l}\right)$	$l = 1.96d$
	$\frac{Wl^3}{48EI}$	$\frac{Wl}{4AE_S}$	$7.5\left(\frac{d}{l}\right)^2$	$l = 2.74d$	$4.8\left(\frac{d}{l}\right)^2$	$l = 2.19d$	$4.8\left(\frac{d}{l}\right)^2$	$9d\ \ l = 2.1$
	$\frac{wl^4}{8EI}$	$\frac{wl^2}{2AE_S}$	$2.5\left(\frac{d}{l}\right)^2$	$l = 1.58d$	$1.6\left(\frac{d}{l}\right)^2$	$l = 1.26d$	$1.6\left(\frac{d}{l}\right)^2$	$l = 1.26d$
	$\frac{Wl^3}{3EI}$	$\frac{Wl^3}{AE_S}$	$1.875\left(\frac{d}{l}\right)^2$	$l = 1.37d$	$1.2\left(\frac{d}{l}\right)^2$	$l = 1.095d$	$1.2\left(\frac{d}{l}\right)^2$	$l = 1.095d$

For steel, $E_S = 0.4E$ For I-beams and channels, $r = 0.4d$ } Shear assumed as uniformly distributed
For wood, $E_S = 0.25E$ For plate girders, $r = 0.5d$ } over section $\therefore N = 1$
For wooden rectangular beams, shear assumed as variable over section

$$\therefore N = \frac{1.2}{A}$$

In the usual beam designed in practice for moment conditions, it will generally be found that the ratio of depth of section to length of beam varies from 1⁄15 to 1⁄20, while in plate girders a ratio of 1⁄10 is common. For these ratios the shearing deflections expressed as a percentage of bending deflections, as given by eqs. (2) to (4), are as follows:

Ratio depth to span length	1/10	1/15	1/20
Plate girders	6.0	2.66	1.5
I-beams and channels	3.84	1.71	0.96
Wooden beams	3.84	1.71	0.96

These shearing deflections are all so small compared to the bending deflection, that they reasonably may, and always are neglected in practice. However, for beams with depth ratios greater than 1/10, the shearing deflection becomes a larger percentage of the bending deflection, and should not in general be neglected.

To determine the span length in terms of the depth of beam for which shearing and bending deflections become equal, place $\frac{D_S}{D_B} = 1$ in eqs. (2) to (4) and solve for l. Thus for a plate girder with a uniform load, we have

$$\frac{D_S}{D_B} = 1 = 6\left(\frac{d}{l}\right)^2$$

Solving for l, we have, $l = 2.45d$. Values for other cases are given in Table 5.

Table 6 gives in convenient form for ready reference the general equations for moment shear and deflection for simple and cantilever beams in common use.

72. Deflection of Concrete Beams.

72a. Maney's Method.[1]—The deflection of a reinforced-concrete beam of whatever shape may be determined by the formula

$$D = c\frac{l^2}{d}(e_c + e_s)$$

where

D = maximum deflection (if desired in inches, the units specified below should be used).

l = span (inches).

d = depth of the beam to the center of the steel (inches).

e_c = unit deformation in extreme fiber for the concrete $= \frac{f_c}{E_c}$.

e_s = unit deformation in extreme fiber for the steel $= \frac{f_s}{E_s}$.

$c = \frac{c_1}{c_2}$ in which

c_1 = the numerical coefficient in the formula for deflection of homogeneous beams, $D = c_1\frac{Wl^3}{EI}$, depending on the loading and on how the ends are supported.

c_2 = the numerical coefficient in the formula for bending moment, $M = c_2wl^2$,

for a simple beam loaded at center, c = 1/12 or 0.0833
uniformly loaded, c = 5/48 or 0.1041
loaded at the third points, c = 23/216 or 0.1065
for a beam with fixed ends, loaded at center, c = 1/24 or 0.0416
uniformly loaded, c = 1/32 or 0.0313
loaded at the third points, c = 5/144 or 0.0347

[1] See paper by G. A. Maney, presented before the seventeenth annual meeting of the American Society for Testing Materials.

TABLE 6.—PROPERTIES OF SIMPLE AND CANTILEVER BEAMS

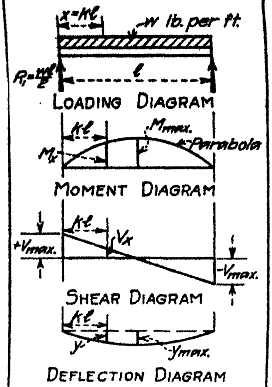
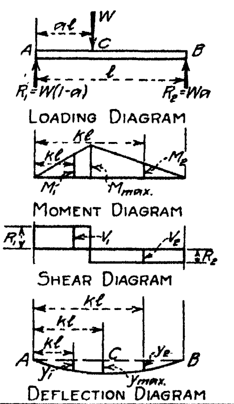
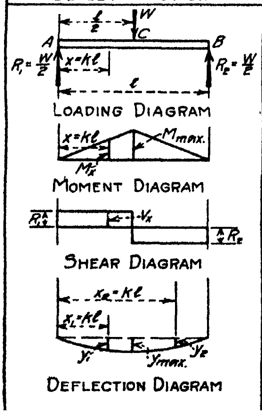
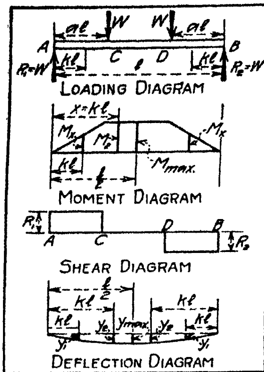
	MOMENT $M_x=\frac{wl^2}{2}k(1-k)$ $M_{max}=\frac{wl^2}{8}$ for $k=\frac{1}{2}$ SHEAR $V_x=\frac{wl}{2}(1-2k)$ $+V_{max.}=+\frac{wl}{2}$ for $k=0$ $-V_{max.}=-\frac{wl}{2}$ for $k=1$ DEFLECTION $y=\frac{wl^4}{24EI}k(1-2k^2+k^3)$ $y_{max.}=\frac{5}{384}\frac{wl^4}{EI}$ for $k=\frac{1}{2}$	FIBER STRESS IN TERMS OF LOADING $f_x=\frac{wl^2c}{2I}k(1-k)$ $f_{max.}=\frac{wl^2c}{8I}$ for $k=\frac{1}{2}$ DEFLECTION IN TERMS OF MAXIMUM FIBER STRESS $y=\frac{fl^2}{3Ec}k(1-2k^2+k^3)$ $y_{max.}=\frac{5}{48}\frac{fl^2}{Ec}$ for $k=\frac{1}{2}$
	MOMENT A to C: $M_1=Wlk(1-a)$ C to B: $M_2=Wla(1-k)$ $M_{max.}=Wla(1-a)$ at C for $k=a$ SHEAR A to C: $V_1=+W(1-a)$ C to B: $V_2=-Wa$ DEFLECTION A to C: $y_1=\frac{Wl^3}{6EI}k(1-a)[(2-a)a-k^2]$ C to B: $y_2=\frac{Wl^3}{6EI}a(1-k)[(2-k)k-a^2]$ At C: $y_c=\frac{Wl^3}{3EI}a^2(1-a)^2$ for $k=a$ At C: $y_{max.}=\frac{Wl^3}{3EI}(1-a)[\frac{a}{3}(2-a)]^{\frac{3}{2}}$ for $k=[\frac{a}{3}(2-a)]^{\frac{1}{2}}$	FIBER STRESS IN TERMS OF LOADING A to C: $f_x=\frac{Wlk(1-a)c}{I}$ C to B: $f_x=\frac{Wla(1-k)c}{I}$ At C: $f_{max.}=\frac{Wla(1-a)c}{I}$ DEFLECTION IN TERMS OF MAXIMUM FIBER STRESS $y_{max.}=\frac{fl^2}{27Ec}[3a(2-a)^3]^{\frac{1}{2}}$
	MOMENT $M_x=\frac{1}{2}Wkl$ $M_{max.}=\frac{1}{4}Wl$ at C SHEAR A to C: $V=+\frac{W}{2}$ C to B: $V=-\frac{W}{2}$ DEFLECTION A to C: $y_1=\frac{Wl^3}{48EI}k(3-4k^2)$ C to B: $y_2=\frac{Wl^3}{48EI}(1-k)[4(2-k)k-1]$ $y_{max.}=\frac{Wl^3}{48EI}$ for $k=\frac{1}{2}$	FIBER STRESS IN TERMS OF LOADING A to C: $f_x=\frac{1}{2I}Wklc$ At C: $f_{max.}=\frac{1}{4I}Wlc$ DEFLECTION IN TERMS OF MAXIMUM FIBER STRESS A to C: $y_1=\frac{fl^2}{12Ec}(3-4k^2)k$ $y_{max.}=\frac{fl^2}{12Ec}$ at C

TABLE 6.—PROPERTIES OF SIMPLE AND CANTILEVER BEAMS—*Continued*



MOMENT

A to C and B to D: $M_x = Wkl$

C to D: $M_2 = Wal$

$M_{max.} = Wal$

SHEAR

A to C: $V = +W$.

C to D: $V = 0$

B to D: $V = -W$

DEFLECTION

A to C and B to D: $y_1 = \frac{Wkl^3}{6EI}[3a(1-a)-k^2]$

C to D: $y_2 = \frac{Wal^3}{6EI}[3k(1-k)-a^2]$

At C and D: $y_c = \frac{Wa^2l^3}{6EI}(3-4a)$ for $k=a$

$y_{max.} = \frac{Wal^3}{24EI}(3-4a^2)$ for $k=\frac{1}{2}$

FIBER STRESS IN TERMS OF LOADING

A to C and B to D: $f = \frac{Wklc}{I}$

C to D: $f = f_{max.} = \frac{Walc}{I}$

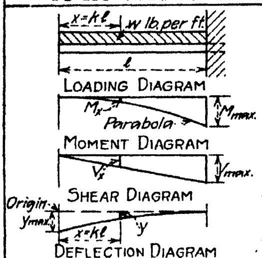
DEFLECTION IN TERMS OF MAXIMUM FIBER STRESS

A to C and B to D:

$y_1 = \frac{fkl^2}{6acE}[3a(1-a)-k^2]$

C to D: $y_2 = \frac{fl^2}{6Ec}[3k(1-k)-a^2]$

$y_{max.} = \frac{fl^2}{24Ec}(3-4a^2)$ for $k=\frac{1}{2}$



MOMENT

$M_x = -\frac{wk^2l^2}{2}$

$M_{max.} = -\frac{wl^2}{2}$ for $k=1$

SHEAR

$V_x = -wkl$

$V_{max.} = -wl$ for $k=1$

DEFLECTION

$y = \frac{wl^4}{24EI}(k^4-4k+3)$

$y_{max.} = \frac{wl^4}{8EI}$ for $k=0$

FIBER STRESS IN TERMS OF LOADING

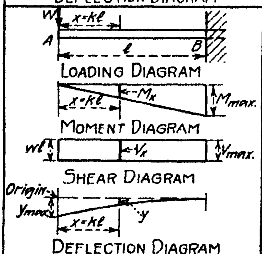
A to B: $f_x = \frac{wk^2l^2c}{2I}$

$f_{max.} = \frac{wl^2c}{2I}$ at B

DEFLECTION IN TERMS OF MAXIMUM FIBER STRESS

A to B: $y = \frac{fl^2}{12Ec}(k^4-4k+3)$

$y_{max.} = \frac{fl^2}{4Ec}$ for $k=0$



MOMENT

$M_x = -Wkl$

$M_{max.} = -Wl$ at B for $k=1$

SHEAR

V_x = Constant $= -W$

$V_{max.} = -W$

DEFLECTION

$y = \frac{Wl^3}{6EI}(2+k)(1-k)^2$

$y_{max.} = \frac{Wl^3}{3EI}$

FIBER STRESS IN TERMS OF LOADING

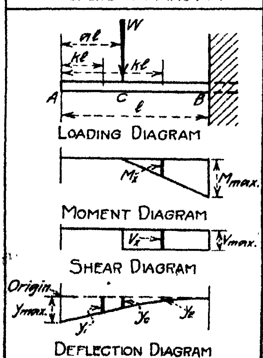
$f_x = \frac{Wklc}{I}$

$f_{max.} = \frac{Wlc}{I}$

DEFLECTION IN TERMS OF MAXIMUM FIBER STRESS

$y = \frac{fl^2}{6Ec}(2+k)(1-k)^2$

$y_{max.} = \frac{fl^2}{3Ec}$



MOMENT

A to C: $M_x = 0$

C to B: $M_x = -Wl(k-a)$

$M_{max.} = -Wl(1-a)$ for $k=1$

SHEAR

A to C: $V_x = 0$

C to B: $V_x = -W$

$V_{max.} = -W$ from C to B

DEFLECTION

A to C: $y_1 = \frac{Wl^3}{6EI}(2+a-3k)(1-a)^2$

C to B: $y_2 = \frac{Wl^3}{6EI}(2+k-3a)(1-k)^2$

$y_c = \frac{Wl^3}{3EI}(1-a)^3$ for $k=a$

$y_{max.} = \frac{Wl^3}{6EI}(2+a)(1-a)^2$ for $k=0$

FIBER STRESS IN TERMS OF LOADING

C to B: $f_x = \frac{Wl(k-a)c}{I}$

$f_{max.} = \frac{Wl(1-a)c}{I}$ at B

DEFLECTION IN TERMS OF MAXIMUM FIBER STRESS

A to C: $y_1 = \frac{fl^2}{6Ec}(2+a-3k)(1-a)$

C to B: $y_2 = \frac{fl^2}{6Ec}\frac{(2+k-3a)(1-k)^2}{(1-a)}$

$y_{max.} = \frac{fl^2}{6Ec}(2+a)(1-a)$ at A

72b. Turneaure and Maurer's Method.[1]—Turneaure and Maurer recommend that 8 to 10 be used for n in the formulas which they have derived, and which are given below. They also state that the formulas presented are the result of modifying the deflection formulas for homogeneous beams in accordance with the following assumptions:

1. The representative or mean section has a depth equal to the distance from the top of the beam to the center of the steel.
2. It sustains tension as well as compression, both following the linear law.
3. The proper mean modulus of elasticity of the concrete equals the average or secant modulus up to the working compressive stress.
4. The allowance for steel in computing the moment of inertia of the mean section should be based on the amount of steel in the mid-sections, since stirrups and bent-up rods do not affect stiffness materially for working loads.

The following are the deflection formulas for rectangular reinforced concrete beams:

$$D = \frac{c_1}{E_s} \cdot \frac{Wl^3}{bd^3} \cdot \frac{n}{\alpha} \tag{1}$$

$$\alpha = \tfrac{1}{3}[k^3 + (1 - k)^3 + 3np(1 - k)^2] \tag{2}$$

$$k = \frac{1 + 2np}{2 + 2np} \tag{3}$$

From eqs. (2) and (3), the value of α for any values of p and n may be computed, and then the deflection from eq. (1). The notation employed in the above formulas is as follows:

D = maximum deflection (if desired in inches, the units specified below should be used).
b = breadth of the beam (inches).
d = depth of the beam to the center of the steel (inches).
W = total load (pounds).
l = span (inches).
p = steel ratio.
E_s = modulus of elasticity of the reinforcing steel (pounds per square inch).
n = ratio of the moduli of elasticity of steel and concrete.
α = a numerical coefficient depending on p and n.
k = proportionate depth of the neutral axis.
c_1 = the numerical coefficient in the formula for deflection of homogeneous, beams, $c_1 \frac{Wl^3}{EI}$, depending on the loading and support. For example,

for a cantilever loaded at the end, $c_1 = \frac{1}{3}$
for a cantilever uniformly loaded, $c_1 = \frac{1}{8}$
for a simple beam loaded at center, $c_1 = \frac{1}{48}$
for a simple beam uniformly loaded, $c_1 = \frac{5}{384}$
for a beam with fixed ends, load at the center, $c_1 = \frac{1}{192}$
for a beam with fixed ends, uniformly loaded, $c_1 = \frac{1}{384}$

[1] "Principles of Reinforced Concrete Construction," 3rd Edition, p. 197.

The following are the deflection formulas for reinforced concrete T-beams (referred to later):

$$D = \frac{c_1}{E_s} \cdot \frac{Wl_3}{bd^3} \cdot \frac{n}{\beta}$$

$$\beta = \tfrac{1}{3}[k^3 - \left(1 - \frac{b'}{b}\right)\left(k - \frac{t}{d}\right)^3 + \frac{b'}{b}(1 - k)^3 + 3pn(1 - k)^2$$

$$k = \frac{np + \tfrac{1}{2}\left[\frac{b'}{b} - \frac{b'}{b}\left(\frac{t}{d}\right)^2 + \left(\frac{t}{d}\right)^2\right]}{np + \frac{b'}{b} - \frac{b'}{b}\left(\frac{t}{d}\right) + \frac{t}{d}}$$

in which β is a coefficient depending upon the steel ratio and n, and other symbols as before.

RESTRAINED AND CONTINUOUS BEAMS

By Chas. A. Ellis

73. General Considerations.—A beam is said to be fixed or *restrained* when, as shown in Figs. 101*a* and *b*, one or both ends are built into a wall in such a manner that the action of external forces tends to cause no rotation of the beam at the fixed end, or ends, as the case may be. Consequently, the neutral plane in its original position, AB, remains tangent to the elastic curve at the fixed end, or ends, when the beam is bent.

Beams are often supported at several points along their length, as shown in Fig. 102. When the elastic line of the deformed beam forms a continuous curve, with the elastic lines in two adjacent spans, as AB and BC, joining at a support B in such a manner that the adjacent tangents DB and BE form a straight line DBE, the beam is said to be a *continuous beam.* In some continuous beams the form of the structure in some of the panels is such that

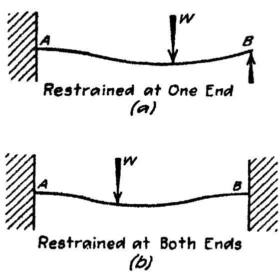


Fig. 101.

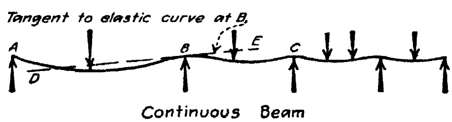


Fig. 102.

the elastic line is not continuous over the supports but forms a "cusp" or sharp point. Such beams are known as *partially continuous beams.* These beams are encountered in certain types of swing bridges.

In the cases shown in Figs. 101 and 102, the presence of restraining moments which fix the ends of the beam or cause adjacent tangents to remain parallel, together with the unknown supporting forces, or reactions, present more unknown quantities than can be determined by the principles of statics. Such beams are said to be *statically indeterminate.*

Problems in the determination of the restraining moments and supporting forces may be solved by any of the general methods for deflection of beams given in the chapter on "Deflection of Beams" and in Appendix C. In the present

chapter the discussion will be confined to the solution by the Area Moment and Elastic Weight methods.

The solution of problems in restrained and continuous beams requires an expression for moment at any point in the beam. This may be obtained from Fig. 103*a*, which shows the beam of Fig. 101*b*, or any span of Fig. 102, removed by cutting sections close to the wall or the supports. The end moments are shown by the arrows M_1 and M_2. At any point distance x from the end of the beam, the moment may be expressed as the effect of the load W and the effect of the end moments M_1 and M_2 considered as acting independently. These values are shown by diagrams of Figs. 103*b* and *c* respectively. The values of the moments are indicated on the diagrams.

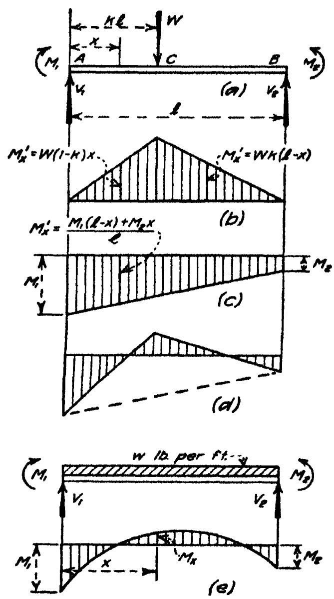


FIG. 103.

After the values of M_1 and M_2 have been determined by the methods given in the articles which follow, the diagrams of Figs. 103*b* and *c* may be combined to form the moment diagram of Fig. 103*d*. It will generally be found that the end restraining moments are negative. The shaded areas show negative moments near the ends of the beam and positive moment near the center.

General equations for moment at any point may be written in the following form:

From A to C

$$M_x = M_1 + V_1x \tag{1}$$

From C to B

$$M_x = M_1 + V_1x - W(x - kl) \tag{2}$$

To determine V_1 take moments about B, from which

$$V_1 = \frac{1}{l}(M_2 - M_1) + W(1 - k) \tag{3}$$

Equation (3) gives the reaction at the left end of the beam of Fig. 101*b* or the shear at the left end of one of the spans of Fig. 102. The reaction at the right end of the beam is

$$V_2 = \frac{1}{l}(M_1 - M_2) + Wk \tag{3a}$$

The shear at any point in the beam is given by the following equations:
From A to C

$$V_x = +V_1 \tag{4}$$

From C to B

$$V_x = +V_1 - W \tag{5}$$

Values of moment and shear may be determined from eqs. (1) to (5) as soon as M_1 and M_2 are known.

Figure 103*e* shows a beam supporting a uniform load and subjected to end moments M_1 and M_2. The values of moments, shears, and reactions are as follows:

$$M_x = M_1 + V_1 x - \frac{wx^2}{2} \tag{6}$$

$$V_x = +V_1 - wx \tag{7}$$

$$V_1 = \frac{1}{l}(M_2 - M_1) + \frac{wl}{2} \tag{8}$$

and

$$V_2 = \frac{1}{l}(M_1 - M_2) + \frac{wl}{2} \tag{9}$$

74. Restrained or Fixed Beams.

74*a*. Analysis by Prof. Greene's Area-moment Method. *Beam Fixed at One End, Free at Other End.*—The beam in Fig. 104 is *fixed* or *restrained* at A, and freely supported at B. Under these conditions the reactions R_1 and R_2, and the resisting moment M_1 at A, present more unknown quantities than can be determined by the principles of statics, and the beam is statically indeterminate. The ease with which problems of this kind are solved, depends to some extent upon the manner in which the M-diagram is drawn, for it may be represented in three ways, as shown in Figs. 104*b*, 104*c*, or 104*d*. Each case will be considered separately.

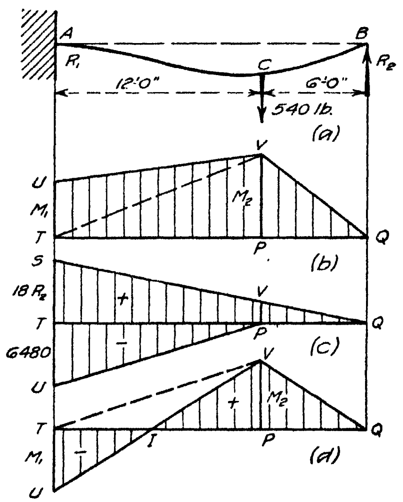


Fig. 104.

In Fig. 104*b*, let M_1 and M_2 represent the bending moments at A and C respectively. The bending moment at B is zero. The line AB, (Fig. 104*a*), is tangent to the elastic curve at A, and the tangential deviation t at B is zero. Therefore

$$t = \frac{1}{EI}\int_A^B Mxdx = 0$$

or

$$\int_A^B Mxdx = 0$$

Hence the area-moment of $QVUTP$, Fig. 104*b* about Q is zero, or

$$M_2(3)(4) + M_2(6)(10) + M_1(6)(14) = 0$$
$$7M_1 + 6M_2 = 0$$

From statics

$$M_1 = -(540)(12) + 18R_2$$

and

$$M_2 = 6R_2$$

Whence

$$-45{,}360 + 126R_2 + 36R_2 = 0$$

and

$$R_2 = 280 \qquad M_1 = -1{,}440$$
$$R_1 = 260 \qquad M_2 = 1{,}680$$

In Fig. 104*c*, the M-diagram is drawn in parts; QST is the M-diagram for the reaction at B, and TPU is the M-diagram for the load at C. The area QST is positive, and the area TPU is negative. The area-moment of the total diagram about Q is zero. Therefore

$$18R_2\,(9)(12) - (6{,}480)(6)(14) = 0$$
$$R_2 = 280$$
$$TS = 18R_2 = 5{,}040$$
$$PV = (\tfrac{1}{3})(5{,}040) = 1{,}680$$
$$SU = 5{,}040 - 6{,}480 = -1{,}440$$

In the two preceding solutions no speculation was made as to the general form of the elastic curve. The curve ACB, Fig. 104*a*, might have had any shape whatsoever, so long as its tangent at A passes through B. It is not always wise to presume upon the general form of the elastic curve before computations are made; but in the present simple case it is quite safe to assume that the curve is concave on the under side near A, and concave on the upper side at C, with a point of contraflexure between. Hence the bending moment is negative at A, positive at C, and zero at an intermediate point I; consequently the M-diagram may be sketched as in Fig. 104*d*. In finding the area-moment, the area TIV, which is not a part of the diagram, can be included as positive area with QVI, and as negative area with TIU.

$$M_2(3)(4) + M_2(6)(10) - M_1(6)(14) = 0$$
$$7M_1 - 6M_2 = 0$$

From statics

$$-M_1 = -(540)(12) + 18R_2$$

and

$$M_2 = 6R_2$$

Whence

$$R_3 = 280 \qquad -M_1 = -1{,}440$$
$$R_1 = 260 \qquad M_2 = 1{,}680$$

When an unknown ordinate in the M-diagram is represented by a symbol, it is generally better to assume that the ordinate is positive, as in Fig. 104*b*. If the solution shows that the ordinate is negative, the M-diagram may be re-drawn if desirable. Frequently the M-diagram may be constructed to advantage as shown in Fig. 104*c*.

Only two independent static equations can be written for the solution of a system of parallel forces. In the present problem there were three unknown quantities to be determined, hence one elastic equation was necessary for a solution.

The beam in Fig. 105*a*, fixed at A and simply supported at B, carries a total load W, uniformly distributed. In Fig. 105*b*, QTU is the M-diagram for the

If the beam were simply supported (not fixed) at A and B, the bending moment at C would be $M_4 = k(1 - k)Pl$. Therefore TSQ is the M-diagram when the beam is not restrained at the ends by M_1 and M_2. In a numerical problem the M-diagram should be sketched as in Fig. 106c, and M_4 computed as for a simple beam. After the negative moments M_1 and M_2 have been determined, the trapezoid $TQPU$ may be revolved about TQ and the diagram drawn to scale as shown in Fig. 106d.

The fixed beam in Fig. 107 supports a total load W, uniformly distributed. Since the loading is symmetrical, the resisting moment and reactions at B

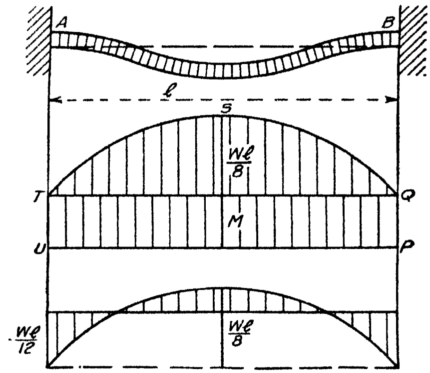


FIG. 107.

are the same as at A. The area TSQ is the M-diagram for a simply supported beam and the area $TUPQ$ represents the resisting moment M at each end. The angle ϕ between the tangents through A and B is zero. Consequently the area of the M-diagram is zero. Therefore

$$Ml + \left(\frac{Wl}{8}\right)\left(\frac{2}{3}\right) = 0$$

$$M = -\frac{Wl}{12}$$

The bending moment at the center is

$$\frac{Wl}{8} - \frac{Wl}{12} = \frac{Wl}{24}$$

and the M-diagram may be drawn to scale as shown.

Illustrative Problem.—The reactions and resisting moments will be determined for the fixed beam in Fig. 108a.

From statics

$$SV = \left(\frac{640}{24}\right)(9)(15) = 3{,}600$$

$$M_1 = -(640)(9) + 24R_2 + M_2$$

or

$$R_2 = \frac{5{,}760 + M_1 - M_2}{24}$$

The angle between the tangents through A and B is

$$\phi = 0$$

or

$$\frac{(M_1 + M_2)24}{2} + \frac{(3,600)(24)}{2} = 0$$

The tangential deviation at B is

$$t_2 = 0$$

or

$$\frac{(3,600)(15)(10)}{2} + \frac{(3,600)(9)(18)}{2} + \frac{24M_2(8)}{2} + \frac{24M_1(16)}{2} = 0$$

Whence

$$M_1 = -2,250 \qquad R_1 = 437.5$$
$$M_2 = -1,350 \qquad R_2 = 202.5$$

These results may be checked by the formulas given above. The M-diagram is drawn to scale in Fig. 108c.

74b. Analysis of Restrained Beams by Method of Elastic Weights.[1]

Beam Fixed at Both Ends.—Figure 109a shows a beam fixed at both ends and supporting a single concentrated load W at a distance kl from the left end of the beam. In Fig. 109b the restraining and supporting effect of the wall is represented by the moments M_1 and M_2 and the forces V_1 and V_2. The directions assumed for these moments and forces will be taken as positive. Let ab of Fig. 109c represent the conjugate beam for the case under consideration. Load this conjugate beam with an M/EI diagram formed by combining the moment diagrams for simple beam effect due to

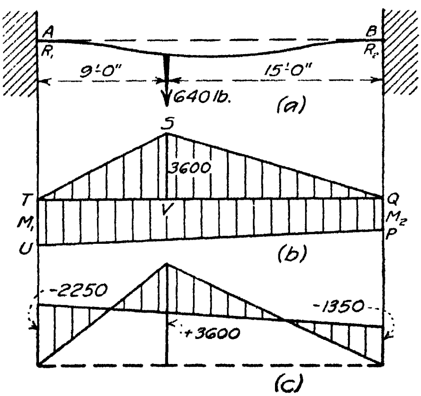


Fig. 108.

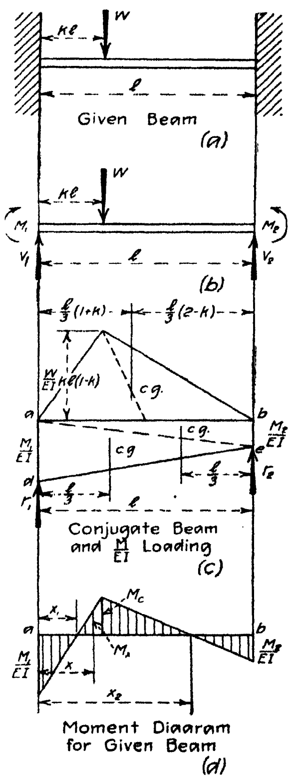


Fig. 109.

the load W and the moment diagram for the effect of M_1 and M_2. Let r_1 and r_2 represent the conjugate beam reactions due to the M/EI loading.

[1] By W. S. Kinne.

A necessary condition for fixed ends in a restrained beam is that the tangents to the elastic curve at the ends of the beam must have zero slope. Since the slope of the tangents at the ends of the given beam are zero, the shear at the ends of the conjugate beam must be zero. But these shears are equal to the end reactions for the conjugate beam. Therefore r_1 and r_2 must be equal to zero. The values of M_1 and M_2 may be determined subject to the condition that r_1 and r_2 are equal to zero.

Values of r_1 and r_2 may be determined for the conditions shown in Fig. 109c. The moment diagram for M_1 and M_2 is divided into two triangles to facilitate the calculations. Distances to the center of gravity of the several triangles are shown on the figure. On placing equal to zero values of r_1 and r_2 obtained by writing moment equations about the ends of the beam we derive the following condition equations:

$$2M_1 + M_2 = -Wkl(1 - k)(2 - k)$$

and

$$M_1 + 2M_2 = -Wkl(1 - k)(1 + k)$$

Solving these equations for M_1 and M_2, we have

$$M_1 = -Wkl(1 - k)^2 \tag{5}$$

$$M_2 = -Wk^2l(1 - k) \tag{6}$$

Equations (5) and (6) give the values of the end moments for the given beam of Fig. 109*a*.

The supporting force or reaction at the left end of the given beam may be determined from eq. (3), Art. **73**, by substituting values of M_1 and M_2 as given by eqs. (5) and (6), from which

$$V_1 = W(1 - k)^2(1 + 2k) \tag{7}$$

From statics

$$V_2 = W - V_1 = Wk^2(3 - 2k) \tag{8}$$

To determine the moment at any point in the given beam substitute values of M_1, M_2 and V_1 in eqs. (1) and (2) of Art. **73**. We then have the following general equations for moments:

From A to C

$$M_x = W(1 - k)^2\,[x(1 + 2k) - kl] \tag{9}$$

From C to B

$$M_x = Wk^2\,[l(2 - k) - (3 - 2k)x] \tag{10}$$

To determine the moment under the load W substitute $x = kl$ in eq. (9) or (10), from which

$$M_C = 2Wk^2l(1 - k)^2 \tag{11}$$

The moment diagram for the given beam as plotted from eqs. (9) to (11) is shown in Fig. 109*d*. Note that the end moments, which are negative, are plotted below the base line *ab* while the positive moment at the load is plotted above the base line. The shaded area shows the complete moment diagram.

From the moment diagram of Fig. 109*d* it can be seen that the moment is zero at two points, one on either side of the load. To locate these points, place

M_x from eqs. (9) and (10) equal to zero and solve for the values of x. Let x_1 and x_2 represent these values of x. From eq. (9)

$$x_1 = \frac{kl}{1 + 2k} \tag{12}$$

and from eq. (10)

$$x_2 = \frac{2 - k}{3 - 2k} l \tag{13}$$

Equations (12) and (13) locate the *points of inflection*, or the points at which the moment changes from positive to negative. It can also be shown that at these points there is a reversal of curvature in the elastic curve of the beam.

The deflection at any point in the given beam may be determined by calculating the moment at that point in the conjugate beam due to the M/EI loading. In Fig. 110 let ab represent the conjugate beam and let the figure $adceb$ represent the M/EI diagram taken from the moment diagram of Fig. 109d. Note that the moment areas amd and nbe are negative and are represented by forces which act upward and that area mcn is positive and is represented by a force which acts downward.

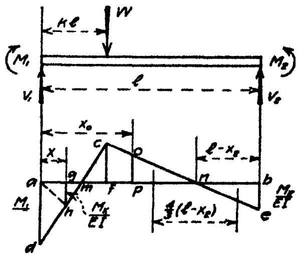


FIG. 110.

To determine the general equation for deflection at a point distance x from the left end of the given beam, determine the moment of area $adhg$ of Fig. 110 about point g. On dividing this area into triangles and taking moments about point g, we have

$$y = \frac{1}{6EI}(2M_1 + M_x)$$

from which finally

$$y = \frac{W}{6EI}(1 - k)^2 x^2[x(1 + 2k) - 3kl] \tag{14}$$

Equation (14) is the equation of the elastic curve for a point to the left of the load W. For a point to the right of the load and at a distance x from the left end of the beam, we derive

$$y = \frac{W}{6EI} k^2(l - x)^2[(3 - 2k)x - kl] \tag{15}$$

which is the equation of the elastic curve for a point to the right of the load W.

The deflection of the given beam at the position of the load W may be determined by substituting $x = kl$ in eqs. (14) or (15), or the moment area on either side of the load may be taken about point f. In any event we derive

$$y_c = \frac{W}{3EI} k^3l^3(1 - k)^3 \tag{16}$$

The maximum deflection in the given beam due to the load W occurs at the point where the conjugate beam shear due to the load W is equal to zero. From Fig. 110 it can be seen that zeor shear occurs at point p which is so located that area nop = area nbe. Let x_o = distance from left end of beam to point of maximum deflection. Then

$$x_o = l - 2(l - x_2) = \frac{l}{3 - 2k} \tag{17}$$

The maximum deflection may be determined by substituting x_o from eq. (17) in eq. (15), or by calculating the moment area to the right of point p. Note that this moment area is a couple which is equal to area nop multiplied by the distance between the centers of gravity of the triangles nop and nbe. Performing the operation indicated, we derive

$$y_{max} = \frac{2}{3} W \frac{k^2 l^3}{EI} \frac{(1-k)^3}{(3-2k)^2} \tag{18}$$

Equation (18) gives the maximum deflection of the given beam for any position of the load W.

The maximum deflection given by eq. (18) varies with the position of the load W. It can be shown that the greatest value of y_{max} will occur at the beam center when the load W is placed at that point. Let Δ represent this deflection. Substituting $k = \frac{1}{2}$ in eq. (18) we have

$$\Delta = \frac{1}{192} \frac{Wl^3}{EI} \tag{19}$$

which is the greatest deflection of the given beam.

When the given beam supports a uniform load the process is similar to that given above. In this case the simple beam moment diagram is a parabola. Due to symmetry, the restraining moments M_1 and M_2 are equal. The values of moments and deflection for this case are given in Table 1, Art. **74c**.

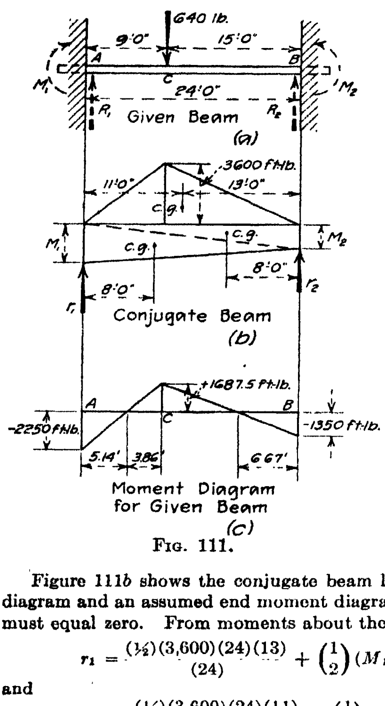


FIG. 111.

Illustrative Problem.—Determine the end moments and reactions for the fixed beam shown in Fig. 111.

Figure 111b shows the conjugate beam loaded with the known simple beam moment diagram and an assumed end moment diagram. The conjugate beam reactions, r_1 and r_2 must equal zero. From moments about the ends of the beam

$$r_1 = \frac{(\frac{1}{2})(3{,}600)(24)(13)}{(24)} + \left(\frac{1}{2}\right)(M_1)(24)\frac{(16)}{(24)} + \left(\frac{1}{2}\right)(M_2)(24)\frac{(8)}{(24)} = 0$$

and

$$r_2 = \frac{(\frac{1}{2})(3{,}600)(24)(11)}{(24)} + \left(\frac{1}{2}\right)(M_1)(24)\frac{(8)}{(24)} + \left(\frac{1}{2}\right)(M_2)(24)\frac{(16)}{(24)} = 0$$

From these equations we have

$$M_1 + 0.5M_2 = -2{,}925$$
$$M_1 + 2.0M_2 = -4{,}950$$

Solving for M_1 and M_2, we have

$$M_1 = -2{,}250 \text{ ft.-lb.}$$
$$M_2 = -1{,}350 \text{ ft.-lb.}$$

Figure 111c shows the bending moment diagram.

The reactions may be determined from moment equations taken about the ends of the beam for the conditions shown in Fig. 111a. Noting that M_1 and M_2 are negative moments, moments about the right end of the beam give

$$+24R_1 - (640)(15) - 2{,}250 + 1{,}350 = 0$$

from which

$$R_1 = 437.5 \text{ lb.}$$

From summation of vertical forces or moments about the left end of the beam

$$R_2 = 202.5 \text{ lb.}$$

Illustrative Problem.—Calculate the deflection under the load and the maximum deflection for the beam of Fig. 111.

The deflection under the load is given by area moments for the portion of the moment diagram of Fig. 111c which lies to the left of point C. Whence

$$y_c = \frac{1}{EI}\left\{\left(\frac{1}{2}\right)(2{,}250)[3.86 + \left(\frac{2}{3}\right)(5.14)] - \left(\frac{1}{2}\right)(1{,}687.5)(3.86)^2\left(\frac{1}{3}\right)\right\}$$

$$y_c = \frac{1}{EI}(4{,}010) \text{ ft.}$$

The maximum deflection occurs at the point for which the conjugate beam shear is zero. From Fig. 111c it can be seen that the shear is zero at a point $(2)(6.67) = 13.34$ ft. to the left of point B. The maximum deflection, equal to the area moment about this point for areas to the right, is

$$y_{max} = \frac{1}{EI}\left(\frac{4}{3}\right)(6.67)\left(\frac{1}{2}\right)(6.67)(1{,}350) = \frac{1}{EI}(4{,}020) \text{ ft.}$$

Illustrative Problem.—Calculate the end moments, reactions, and maximum deflection for the beam shown in Fig. 112.

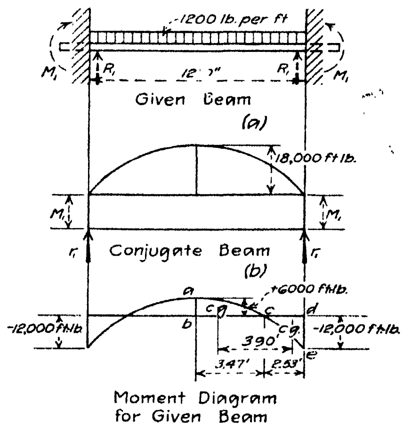


Fig. 112.

The simple beam moment diagram is a parabola, as shown in Fig. 112b, and the end moment diagram is a rectangle, since from symmetry of loading, the end moments are equal. Placing the value of the conjugate beam reaction r_1 equal to zero, we have

$$r_1 = (\tfrac{1}{2})(\tfrac{2}{3})(18{,}000)(12) + (\tfrac{1}{2})(M_1)(12) = 0$$

from which

$$M_1 = -12{,}000 \text{ ft.-lb.}$$

Figure 112c shows the complete moment diagram.

Since the end moments are equal, eqs. (8) and (9), Art. 27, show that the end reactions are equal to each other and each is equal to half the applied load. Hence

$$R_1 = 7{,}200 \text{ lb.}$$

The maximum deflection occurs at the center of the beam due to symmetry of loading. It can be shown that the areas *abc* and *cde* are equal and that the distances between their centers of gravity is 3.9 ft. Hence

$$y_{max} = \frac{1}{EI}\left(\frac{2}{3}\right)(6{,}000)(3.47)(3.9) = \frac{1}{EI}(54{,}100)$$

Beam Fixed at One End, Free at Other End.—The beam of Fig. 113 may be analyzed by the methods used in the preceding article. Since the beam is freely supported at the left end, the moment at that point is zero. Let M_2 represent the restraining moment at the right end of the beam.

Figure 113*c* represents the conjugate beam with the M/EI loading in position. The simple beam effect is shown by the triangle *acb* and the effect of M_2 is shown by the triangle *abd*. Since the slope of the tangent to the elastic curve at the right end of the given beam is zero, the conjugate beam reaction r_2 must be zero. In this case r_1 is not zero, for the tangent to the elastic curve at this point is not horizontal. Figure 113*d* shows the completed moment diagram. General equations for moments, reactions, and equations of the elastic curve are given in Table 1, Art. 74*c*.

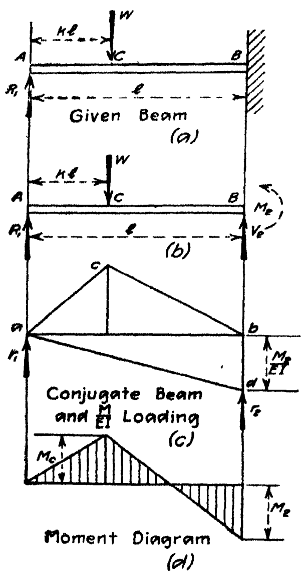


FIG. 113.

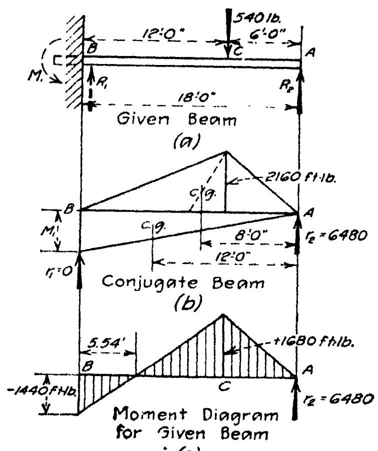


FIG. 114.

Illustrative Problem.—Calculate moments, reactions, deflection under the load, and maximum deflection for the beam of Fig. 114.

Figure 114*b* shows the conjugate beam with the simple beam moment diagram in place. Since the right end of the given beam is freely supported, the end moment diagram is a triangle, as shown in Fig. 114*b*. To determine M_1, place the value of r_1, the left hand conjugate beam reaction, equal to zero, whence,

$$r_1 = \frac{\frac{1}{2}(M_1)(18)(12)}{(18)} + \frac{\frac{1}{2}(2{,}160)(18)(8)}{(18)} = 0$$

from which

$$M_1 = -1{,}440 \text{ ft.-lb.}$$

The right reaction R_2 may be determined from moments about point B, Fig. 114a from which

$$18R_2 - (540)(12) = -1{,}440$$

Hence

$$R_2 = 280 \text{ lb.}$$

From summation vertical forces

$$R_1 = 540 - R_2 = 260 \text{ lb.}$$

The momnet under the load is

$$M_C = 6R_2 = 1{,}680 \text{ ft.-lb.}$$

Figure 114c shows the complete moment diagram.

The deflection under the load may be determined from area moments to the right of point C of Fig. 114c. In this case r_2, the right hand conjugate beam reaction is not zero, for end A is not restrained. To determine r_2 take moments about point B of Fig. 114b, using the value of M_1 calculated above. The value of r_2 is found to be 6,480, as shown on Fig. 114c. Hence

$$y_c = \left[\frac{1}{EI}(6{,}480)(6) - \tfrac{1}{2}(1{,}680)(6)^2\left(\frac{1}{3}\right)\right]$$

$$y_c = \frac{1}{EI}(28{,}800)$$

The maximum deflection occurs at the point where the conjugate beam shear is zero From Fig. 114c it can be seen that this point is located (2)(5.54) = 10.08 from the left end of the beam. Hence

$$y_{max} = \frac{1}{EI}\left(\frac{1}{2}\right)(1{,}440)(5.54)^2\left(\frac{4}{3}\right) = \frac{1}{EI}(29{,}500)$$

74c. Properties of Restrained Beams.—Table 1 on pp. 111*a* and 111*b* gives in convenient form the principal properties of the common forms of restrained beams. Substitution in the general formulas will give the moments, shears, and deflection in any desired case.

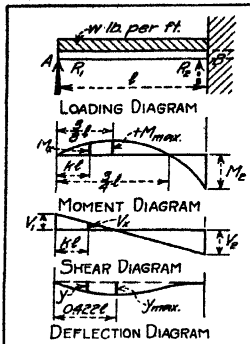
75. Continuous Beams.

75a. Analysis of Continuous Beams by the Area-moment Method. (*Prof. Greene's Method*).—Two general methods of procedure may be adopted in the analysis of continuous beams by this method. In the first method, which is known as the Conventional Method, each problem is treated as a separate case. The results may be expressed in general formulas or numerical values may be obtained for moments and reactions. The second method, which is more general in nature, makes use of a relation which exists between the moments at three consecutive supports. This relation, which is known as the Theorem of Three Moments, is expressed in the form of an equation. In the following articles these methods will be given in detail.

75b. The Conventional Method.—The ease with which problems may be solved by this method depends upon the manner in which the moment diagrams are drawn and the selection of the tangent for which the intercepts are to be determined. For convenience in solving problems, all reactions will be assumed to act upwards and all moments will be assumed to act in a positive direction. Hence a negative result indicates that the reaction or moment acts in a direction opposite to that assumed.

Beam on Three Supports—Single Concentrated Load.—In Fig. 115a let FG represent the tangent to the elastic curve drawn through C. The elastic curve is not shown and no speculation with reference to its form will be made. Its slope is unknown; it may be positive or negative. One thing is certain. Since the two spans are equal in length, the tangential deviations t_1 and t_2 are equal in magnitude.

TABLE 1.—PROPERTIES OF RESTRAINED BEAMS



REACTIONS

$R_1=\frac{3}{8}wl$ $R_2=\frac{5}{8}wl$

MOMENTS

$M_x=\frac{wl^2}{8}k(3-4k)$

$+M_{max.}=+\frac{9}{128}wl^2$ at $k=\frac{3}{8}$

$M_2=-\frac{1}{8}wl^2$ at B

SHEAR

$V_x=\frac{1}{8}wl(3-8k)$

$V_1=+\frac{3}{8}wl$ $V_2=-\frac{5}{8}wl$

DEFLECTION

$y=\frac{wl^4}{48EI}k(1+2k)(1-k)^2$

$y_{max.}=0.00542\frac{wl^4}{EI}$ for $k=0.422$

FIBER STRESS IN TERMS OF LOADING

$f_x=\frac{wl^2c}{8I}k(3-4k)$

$f_2=\frac{wl^2c}{8I}$ $+f_{max.}=\frac{9wl^2c}{128I}$

DEFLECTION IN TERMS OF FIBER STRESS

y in terms of $+f_{max.}$

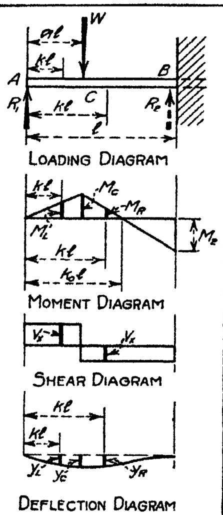
$y=\frac{8}{27}\frac{f_{max.}l^2}{Ec}k(1+2k)(1-k)^2$

$y_{max.}$ in terms of $+f_{max.}$

$y_{max.}=0.0722\frac{f_{max.}l^2}{Ec}$

$y_{max.}$ in terms of f_2

$y_{max.}=0.0434\frac{f_2l^2}{Ec}$



REACTIONS

$R_1=\frac{W}{2}(2+a)(1-a)^2$

$R_2=\frac{W}{2}a(3-a^2)$

MOMENTS

$M_L=\frac{Wl}{2}k(2+a)(1-a)^2$

$M_R=\frac{Wl}{2}a[2-k(3-a^2)]$

At the load: $M_c=\frac{Wl}{2}a(2+a)(1-a)^2$

$M_{c_{max.}}=+0.174Wl$ for $a=0.375$

At the fixed end: $M_2=-\frac{Wl}{2}a(1-a^2)$

$M_{2_{max.}}=-0.192Wl$ for $a=0.577$

DEFLECTION

A to C: $y_L=\frac{Wl^3}{12EI}[3a-(2+a)k^2](1-a)^2k$

When $a>0.414$ max. y is in AC

$y_{L_{max.}}=\frac{Wl^3}{6EI}a(1-a)^2\left(\frac{a}{2+a}\right)^{\frac{1}{2}}$ at $k=\left(\frac{a}{2+a}\right)^{\frac{1}{2}}$

C to B: $y_R=\frac{Wl^3}{12EI}[(3-a^2)ak^3-6ak^2+3(1+a^2)ak-2a^2]$

When $a<0.414$ max. y is in CB

$y_{R_{max.}}=\frac{Wl^3}{3EI}\frac{a(1-a^2)^3}{(3-a^2)^2}$ at $k=\left(\frac{1+a^2}{3-a^2}\right)$

Deflection under load

$y_c=\frac{Wl^3}{12EI}(3-2a+2a^2)(1-a)^2a^2$

$y_{c_{max.}}=0.0098\frac{Wl^3}{EI}$ for $a=0.414$

(Greatest possible deflection in beam)

POINT OF INFLECTION

$K_0=\left(\frac{2}{3-a^2}\right)$

FIBER STRESS IN TERMS OF LOADING

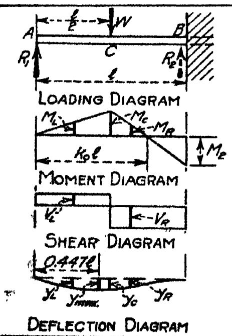
$f_c=\frac{Wlc}{2I}a(2+a)(1-a)^2$

$f_2=\frac{Wlc}{2I}a(1-a^2)$

DEFLECTION IN TERMS OF FIBER STRESS

$y_c=\frac{f_cl^2a}{6Ec}\frac{(3-2a+2a^2)}{2+a}$

$y_c=\frac{f_2l^2a}{6Ec}\frac{(3-2a+2a^2)(1-a)}{(1+a)}$



REACTIONS

$R_1=\frac{5}{16}W$ $R_2=\frac{11}{16}W$

MOMENTS

$M_L=\frac{5}{16}Wlk$

$M_R=\frac{Wl}{2}(1-\frac{11}{4}k)$

$M_c=\frac{5}{32}Wl$ $M_2=-\frac{3}{16}Wl$

SHEARS

$V_L=+\frac{5}{16}W$ $V_R=-\frac{11}{16}W$

DEFLECTION

A to C: $y_L=\frac{1}{96}\frac{Wl^3}{EI}(3-5k^2)k$

C to B: $y_R=\frac{1}{48}\frac{Wl^3}{EI}[\frac{11}{2}k^3-12k^2+\frac{15}{2}k-1]$

$y_{max.}=0.00931\frac{Wl^3}{EI}$ for $k=0.447$

FIBER STRESS IN TERMS OF LOADING

$f_c=\frac{5}{32}\frac{Wlc}{I}$

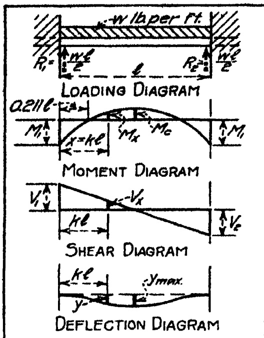
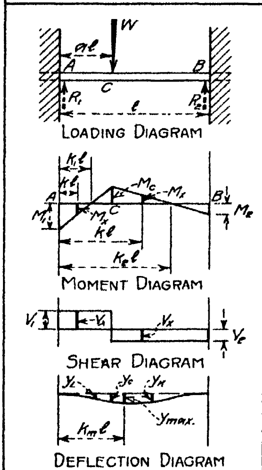
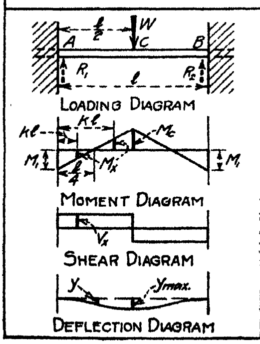
$f_2=\frac{3}{16}\frac{Wlc}{I}$

DEFLECTION IN TERMS OF FIBER STRESS

$y_c=\frac{f_cl^2}{12Ec}$

$y_c=\frac{5}{72}\frac{f_2l^2}{Ec}$

TABLE 1.—PROPERTIES OF RESTRAINED BEAMS—*Continued*

	<u>MOMENT</u> $M_x = -\frac{wl^2}{12}(1-6k+6k^2)$ $M_1 = -\frac{1}{12}wl^2$ for $k=0$ and $k=1$ $M_c = +\frac{1}{24}wl^2$ for $k=\frac{1}{2}$ <u>SHEAR</u> $V_x = \frac{wl}{2}(1-2k)$ $V_1 = +\frac{wl}{2}$ $V_2 = -\frac{wl}{2}$ <u>DEFLECTION</u> $y = \frac{wl^4}{24EI}k^2(1-k)^2$ $y_{max.} = \frac{wl^4}{384EI}$ for $k=\frac{1}{2}$ <u>POINTS OF INFLECTION</u> $k_0 = \frac{1}{12}(6 \pm \sqrt{12}) = 0.211$ or 0.789	<u>FIBER STRESS IN TERMS OF LOADING</u> At end: $f_1 = \frac{1}{12}\frac{wl^2c}{I}$ At center: $f_c = \frac{1}{24}\frac{wl^2c}{I}$ <u>DEFLECTION IN TERMS OF FIBER STRESS</u> In terms of fiber stress at end $y = \frac{f_1 l^2}{2Ec}k^2(1-k)^2$ $y_{max.} = \frac{f_1 l^2}{32Ec}$ In terms of fiber stress at center $y = \frac{f_c l^2}{Ec}k^2(1-k)^2$ $y_{max.} = \frac{f_c l^2}{16Ec}$
	<u>REACTIONS</u> $R_1 = W(1-3a^2+2a^3)$ $R_2 = Wa^2(3-2a)$ <u>MOMENT</u> A to C: $M_x = Wl(1-a)[(1+a-2a^2)k-(1-a)a]$ C to B: $M_x = Wla^2[(2-a)-(3-2a)k]$ $M_{max.} = 2Wla^2(1-a)^2$ at C $M_1 = -Wla(1-a)^2$ $M_2 = -Wla^2(1-a)$ <u>SHEAR</u> A to C: $V_x = R_1$ C to B: $V_x = -R_2$ <u>DEFLECTION</u> A to C $y_L = \frac{Wl^3}{6EI}k^2(1-a)[3a(1-a)-(1+a-2a^2)k]$ C to B $y_R = \frac{Wl^3}{6EI}a^2[(3-2a)k^3-3(2-a)k^2+3k-a]$ $y_c = \frac{Wl^3}{3EI}a^3(1-a)^3$ for $k=a$ $y_{max.} = \frac{2Wl^3}{3EI}\frac{a^2(1-a)^3}{(3-2a)^2}$ for $k_m = \frac{1}{3-2a}$ <u>POINTS OF INFLECTION</u> $k_1 = \frac{a}{1+2a}$ $k_2 = \frac{2-a}{3-2a}$	<u>FIBER STRESS IN TERMS OF LOADING</u> At left end: $f_1 = \frac{Wl}{I}a(1-a)^2c$ At load: $f_{max.} = \frac{2Wl}{I}a^2(1-a)^2c$ At right end: $f_2 = \frac{Wl}{I}a^2(1-a)c$ <u>DEFLECTION IN TERMS OF FIBER STRESS</u> In terms of f.s. at left end $y_c = \frac{f_1 l^2 a^2(1-a)}{3Ec}$ In terms of fiber stress at C $y_c = \frac{f_{max.} l^2 a(1-a)}{6Ec}$ $y_{max.} = \frac{f_{max.} l^2(1-a)}{3Ec\,(3-2a)^2}$ In terms of f.s. at right end $y_c = \frac{f_2 l^2}{3Ec}a(1-a)^2$ $y_{max.} = \frac{2}{3}\frac{f_2 l^2(1-a)^2}{Ec\,(3-2a)^2}$ <u>ABSOLUTE MAXIMUM VALUES FOR A MOVABLE LOAD</u> <u>MOMENT</u> $M_1(\text{abs.max.}) = -\frac{4}{27}Wl$ for $a=\frac{1}{3}$ $M_c(\text{abs.max.}) = +\frac{1}{8}Wl$ for $a=\frac{1}{2}$ $M_2(\text{abs.max.}) = -\frac{4}{27}Wl$ for $a=\frac{2}{3}$ <u>DEFLECTION</u> $y_{max.} = \frac{1}{192}\frac{Wl^3}{EI}$ for $a=\frac{1}{2}$
	<u>MOMENT</u> A to C: $M_x = \frac{Wl}{8}(4k-1)$ C to B: $M_x = \frac{Wl}{8}(3-4k)$ $M_c = M_{max.} = +\frac{1}{8}Wl$ $M_1 = -\frac{1}{8}Wl$ at A and B <u>SHEAR</u> A to C: $V_x = +W$ C to B: $V_x = -W$ <u>DEFLECTION</u> A to C: $y_L = \frac{Wl^3}{48EI}k^2(3-4k)$ C to B: $y_R = \frac{Wl^3}{48EI}(4k^3-9k^2+6k-1)$ $y_{max.} = \frac{Wl^3}{192EI}$ for $k=\frac{1}{2}$	<u>FIBER STRESS IN TERMS OF LOADING</u> At A and B: $f_1 = \frac{1}{8}\frac{Wlc}{I}$ At C: $f_{max.} = \frac{1}{8}\frac{Wlc}{I}$ <u>DEFLECTION IN TERMS OF FIBER STRESS</u> $y_{max.} = \frac{1}{24}\frac{fl^2}{Ec}$

They have opposite signs, since one is measured above the line AB and the other below it. Hence $t_1 = -t_2$. Figures 115*b*, *c*, and *d* show three methods of drawing the moment diagrams. All ordinates will be assumed as positive.

In Fig. 115*b*, all moments are expressed in terms of the end reactions. The bending moment is known only at A and B. At these points the moment is zero since the beam is assumed to be freely supported at the ends. We then have

$$EIt_1 = 12R_1(6)(8) + 12R_1(9)(18) + 30R_3(9)(24)$$

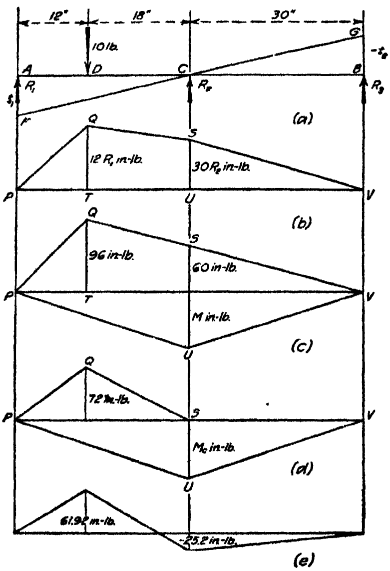


FIG. 115.

and

$$EIt_2 = 30R_3(15)(20)$$

But

$$t_1 = -t_2$$

Whence

$$7R_1 = -43R_3 \qquad \text{(a)}$$

From statics

$$30R_1 - 180 = 30R_3$$

Whence

$$R_1 = R_3 + 6 \qquad \text{(b)}$$

From (a) and (b)

$$R_1 = 5.16 \text{ lb.}$$
$$R_2 = 5.68 \text{ lb.}$$
$$R_3 = -0.84 \text{ lb.}$$

From statics, the moment at point C is

$$M_C = 30R_3 = -25.2 \text{ in.-lb.}$$

The complete moment diagram is shown in Fig. 115*e*.

In Fig. 115*c*, the M-diagrams for the load at D and for the reaction at C are sketched separately. PQV is the M-diagram when the center reaction is removed and AB considered as a simple beam, supporting the load at D. PUV is the M-diagram when the load at D is removed, and AB considered as a simple beam held in equilibrium by the forces at A, B, and C.

From area-moments about P and V we have

$$EIt_1 = (96)(6)(8) + (96)(9)(18) + (60)(9)(24) + M(15)(20)$$

and

$$EIt_2 = (60)(15)(20) + M(15)(20)$$

But

$$t_1 = -t_2$$

Therefore

$$M = -85.2 \text{ in.-lb.}$$

The bending moment at C is

$$M_C = SU = 60 + M = -25.2 \text{ in.-lb.}$$

To determine the reactions note that moments about C taken for forces on either side of this point must equal M_C. Hence, for the section to the left of C,

$$30R_1 - (10)(18) = M_C = -25.2$$

and

$$R_1 = 5.16 \text{ lb.}$$

Also, for the section to the right of C,

$$30R_3 = M_C = -25.2$$

and

$$R_3 = -0.84 \text{ lb.}$$

Placing a summation of vertical forces equal to zero, to determine R_2, we have

$$R_1 + R_2 + R_3 = 10$$

from which

$$R_2 = 5.68 \text{ lb.}$$

The value of R_2 may also be determined from a moment equation about points A or B.

In Fig. 115d, the M-diagram is drawn by first considering that AC and CB are simple beams, i.e., by assuming no continuity and no bending moment at C. Thus PQS is the M-diagram for the simple beam AC supporting 10 lb. at D. There is no corresponding diagram for CB, since there is no load in that span. The area PUV is then added to provide for the bending moment on account of continuity at C.

$$\begin{aligned} EIt_1 &= (72)(6)(8) + (72)(9)(18) + M_C(15)(20) \\ EIt_2 &= M_C(15)(20) \\ t_1 &= -t_2 \end{aligned}$$

Whence

$$M_C = -25.2 \text{ as before.}$$

After the reactions have been determined, the bending moments at C and D, and the deflection at any point may be computed. Figure 115e shows the completed moment diagram. The elastic curve when drawn will have the general configuration shown in Fig. 116.

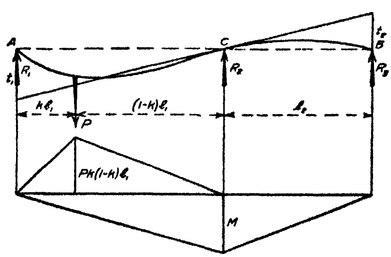


Fig. 116.

The general expressions for R_1, R_2, and R_3 will now be developed in connection with Fig. 116. Let the tangent to the elastic curve be drawn through C and let t_1 and t_2 represent the tangential deviations at A and B respectively. Then, from area-moments about A and B, we have

$$\begin{aligned} EIt_1 &= Pk(1-k)l_1\left(\frac{1}{2}l_1\right)\frac{2}{3}\left[kl_1 + \frac{1}{2}\left(1-k\right)l_1\right] + M\left(\frac{1}{2}l_1\right)\left(\frac{2}{3}l_1\right) \\ &= \frac{Pl^3}{6}(k - k^3) + \frac{Ml_1^2}{3} \\ EIt_2 &= M\left(\frac{1}{2}l_2\right)\left(\frac{2}{3}l_2\right) = \frac{Ml_2^2}{3} \end{aligned}$$

From similar triangles we have the proportion

$$t_1 : -t_2 :: l_1 : l_2$$

Whence

$$t_1 l_2 = t_2 l_1$$

Therefore

$$M = -\frac{Pl_1^2(k - k^3)}{2(l_1 + l_2)} \tag{1}$$

The values of the reactions may be determined by statics as in the problem given above, whence

$$R_1 = P(1 - k) - \frac{Pl_1(k - k^3)}{2(l_1 + l_2)} \tag{2}$$

$$R_2 = Pk + \frac{Pl_1(k - k^3)}{2l_2} \tag{3}$$

$$R_3 = -\frac{Pl_1^2(k - k^3)}{2(l_1 + l_2)l_2} \tag{4}$$

Since k is less than unity, k-k^3 is positive; hence M is a negative bending moment, and R_3 is a negative reaction acting downward.

When the two spans are of equal length l, the reactions are

$$R_1 = \frac{P}{4}(k^3 - 5k + 4) \tag{5}$$

$$R_2 = \frac{P}{4}(-2k^3 + 6k) \tag{6}$$

$$R_3 = \frac{P}{4}(k^3 - k) \tag{7}$$

Beam on Three Supports—Uniform Loads.—In Fig. 117*a* the parabola PQS is the M-diagram, when AC is considered as a simple beam, and the parabola STV is the M-diagram when CB is considered as a simple beam. The triangle PUV is added to represent the bending moment on account of the continuity. If the tangent to the elastic curve be drawn through C, and t_1 and t_2 represent the tangential deviations at A and B respectively, then

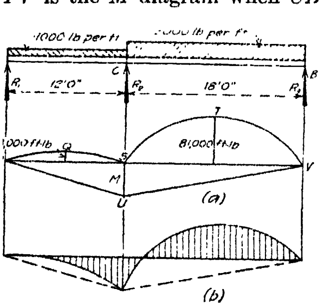


Fig. 117.

$$\begin{aligned}
EIt_1 &= (18{,}000)(12)(\tfrac{2}{3})(6) + M(6)(8) \\
&= 864{,}000 + 48M \\
EIt_2 &= (81{,}000)(18)(\tfrac{2}{3})(9) + M(9)(12) \\
&= 8{,}748{,}000 + 108M \\
3t_1 &= -2t_2 \\
M &= -55{,}800
\end{aligned}$$

The M-diagram may now be drawn to scale as shown in Fig. 117*b*. From statics

$$-55{,}800 = 12R_1 - (12{,}000)(6) = 18R_3 - (36{,}000)(9)$$

$$\begin{aligned}
R_1 &= 1{,}350 \\
R_2 &= 31{,}750 \\
R_3 &= 14{,}900
\end{aligned}$$

The general expressions for R_1 and R_2, and R_3 for a continuous beam of two unequal spans l_1 and l_2, supporting unequal uniform loads w_1 and w_2 per unit of length, will now be developed in connection with Fig. 118. The M-diagram is drawn as in the preceding problem. If the tangent to the elastic curve is drawn through C, and t_1 and t_2 represent the tangential deviations at A and B respectively, then

$$EIt_1 = \left(\frac{w_1 l_1^2}{8}\right)\left(\frac{2}{3}l_1\right)\left(\frac{1}{2}l_1\right) + \left(\frac{Ml_1}{2}\right)\left(\frac{2}{3}l_1\right)$$

$$= \frac{w_1 l_1^4}{24} + \frac{Ml_1^2}{3}$$

$$EIt_2 = \frac{w_2 l_2^2}{24} + \frac{Ml_2^2}{3}$$

$$\left.\begin{aligned} t_1 l_2 &= -t_2 l_1 \\ M &= -\frac{w_1 l_1^3 + w_2 l_2^3}{8(l_1 + l_2)} \end{aligned}\right\} \qquad (8)$$

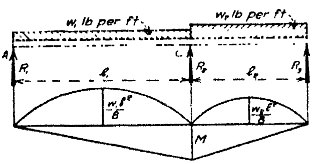


Fig. 118.

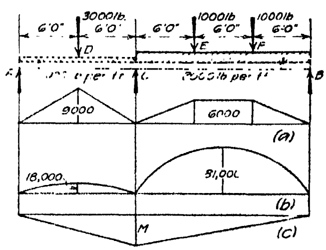


Fig 119.

When the spans are equal in length l and the uniform load w per unit of length is the same in both spans, eq. (8) reduces to

$$M = -\frac{wl^2}{8} \qquad (9)$$

Beam on Three Supports—Special Cases.—When a continuous beam supports a combination of uniform and concentrated loads, it will be found expedient to sketch the M-diagram in parts as shown in Fig. 119. The portion (*a*) is the M-diagram for the concentrated loads when no continuity is considered at C; and the portion (*b*) is a similar diagram for the uniform loads. The continuity is provided for by the portion (*c*). If the tangent to the elastic curve is drawn through C, and t_1 and t_2 represent the tangential deviations at A and B, then

$$EIt_1 = (9{,}000)(6)(6) + (18{,}000)(12)(\tfrac{2}{3})(6) + M(6)(8)$$

$$= 1{,}188{,}000 + 48M$$

$$EIt_2 = (6{,}000)(3)(4) + (6{,}000)(6)(9) + (6{,}000)(3)(1\ ) + (81{,}000)(18)(\tfrac{2}{3})(9) + M(9)(12)$$

$$= 9{,}396{,}000 + 108M$$

$$3t_1 = -2t_2$$

$$M = -62{,}100$$

The reactions may now be determined by the principles of statics.

The value of M may also be determined from eqs. (1) and (8). Equation (1) is applicable to the concentrated loads. For the load at D, $P = 3{,}000$, $k = \frac{1}{2}$, $l_1 = 12$, and $l_2 = 18$. Hence

$$M = -\frac{(3{,}000)(144)(\frac{1}{2} - \frac{1}{8})}{2(12 + 18)} = -2{,}700$$

For the load at F, $P = 1{,}000$, $k = \frac{1}{3}$, $l_1 = 18$, and $l_2 = 12$. Hence

$$M = -\frac{(1{,}000)(324)(\frac{1}{3} - \frac{1}{27})}{2(12 + 18)} = -1{,}600$$

For the load at E, $P = 1{,}000$, $k = \frac{2}{3}$, $l_1 = 18$, and $l_2 = 12$. Hence

$$M = -\frac{(1{,}000)(324)(\frac{2}{3} - \frac{8}{27})}{2(12 + 18)} = -2{,}000$$

Hence the bending moment at C, due to the three concentrated loads is

$$M = -2{,}700 - 1{,}600 - 2{,}000 = -6{,}300$$

Equation (8) is applicable to the uniform loads, where $w_1 = 1{,}000$, $w_2 = 2{,}000$, $l_1 = 12$, and $l_2 = 18$. Hence

$$M = -\frac{(1{,}000)(12^3) + (2{,}000)(18^3)}{8(12 + 18)} = -55{,}800$$

which agrees with the bending moment at C for the beam in Fig. 117. The total bending moment at C for the combined uniform and concentrated loads is

$$M = -6{,}300 - 55{,}800 = -62{,}100$$

as previously determined.

The continuous beam in Fig. 120 supports a uniform load of 1,000 lb. per foot over a part of the span AC. The area $PQSTW$ is the M-diagram, when AC is considered as a simple span. Let the tangent to the elastic curve be drawn through C, and let t_1 and t_2 represent the tangential deviations at A and B respectively. In finding the area-moment of $PQSTW$ about P, the parabolic area QST is encountered. This area has all the properties of the area $Q'S'T'$, which is the M-diagram for a simple beam 12 ft. long supporting a uniform load of 1,000 lb. per foot over its entire length; hence the area-moment of $PQSTW$ about P may be found as follows:

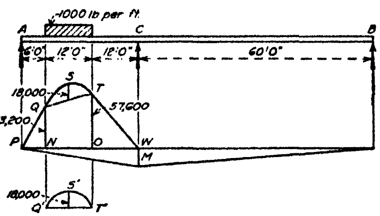


FIG. 120.

$$\begin{aligned}
\text{Area } PQN &\quad (43{,}200)(3)(4) = 518{,}400 \\
QNO &\quad (43{,}200)(6)(10) = 2{,}592{,}000 \\
QTO &\quad (57{,}600)(6)(14) = 4{,}838{,}400 \\
TOW &\quad (57{,}600)(6)(22) = 7{,}603{,}200 \\
QST &\quad (18{,}000)(12)(\tfrac{2}{3})(12) = \underline{1{,}728{,}000} \\
&\qquad 17{,}280{,}000
\end{aligned}$$

$$EIt_1 = 17{,}280{,}000 + M(15)(20)$$

$$EIt_2 = M(30)(40)$$

$$2t_1 = -t_2$$

$$M = -19{,}200$$

The reactions are statically determinate when M is known. The value of M may also be determined by the use of eq. (1) in which P is a concentrated load at the distance kl_1 from A. In the present case let P represent the weight of an element of length dkl_1 at the distance kl_1 from A. Then

$$P = 1{,}000\, l_1 dk$$

Whence

$$dM = -\frac{1{,}000\, l_1^3(k - k^3)dk}{2(l_1 + l_2)}$$

The value of M may be found by integrating between the limits $k = 0.2$ and $k = 0.6$, hence

$$M = -\int_{0.2}^{0.6} \frac{(1{,}000)(30^3)}{2(30 + 60)}(k - k^3)dk$$
$$= -150{,}000\left[\frac{k^2}{2} - \frac{k^4}{4}\right]_{0.2}^{0.6}$$
$$= -19{,}200$$

Beam on Four Supports.—The beam in Fig. 121 is continuous over four supports. Two elastic equations are required, in addition to the two static equations which may be written, for the determination of R_1, R_2, R_3, and R_4. Let t_1 and t_3 represent the tangential deviations at A and D, for the tangent to the elastic curve at C; and let t_2 and t_4 represent the tangential deviations at C and B, for the tangent to the elastic curve at D. Then

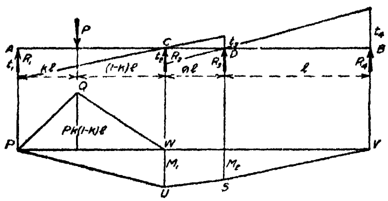


FIG. 121.

$$at_1 = -t_3$$

and

$$t_2 = -at_4$$

PQW is the M-diagram when AC is considered as a simple span, to which the diagram $PUSV$ is added to provide for continuity.

$$EIt_1 = Pk(1 - k)l\left(\frac{1}{2}l\right)\frac{2}{3}\left[kl + \frac{1}{2}(1 - k)l\right] + M_1\left(\frac{1}{2}l\right)\left(\frac{2}{3}l\right)$$
$$= \frac{Pl^3}{6}(k - k^3) + \frac{M_1l^2}{3}$$
$$EIt_3 = M_1\left(\frac{1}{2}al\right)\left(\frac{2}{3}al\right) + M_2\left(\frac{1}{2}al\right)\left(\frac{1}{3}al\right)$$
$$= \frac{a^2l^2}{6}(2M_1 + M_2)$$
$$EIt_2 = M_1\left(\frac{1}{2}al\right)\left(\frac{1}{3}al\right) + M_2\left(\frac{1}{2}al\right)\left(\frac{2}{3}al\right)$$
$$= \frac{a^2l^2}{6}(M_1 + 2M_2)$$
$$EIt_4 = M_2\left(\frac{1}{2}l\right)\left(\frac{2}{3}l\right) = \frac{M_2l^2}{3}$$

Whence

$$M_1 = \frac{-2Pl(k - k^3)(a + 1)}{3a^2 + 8a + 4}$$
$$M_2 = \frac{Pl(k - k^3)a}{3a^2 + 8a + 4}$$

To determine the reactions, note that the moments to the right or left of any support is equal to the moment at that support, as given above. We then have

$$R_1 = P(1-k) - \frac{2P(k-k^2)(a+1)}{3a^2+8a+4} \qquad R_3 = -\frac{P(k-k^3)(a^2+3a+2)}{(3a^2+8a+4)a}$$

$$R_2 = Pk + \frac{P(k-k^3)(2a^2+5a+2)}{(3a^2+8a+4)a} \qquad R_4 = +\frac{P(k-k^3)a}{3a^2+8a+4}$$

75c. The Theorem of Three Moments. *Area-Moment Method.*—This theorem establishes a relation between the bending moments at any three consecutive supports of a beam. In Fig. 122 let l_1 and l_2 represent the lengths of two adjacent spans which support uniform loads of w_1 and w_2 per unit of length respectively. Let M_0, M_1, and M_2 represent the bending moments at the three supports, and let I_1 and I_2 represent the moments of inertia of the cross-sections of the two spans. Let the tangent to the elastic curve be drawn through the middle support and let t_0 and t_2 represent the tangential deviations at the left and right supports respectively. From moment areas about points 0 and 2, we have

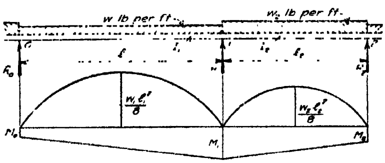


FIG. 122.

$$EI_1t_0 = \left(\frac{w_1l_1^2}{8}\right)\left(\frac{2}{3}l_1\right)\left(\frac{1}{2}l_1\right) + M_0\left(\frac{1}{2}l_1\right)\left(\frac{1}{3}l_1\right) + M_1\left(\frac{1}{2}l_1\right)\left(\frac{2}{3}l_1\right)$$

$$= \frac{w_1l_1^4}{24} + \frac{M_0l_1^2}{6} + \frac{M_1l_1^2}{3}$$

$$EI_2t_2 = \left(\frac{w_2l_2^2}{8}\right)\left(\frac{2}{3}l_2\right)\left(\frac{1}{2}l_2\right) + M_1\left(\frac{1}{2}l_2\right)\left(\frac{2}{3}l_2\right) + M_2\left(\frac{1}{2}l_2\right)\left(\frac{1}{3}l_2\right)$$

$$= \frac{w_2l_2^4}{24} + \frac{M_1l_2^2}{3} + \frac{M_2l_2^2}{6}$$

From similar triangles

$$t_0l_2 = -t_2l_1$$

Whence

$$M_0\frac{l_1}{I_1} + 2M_1\left(\frac{l_1}{I_1} + \frac{l_2}{I_2}\right) + M_2\frac{l_2}{I_2} = -\frac{1}{4}\left(w_1\frac{l_1^3}{I_1} + w_2\frac{l_2^3}{I_2}\right) \qquad (10)$$

Equation (10) is the general equation for the Theorem of Three Moments for uniform loads.

When the continuous beam supports concentrated loads W and W_2 as shown in Fig. 123, the parabolic simple beam moment diagrams are replaced by triangular moment diagrams. By a process similar to the one given above, we derive

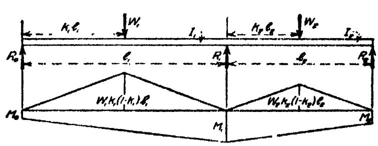


FIG. 123.

$$\frac{M_0l_1}{I_1} + 2M_1\left(\frac{l_1}{I_1} + \frac{l_2}{I_2}\right) + \frac{M_2l_2}{I_2} = -\frac{W_1}{I_1}k_1(1-k_1^2)l_1^2$$

$$-\frac{W_2}{I_2}k_2(1-k_2)(2-k_2)l_2^2 \qquad (11)$$

Equation (11) is the general equation for the Theorem of Three Moments for concentrated loads.

When several loads are included on any span a term must be added to the right hand side of eq. (11) to account for each load.

To determine the moments at the supports of a continuous beam by means of the Theorem of Three Moments, eq. (10) or (11) must be written for pairs of adjacent spans beginning at one end of the beam. Thus in Fig. 124 apply the Theorem of Three Moments to spans 1 and 2; then to spans 2 and 3; and finally to spans 3 and 4. In this manner three independent equations are derived.

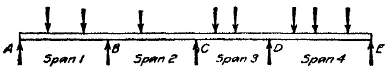


Fig. 124.

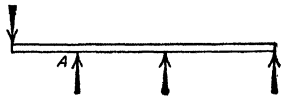


Fig. 125.

From Fig. 124 it can be seen that there are five unknown moments to be determined. However, it is generally possible from the conditions of the problem to determine values of the moments at the two ends of the span. The number of unknowns may then be reduced, becoming equal to the number of condition equations derived by the application of the Theorem of Three Moments.

As stated above, the moments at the end supports may be determined from the conditions of the problem. If the beam is freely supported at the ends, the moments at these points are zero. When one or both ends overhang the end support, as shown in Fig. 125, the end moments are determined as for a cantilever beam of the same dimensions. If one end of the beam is fixed, as shown in Fig. 126, the moment at A may be determined by assuming the conditions of restraint to be replaced by a span of zero length with a free support at the outer end of this span. On applying the Theorem of Three Moments to the given spans and the span of zero length, the moment at A is readily determined.

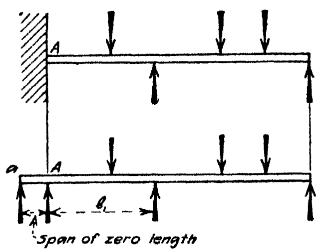


Fig. 126.

The reactions at the supports may be determined by the principles of statics as soon as the moments at the supports are known. Similar solutions are given in the preceding article.

Illustrative Problem.—Determine the moment at the center support of the beam shown in Fig. 115, p. 112, using the Theorem of Three Moments.

Since the beam carries a concentrated load, eq. (11) is to be used. When the moment of inertia of the beam is constant, $I_1 = I_2$. Since I appears in the denominator of each term, it divides out. For the conditions shown, M_1 and M_3 are zero and $k = {}^{12}\!/_{30} = 0.4$. Substituting in eq. (11) we have

$$120M_2 = -(10)(0.4)(1 - 0.4^2)(30)^2 = -3{,}024$$

from which

$$M_2 = -25.2 \text{ in.-lb.}$$

The reactions may be determined by statics, as in the discussion given on p. 113.

Illustrative Problem.—Determine the moment at the center support of the beam of Fig. 119, p. 115, using the Theorem of Three Moments. Calculate the reactions due to the given loading.

Since the beam carries both uniform and concentrated loads, eqs. (10) and (11) may be combined and written in the following form. Noting that $I_1 = I_2$ for uniform cross-section, we have

$$M_0l_1 + 2M_1(l_1 + l_2) + M_2l_2 = -\tfrac{1}{4}(w_1l_1^3 + w_2l_2^3) - W_1k_1(1 - k_1^2)l_1^2 - W_2k_2(1 - k_2)(2 - k_2)l_2^2.$$

For the conditions shown in Fig. 119, the several terms have the following values: $M_0 = 0$; $M_2 = 0$; $M_1 = M_C$; $l_1 = 12$; $l_2 = 18$; $w_1 = 1{,}000$; $w_2 = 2{,}000$; $W_1 = 3{,}000$; $W_2 = 1{,}000$; $k_1 = 0.5$; $k_2 = \tfrac{1}{3}$ and $\tfrac{2}{3}$.

Substituting these values in the above equation, we have

$$60M_C = -\tfrac{1}{4}[(1{,}000)(12)^3 + (2{,}000)(18)^3] - (3{,}000)(\tfrac{1}{2})(1 - \tfrac{1}{4})(12)^2 - (1{,}000)(\tfrac{1}{3})(\tfrac{2}{3})(\tfrac{5}{3})(18)^2 - 1{,}000(\tfrac{2}{3})(\tfrac{1}{3})(\tfrac{4}{3})(18)^2$$

from which finally

$$M_C = -62{,}100 \text{ ft.-lb.}$$

To determine the reactions, note that the moment of forces to the left of the center support must equal M_C. We then have

$$12R_A - (3{,}000)(6) - (12{,}000)(6) = -62{,}100$$

from which

$$R_A = 2{,}325 \text{ lb.}$$

A moment equation for forces to the right of the center support gives

$$18R_B - 1{,}000(6 + 12) - (36{,}000)(9) = -62{,}100$$

from which

$$R_B = 15{,}550 \text{ lb.}$$

From a summation of vertical forces,

$$R_A + R_B + R_C = 3{,}000 + 12{,}000 + (2)(1{,}000) + 36{,}000$$

Solving for R_C we have,

$$R_C = 35{,}125 \text{ lb.}$$

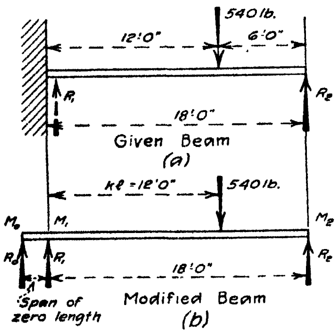


FIG. 127.

Illustrative Problem.—Determine the moment at the left end of the beam of Fig. 114, p. 110, using the Theorem of Three Moments.

Figure 127a shows the given beam, restrained at the left end and freely supported at the right end. In Fig. 127b, the given beam is modified by substituting for the wall, a span of zero length. Let M_0, M_1 and M_2 be the moments at the several supports. From the conditions of the problem, $M_2 = 0$. Assume the beam freely supported at R_0, from which $M_0 = 0$. Applying eq. (11), noting that the several terms have the values $M_0 = 0$, $M_2 = 0$, $l_1 = 0$, $l_2 = 18$, $W_1 = 0$, $W_2 = 540$, $k_1 = 0$, $k_2 = {}^{12}\!/_{18} = \tfrac{2}{3}$, and $I_1 = I_2$, we have

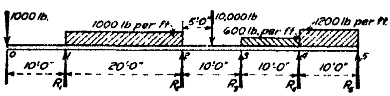


FIG. 128.

$$(2)(M_1)(18) = -(540)(\tfrac{2}{3})(1 - \tfrac{2}{3})(2 - \tfrac{2}{3})(18)^2$$

from which

$$M_1 = -1{,}440$$

This agrees with the result obtained on p. 110.

Illustrative Problem.—Determine the moments and reactions at the supports of the continuous beam shown in Fig. 128. Assume uniform moment of inertia.

Successive application of the Theorem of Three Moments as given by eqs. 10) and (11) to pairs of adjacent spans gives the following independent equations:

$$20M_1 + 60M_2 + 10M_3 = -(\tfrac{1}{4})(1{,}000)(20)^3 - (10{,}000)(\tfrac{1}{2})(\tfrac{1}{2})(\tfrac{3}{2})(10)^2$$
$$10M_2 + 40M_3 + 10M_4 = -(10{,}000)(\tfrac{1}{2})(1 - \tfrac{1}{4})(10)^2 - (\tfrac{1}{4})(600)(10)^3$$
$$20M_3 + 40M_4 + 10M_5 = -(\tfrac{1}{4})(600)(10)^3 - (\tfrac{1}{4})(1{,}200)(10)^3$$

The left end of the beam of Fig. 128 forms a cantilever. Hence $M_1 = -(1,000)(10) = -10,000$ ft.-lb. Since the right end of the beam is freely supported, $M_5 = 0$. Reducing the above condition equations to their simplest form, we have

$$M_2 + 0.167M_3 = -36,250$$
$$M_2 + 6M_3 + 2M_4 = -52,500$$
$$M_3 + 3M_4 = -45,000$$

On solving these simultaneous equations, the values of the moments are found to be

$$M_1 = -10,000 \qquad M_2 = -35,830$$
$$M_3 = -1,446 \qquad M_4 = -10,890$$
$$M_5 = 0$$

All values are given in foot pounds.

The reactions may be determined from the condition that the summation of moments to the right or left of any support must be equal to the moment given above. To determine R_1, take moments about point 2, Fig. 128, for the forces to the left, from which

$$+20R_1 - (1,000)(20 + 10) - (1,000)(20)(10) = M_2 = -35,830$$

Solving for R_1, we have $R_1 = 9,800$ lb.

The other reactions may be determined in a similar manner. In determining R_5 and R_4, it will be found best to take moments for forces to the right of points 4 and 3. This will effect a considerable reduction in the work required. The reactions determined by this process are as follows

$$R_1 = 9,710 \qquad R_2 = 19,730$$
$$R_3 = 3,620 \qquad R_4 = 11,030$$
$$R_5 = 4,910$$

All values are in pounds.

75d. General Methods for Determination of Reactions in Continuous Beams.—The reactions for continuous beams may be determined directly by means of the principles of statics. This method of determining reactions has been adopted in the preceding articles. The reactions may also be determined from general formulas expressed in terms of the moments at the supports. The determination of reactions by formulas has some advantages over a solution by means of the principles of statics. Thus when a formula is used any reaction may be determined without reference to the reaction at any other support. When the principles of statics are used, the value of any reaction is dependent on the reactions at other supports. Errors made in the determination of any reaction therefore affect the values of reactions at other supports.

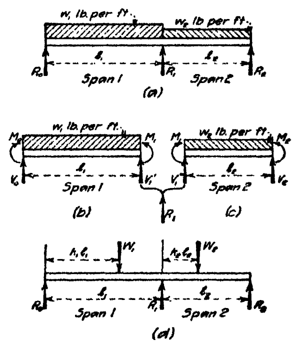


FIG. 129.

A general formula for reactions will now be derived for the conditions shown in Fig. 129a. Let the lengths of two adjacent spans be l_1 and l_2 and let w_1 and w_2 be the loads per foot on these spans. Figure 129b and c show the two spans removed and all external forces shown in position. The reaction R_1 at the junction of the two spans is

$$R_1 = V'_1 + V''_1$$

On substituting in this equation values of V'_1 and V''_1 determined from eqs. (3 and (3a) of Art. 73, modified to fit the conidtions of Fig. 129, we have

$$R_1 = \frac{(M_0 - M_1)}{l_1} + \frac{(M_2 - M_1)}{l_2} + \frac{w_1 l_1}{2} + \frac{w_2 l_2}{2} \qquad (12$$

Equation (12) is a general formula for reactions for uniform loads. When the beam supports concentrated loads, as shown in Fig. 129*d*, the general formula becomes

$$R_1 = \frac{(M_0 - M_1)}{l_1} + \frac{(M_2 - M_1)}{l_2} + W_1k_1 + W_2(1 - k_2) \tag{13}$$

On substituting in eqs. (12) and (13), the sign of the moment as determined from the Theorem of Three Moments must also be taken into account.

The reactions at end supports in a continuous beam may be determined directly from eqs. (3) and (3*a*) of Art. **73,** modified to fit the conditions of Fig. 129. If R_0 represents the left end reaction, we have, for uniform loads,

$$R_0 = \frac{M_1 - M_0}{l_1} + \frac{w_1l_1}{2} \tag{14}$$

and for concentrated loads

$$R_0 = \frac{M_1 - M_0}{l_1} + W_1k_1 \tag{15}$$

If R_2 represents a right end reaction, we have, for uniform loads,

$$R_2 = \frac{M_1 - M_2}{l_2} + \frac{w_2l_2}{2} \tag{16}$$

and for concentrated loads

$$R_2 = \frac{M_1 - M_2}{l_2} + W_2(1 - k_2) \tag{17}$$

Illustrative Problem.—Determine the reactions for the beam of Fig. 119, p. 115, using eqs. (12) and (13) and the moments calculated on p. 120.

The moment at the center support is $M_C = -62{,}000$ ft.-lb. This moment is represented by M_1 in eqs. (12) and (13). Since the ends of the beam are freely supported, M_0 and M_2 are equal to zero. To determine the reaction at the center support of Fig. 119, eqs. (12) and (13) may be combined and written in the form

$$R_C = -\frac{M_1}{l_1} - \frac{M_1}{l_2} + \frac{w_1l_1}{2} + \frac{w_2l_2}{2} + W_1k_1 + W_2(1 - k_2)$$

The several terms of this equation have the following values: $M_1 = M_C = -62{,}100$, $w_1 = 1{,}000$, $w_2 = 2{,}000$, $W_1 = 3{,}000$, $W_2 = 1{,}000$, $k_1 = 0.5$, $k_2 = \frac{1}{3}$ and $\frac{2}{3}$, $l_1 = 12$, $l_2 = 18$. Substituting these values in the above equation, we have

$$R_C = +\frac{62{,}100}{12} + \frac{62{,}100}{18} + (\tfrac{1}{2})(1{,}000)(12) + (\tfrac{1}{2})(2{,}000)(18) + (3{,}000)(\tfrac{1}{2}) + (1{,}000)(\tfrac{2}{3}) + (1{,}000)(\tfrac{1}{3})$$

from which

$$R_C = 35{,}125 \text{ lb.}$$

The reaction at the left end of the beam may be determined by combining eqs. (14) and (15), noting that $M_0 = 0$. Thus

$$R_A = -\frac{62{,}100}{12} + (\tfrac{1}{2})(1{,}000)(12) + (3{,}000)(\tfrac{1}{2}) = 2{,}325 \text{ lb}$$

To determine the reaction at the right end of the beam, combine eqs. (16) and (17), noting that $M_2 = 0$, thus,

$$R_B = -\frac{62{,}100}{18} + (\tfrac{1}{2})(2{,}000)(18) + (1{,}000)(\tfrac{2}{3}) + (1{,}000)(\tfrac{1}{3}) = 15{,}550 \text{ lb.}$$

75*c*. The Theorem of Three Moments, Effect of Settlement of Supports on Moments and Reactions.—The formulas of the preceding articles are based on the assumption that the supports for the beam are rigid. If, due to any cause, the relative elevation of the supports is changed after the beam is placed in position, the relation between the intercepts t_0 and t_2 stated in Art. **75*c*** must be modified to meet the actual conditions.

In Fig. 130 let A, B, and C show the original position of three consecutive supports of a continuous beam. Due to settlement of supports, these points take the positions shown at a, b, and c respectively. Let h_0, h_1 and h_2 respectively represent the actual settlement of each support. At point b draw fbd tangent to the elastic curve. Also, connect points b and c by a straight line and produce this line to an intersection at e with the vertical through point a. The distance ae shows the effect of settlement of supports on the position of point a. If the settlement of supports is proportional, or equal, ae will be zero and settlement will have no effect on the moments or reactions.

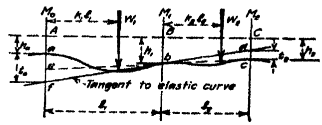


FIG. 130.

For the conditions shown in Fig. 130, we have

$$\frac{ef}{l_1} = -\frac{dc}{l_2}$$

But $ef = t_0 - ac$ and $dc = t_2$, where t_0 and t_2 have the values given in Art. **75c**. From similar triangles in Fig. 130 it can be shown that

$$ae = (h_1 - h_0) + (h_1 - h_2)\frac{l_1}{l_2}.$$

On substituting in the above equation, the value of ae and values of t_0 and t_2 as given in Art. **75c**, we derive finally,

For concentrated loads

$$M_0l_1 + 2M_1(l_1 + l_2) + M_2l_2 = -W_1k_1(1 - k_1^2)l_1^2 - W_2k_2(1 - k_2)(2 - k_2)l_2^2 - 6EI\left(\frac{h_0 - h_1}{l_1} + \frac{h_2 - h_1}{l_2}\right) \quad (18)$$

For uniform loads

$$M_0l_1 + 2M_1(l_1 + l_2) + M_2l_2 = -\tfrac{1}{4}(w_1l_1^3 + w_2l_2^3) - 6EI\left(\frac{h_0 - h_1}{l_1} + \frac{h_1 - h_2}{l_2}\right) \quad (19)$$

Equations (18) and (19) give the Theorem of Three Moments for a beam of uniform moment of inertia I when settlement of the supports has taken place.

Illustrative Problem.—A 15-in. 42.9-lb. I-beam supported at three points 20 ft. apart forms a two-span continuous girder supporting a uniform load of 1,500 lb. per ft. Levels taken at the supports show that settlement of the supports has caused the following changes in elevation: left support 0.010 ft.; center support 0.020 ft.; right support 0.015 ft. Determine the change in moment at the center support due to settlement.

The required moment may be determined from eq. (19). Since the beam is freely supported at the ends, M_0 and M_2 are zero. The terms in eq. (19) have the following values: $l_1 = l_2 = 20$ ft., $w_1 = w_2 = 1{,}500$ lb., $E = 30{,}000{,}000$ lb. per sq. in.; $I = 441.8$ in.^{4}, $h_0 = 0.010$, $h_1 = 0.030$, and $h_2 = 0.015$. Substituting these values in eq. (19), noting that E and I must be reduced to foot units, we have

$$80M_1 = -\left(\frac{1}{4}\right)(1{,}500)(20)^3(2) - (6)(30{,}000{,}000)(144)\left(\frac{441.8}{12^4}\right)\left(\frac{0.010 - 0.030}{20} + \frac{0.015 - 0.030}{20}\right)$$

from which

$$M_1 = -62{,}100 \text{ ft.-lb.}$$

For a similar beam on rigid supports, the center moment is $M_1 = -75{,}000$ ft.-lb. Settlement of supports has decreased the center moment 16 per cent. It is to be noted that the positive moments will be increased to compensate for the decrease in center moment. · If in any case, the center moment is entirely relieved, the beam becomes a simple beam with positive moment at all points.

75f. Continuous Girders on Elastic Supports.—When a continuous girder is supported by posts or columns the relative elevations of the several supports may be effected by the distortion of these members. These changes in elevation have the same effect on moments and reactions as a settlement of the supports.

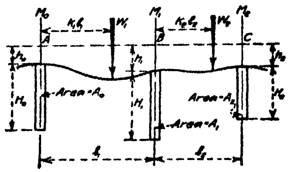


FIG. 131.

Figure 131 shows two adjacent spans of a continuous beam supported by columns of height H_0, H_1 and H_2 whose areas are A_0, A_1, and A_2. Let R_0, R_1, and R_2 represent the final values of the stresses in these columns, or the reactions at the several points. For the conditions shown in Fig. 131, the deformation of the several supports is

$$h_0 = \frac{R_0H_0}{A_0E_0} \qquad h_1 = \frac{R_1H_1}{A_1E_1} \qquad h_2 = \frac{R_2H_2}{A_2E_2}$$

where E_0, E_1, and E_2 denote the modulii of elasticity of the material composing the several columns. These deformations may be considered as settlements of the supports. On substituting these values in eqs. (18) and (19), we have

For concentrated loads

$$M_0l_1 + 2M_1(l_1 + l_2) + M_2l_2 = -W_1k_1(1 - k_1^2)l_1^2 - W_2k_2(1 - k_2)(2 - k_2)l_2^2 - 6EI\left[\frac{R_0H_0}{l_1A_0E_0} - \frac{R_1H_1}{l_1l_2A_1E_1}(l_1 + l_2) + \frac{R_2H_2}{l_2A_2E_2}\right] \quad (20)$$

for uniform loads

$$M_0l_1 + 2M_1(l_1 + l_2) + M_2l_2 = -\tfrac{1}{4}(w_1l_1^3 + w_2l_2^3) - 6EI\left[\frac{R_0H_0}{l_1A_0E_0} - \frac{R_1H_1}{l_1l_2A_1E_1}(l_1 + l_2) + \frac{R_2H_2}{l_2A_2E_2}\right] \quad (21)$$

Equations (20) and (21) give the Theorem of Three Moments for a continuous beam of uniform moment of inertia when the beam is placed on elastic supports. These equations are written in terms of the moments and reactions at the several supports. Values of the reactions, as given by eqs. (16) or (17), may be substituted in the above equations if desired. However, the resulting equations are very cumbersome. An application of the above equations to a problem is given below.

Illustrative Problem.—Calculate the moment at the center support and the reactions for the continuous beam shown in Fig. 132.

Since the columns shown in Fig. 132 are relatively small, deflection under load may be expected and the supports considered as elastic. For the conditions shown in Fig. 132, the terms in eq. (21) have the following values: $w_1 = w_2 = 2{,}400$ lb. per ft., $l_1 = l_2 = 12$ ft. $= 120$ in., $H_0 = H_1 = H_2 = 8' - 4'' = 100$ in., $A_0 = A_1 = A_2 = 6$ sq. in., $I = 122.1$ in.^{4}, $E = E_0 = E_1 = E_2 = 30{,}000{,}000$ lb. per sq. in., M_0 and M_2 are zero. Substituting these values in eq. (21), using inch units, and reducing the resulting expression to its simplest form, noting that R_0 and R_2 are equal because of symmetry of loading, we have finally

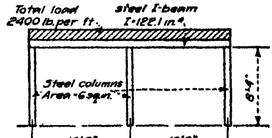


FIG. 132.

$$M_1 = -360{,}000 - 0.424(R_0 - R_1)$$

The reactions may be determined by placing moments about the center support equal to M_1. Thus, using inch units, moments to the left of R_1 gives

$$120R_0 - 1{,}440{,}000 = M_1 = -360{,}000 - 0.424R_0 + 0.424R_1$$

from which

$$R_0 - 0.00352R_1 = 8{,}960$$

Placing summation of vertical forces equal to zero, we have

$$R_0 + 0.5R_1 = 24{,}000$$

Solving these equations

$$R_1 = 29{,}830 \text{ lb.} \qquad R_0 = 9{,}085 \text{ lb}$$

The moment at the center support is

$$M_1 = 29{,}250 \text{ ft.-lb.} = 351{,}000 \text{ in.-lb.}$$

For a beam on rigid supports, we find $M_1 = 30{,}000$ ft.-lb. $= 360{,}000$ in.-lb.; $R_1 = 30{,}000$ lb. and $R_0 = 9{,}000$ lb.

75g. Analysis of Continuous Beams by the Method of Elastic Weights.[1]—To analyze a continuous beam by the method of Elastic Weights, consider the several spans as independent beams restrained at the ends by the moments which exist at the supports of the continuous beam. Since the tangents to the elastic curves for these restrained beams which meet at any support must have the same slope, the M/EI loads for adjacent spans are connected by the relation that the conjugate beam shears for the ends of beams meeting at any support must be equal.

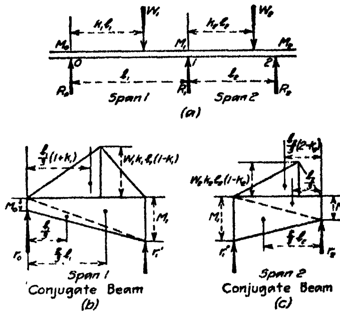


FIG. 133.

Figure 133 shows any two adjacent spans in a continuous beam, and Figs. 133 (*b*) and (*c*) show the conjugate beam and its loading for these spans. At any support as 1, the conjugate beam end shears must be equal, that is $r'_1 = r''_1$. For the conditions shown in Fig. 133*b*

$$r'_1 = \frac{1}{2}W_1k_1l_1(1 - k_1)\left(\frac{l_1}{3}\right)(1 + k_1) + \frac{2}{6}M_1l_1 + \frac{1}{6}M_0l_1$$

and

$$r''_1 = \frac{1}{2}w_2k_2l_2(1 - k_2)\left(\frac{l_2}{3}\right)(2 - k_2) + \frac{2}{6}M_1l_2 + \frac{1}{6}M_2l_2$$

Equating these values, as indicated above, we derive finally

$$M_0l_1 + 2M_1(l_1 + l_2) + M_2l_2 = -W_1k_1(1 - k_1^2)l_1^2 - W_2k_2(1 - k_2)(2 - k_2)l_2^2$$

which is the Theorem of Three Moments for beam of uniform cross section under concentrated loads.

On comparing the analysis given above with the similar solution given in Art. 75*c*, using the Area Moment Method, it will be found that the operations in the two methods are practically identical. The same fact will be noted on comparing numerical problems. Since the two methods are similar, the solutions given in the preceding articles will answer for both methods.

75h. Coefficients for Moments and Reactions for Continuous Beams. The solution of prob ems in continuous beams is greatly facilitated by the use of coefficients for moments and reactions at the supports. These will be given for beams carrying uniform and concentrated loads.

Uniform Loads.—Figures 134*A* and 134*B* give the coefficients for moments and reactions in continuous beams over several supports due to a uniform load.

[1] By W. S. KINNE.

The values to the left and right of any support in Fig. 134*A* represent the shears at these points and the sum of these shears represents the total reaction at any support.

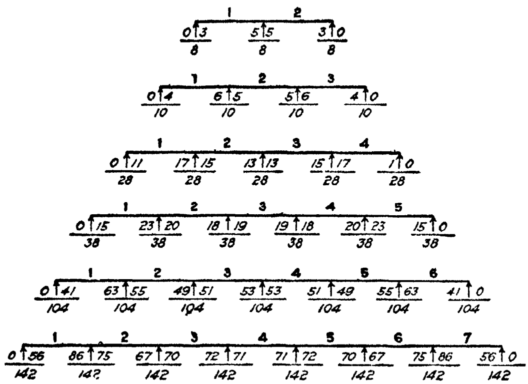


Fig. 134*A*.—Shears in continuous beams; supported ends; uniform loads on all spans; spans all equal. Coefficients of (wl).

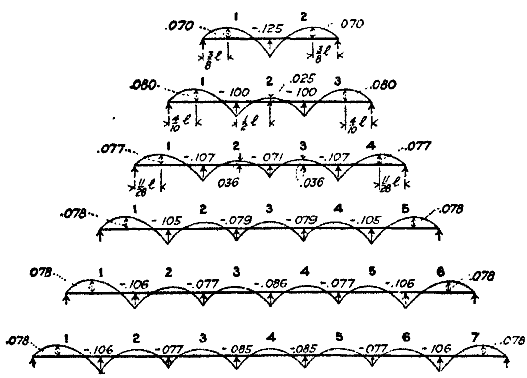


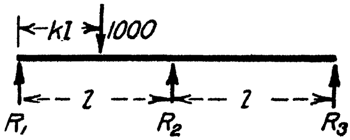
Fig. 134*B*.—Moments in continuous beams; supported ends; uniform load on all spans; spans all equal. Coefficients of (wl^2).

Illustrative Problem.—Determine the moment and reaction at the second support from the left end of a continuous beam of five spans of 10 ft. each due to a uniform load of 1,000 lb. per ft.

On the diagram for the five span beam, the moment at the second support is $-0.105\ wl^2$ and the reaction is $\frac{1}{38}(23 + 20)\ wl$. Hence $M = -(0.105)(1{,}000)(100) = -10{,}500$ ft.-lb. and $R = (\frac{1}{38})(43)(1{,}000)(10) = 11{,}320$ lb.

Concentrated Loads.—Coefficients for reactions for concentrated loads will be given only for a beam continuous over two equal spans carrying a single load on one of the spans. To fix the position of the load, it will be assumed that the span is divided into a varying number of equal panels. Coefficients will be given for each load position. If desired, similar tables may be calculated for beams containing a greater number of spans.

REACTIONS FOR A TWO-SPAN CONTINUOUS BEAM
(1000-lb. load on left hand span)



Number of panels	Values of k	R_1	R_2	R_3
2	1/2	406.25	687.5	− 93.75
3	1/3	592.6	481.5	− 74.1
	2/3	240.7	851.9	− 92.6
		833.3	1,333.4	−166.7
4	1/4	691.4	367.2	− 58.6
	1/2	406.3	687.5	− 93.8
	3/4	168.0	914.0	− 82.0
		1,265.7	1,968.7	−234.4
5	1/5	752.0	296.0	− 48.0
	2/5	516.0	568.0	− 84.0
	3/5	304.0	792.0	− 96.0
	4/5	128.0	944.0	− 82.0
		1,700.0	2,600.0	−300.0
6	1/6	792.8	257.7	− 40.5
	1/3	592.6	471.5	− 74.1
	1/2	406.25	687.5	− 93.75
	2/3	240.75	851.8	− 92.55
	5/6	103.0	960.7	− 63.7
		2,135.4	3,229.2	−364.60
6	1/7	822.2	213.8	− 35.0
	2/7	648.7	416.9	− 65.6
	3/7	484.0	603.5	− 87.5
	4/7	332.4	763.8	− 96.2
	5/7	198.2	889.3	− 87.5
	6/7	86.0	970.9	− 56.9
		2,571.5	3,857.2	−428.7
8	1/8	844.3	186.2	− 30.5
	1/4	691.4	367.2	− 58.6
	3/8	544.5	536.1	− 80.6
	1/2	406.3	687.5	− 93.8
	5/8	279.8	815.4	− 95.2
	3/4	168.0	914.0	− 82.0
	7/8	73.7	977.6	− 51.3
		3,008.0	4,484.0	−492.0
9	1/9	861.5	166.0	− 27.5
	2/9	725.0	327.8	− 52.8
	1/3	592.6	481.5	− 74.1
	4/9	466.4	622.8	− 89.2
	5/9	348.4	747.6	− 96.0
	2/3	240.7	851.9	− 92.6
	7/9	145.4	931.4	− 76.8
	8/9	64.5	982.2	− 46.7
		3,444.5	5,111.2	−555.7
10	1/10	875.25	149.5	− 24.75
	1/5	752.0	296.0	− 48.0
	3/10	631.75	436.5	− 68.25
	2/5	516.0	568.0	− 84.0
	1/2	406.25	687.5	− 93.75
	3/5	304.0	792.0	− 96.0
	7/10	210.75	878.5	− 89.25
	4/5	128.0	944.0	− 72.0
	9/10	57.25	985.5	− 42.75
		3,881.25	5,737.5	−618.75

Negative values indicate downward reactions.

Illustrative Problem.—A beam continuous over two 12-ft. spans supports the following loads: Left hand span—a 10,000-lb. load 4 ft. from the left end and a 20,000-lb. load 8 ft. from the left end. Right hand span—a 30,000-lb. load at the center of the span. Determine the reactions at the supports, using the above table of reactions.

For loads on the left hand span use the values for three panels. The 10,000-lb. load is at the 1/3 point and the 20,000-lb. load is at the 2/3 point. For the load on the right hand span use the values for two panels, interchanging values for R_1 and R_3. The results are as follows:

LOAD	R_1	R_2	R_3
10,000	(10)(592.6) = +5,926.0	(10)(481.5) = + 4,815	(10)(−74.1) = − 741.0
20,000	(20)(240.7) = +4,814.0	(20)(851.9) = +17,038	(20)(−92.6) = − 1,852.0
30,000	(30)(−93.75) = −2,812.5	(30)(687.5) = +20,625	(30)(+406.25) = +12,187.5
Totals	+7,927.5	+42,478	+9,594.5

Illustrative Problem.—A beam continuous over two 20-ft. spans supports six 1,000-lb. loads. The loads are spaced 5 ft. apart, three loads being carried by the left hand span and three by the right hand span. Determine the reactions by means of the above table.

The given loading divides each span into four panels. Hence, use the values for four panels. The value of R_1 for loads on the left hand span is given by the summation of values of R_1 as given in the table. For loads on the right hand span the desired left hand reaction is given in the table under the values for R_3. The results are as follows:

	R_1	R_2	R_3
Loads in left hand span..................	+1,265.7	+1,968.7	− 234.4
Loads in right hand span................	− 234.4	+1,968.7	+1,265.7
Totals...............................	+1,031.3	+3,937.4	+1,031.3

76. Partially Continuous Beams.—It is frequently desirable to consider a structure in which the continuity is imperfect. A swing truss bridge on four supports, designed with parallel chords and very light web members in the center span, so that no shear can be transmitted between the two inside supports, is a structure of this kind. Such structures are called *partially continuous* and their treatment will be illustrated by the beam in Fig. 135.

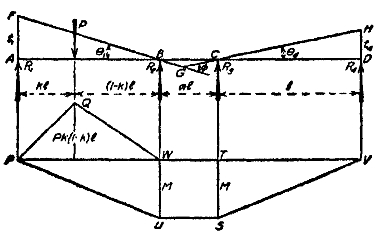


FIG. 135.

It is assumed that bending moment but no shear exists in the center span; hence $R_3 = -R_4$ and the bending moment M at B equals the bending moment at C. Since the continuity of the beam is broken at B and C, the elastic curve is not continuous, but forms cusps at these points; and the tangent FG to the elastic curve for AB at B is *not* tangent to the elastic curve for BC. Similarly the tangent HG to the elastic curve for CD at C is *not* tangent to the elastic curve for BC.

Let θ_1 = Angle ABF, and θ_4 = Angle DCH, then $\theta_1 + \theta_4 = \phi = \dfrac{EI}{\text{area } WTSU}$

The tangential deviations at A and D, being represented as measured above the axis of the beam, are considered negative.

$$-t_1 = \theta_1 l$$
$$-t_4 = \theta_4 l$$
$$t_1 + t_4 = -\phi l$$
$$EI\phi = Mal$$

$$EIt_1 = Pk(1-k)l\left(\frac{1}{2}l\right)\frac{2}{3}\left[kl + \frac{1}{2}(1-k)l\right] + M\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)$$

$$= \frac{Pl^3}{6}(k - k^3) + \frac{Ml^2}{3}$$

$$EIt_4 = \frac{Ml^2}{3}$$

Since

$$EIt_1 + EIt_4 = -EI\phi l.$$

Then

$$\frac{Pl^3}{6}$$

or

$$(k - k^3) + \frac{Ml^2}{3} + \frac{Ml^2}{3} = -Mal^2$$

$$M = \frac{Pl(k - k^3)}{4 + 6a}$$

Therefore

$$R_1 = P(1-k) - \frac{P(k-k^3)}{4+6a} \qquad R_3 = \frac{P(k-k^3)}{4+6a}$$

$$R_2 = Pk + \frac{P(k-k^3)}{4+6a} \qquad R_4 = \frac{-P(k-k^3)}{4+6a}$$

The span in Fig. 136 consists of two restrained beams, connected at mid-span in such a way that shear but no bending moment can be transmitted from one beam to the other. The span, therefore, represents a different phase of partial continuity from that of the previous problem. The principle here involved is employed in the design of a bascule span composed of two leaves connected by a shear lock. The principle must be modified, however, in its application to a bascule span, for the leaves do not as a rule have a constant moment of inertia, nor are they in perfect restraint at the points of support.

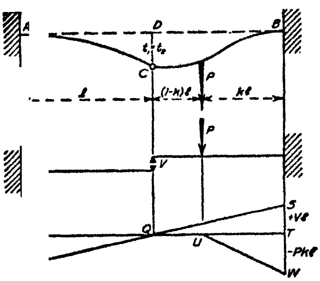


FIG. 136.

A constant moment of inertia and perfect restraint will be assumed in finding the shear V on the pin-connection at C, when the beam CB supports the load P as shown. The M-diagram may be drawn very easily when the partially continuous beam ACB is considered as two restrained beams sketched separately, with the shear at C considered as a force V, acting upward on CB and downward on CA. The bending moment at C is zero. The M-diagram for CB is best sketched in two parts—the area QST representing the bending moment of V, and the area TUW representing the bending moment of P. Since the continuity is broken at C, it cannot be assumed that the total area of the M-diagram is zero, although the angle ϕ between AD and BD is zero. The absurdity of such an assumption is obvious when $k = 1$, and the load P is at C, in which case it is clear that the M-diagram is a negative area throughout and cannot equal zero; neither can the tangential deviation at either A or B be equated to zero for a similar reason.

Let t_1 represent the tangential deviation for the beam AC and t_2 for the beam BC, then

$$t_1 = t_2$$

$$EIt_1 = (-Vl)\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = -\frac{Vl^3}{3}$$

$$EIt_2 = (Vl)\left(\frac{1}{2}l\right)\left(\frac{2}{3}l\right) - (Pkl)\left(\frac{1}{2}kl\right)\left(l - \frac{1}{3}kl\right)$$

Whence

$$V = \frac{P}{4}(3k^2 - k^3)$$

COLUMNS

By J. B. Kommers

A column or strut may be defined as a long compression member. A short block under axial compression has the unit stress uniformly distributed over the cross-section. This is not necessarily true for a column because a column tends to deflect laterally due to lack of straightness, non-homogeneity of the different parts of the cross-section, and eccentricity of the load. The lateral deflection of the column sets up bending stresses so that the column cannot carry as much load as it could if it were very short.

The term "slenderness ratio" is used in discussing columns, and is the ratio of the length of the column to the radius of gyration. The slenderness ratio of practical columns will usually lie between a value of 40 and a value of 125. Columns with a slenderness ratio greater than about 150 are spoken of as long columns.

77. Forms of Columns.—The top chords and end posts of bridges are commonly made up of channels and plates, or angles and plates. These standard shapes may be put together in a variety of ways to produce an efficient column. Web compression members are commonly made of latticed channels, latticed angles, or combinations of angles and plates. For very large columns the sections are made up of combinations of plates and angles, which are usually in the form of an H-section with cover plates on the two open sides.

For the sake of economy it will generally be desirable to have the value of the radius of gyration of the column section practically equal with respect to the two principal axes, for the reason that the lower radius of gyration is the one used to determine the column strength.

In order to obtain a column of great strength a compact box-like section is more desirable than one having unsupported outstanding legs. On the other hand, to obtain a large value of the radius of gyration for a column section the material should be placed as far away as is practical from the axis about which bending takes place.

78. Difference Between Column and Beam Theory.—As will be shown in the later discussion, it is possible to calculate the unit stress in a column under axial load only when it is an ideal long column, and even then the unit stress is known only when the column carries its maximum load. It will be shown that for all columns of practical lengths the unit stress existing cannot be calculated, but that the design depends upon the use of empirical formulas which have in them certain

constants determined from actual tests of columns. This fact leaves the discussion and design of columns in a much more unsatisfactory state than is the case for beams. For all simple cases of beam loading it is possible to calculate the unit stress in the beam with a very satisfactory degree of accuracy, and the formulas used have in them no empirical constants.

79. Euler's Formula for Long Columns.—The discussion of practical columns is made much clearer by discussing first an ideal long column which is supposed to be perfectly straight and homogeneous, and which carries an axial load. It has been found by actual tests that such a column, loaded within a certain critical load, may be deflected laterally, and will straighten again when the lateral push is removed. However, when the critical load has been reached, it is found that the column will remain deflected when the lateral push is removed. Further, it is found that under this critical load the column is in a state of indifferent equilibrium, such that the deflection under this load may be varied within quite wide limits. Euler's formula[1] determines this critical load.

[1] In Fig. 137*A* consider an ideal long column having round ends. The origin of coördinates is taken at the upper end, y being measured horizontally, and x vertically. In the discussion of the deflection of simple beams the formula $M = EI\frac{d^2y}{dx^2}$ is developed (see Art. **1**, Appendix C, p. 582). This formula may be applied to all cases of flexure within the elastic limit of the material. In the present case the moment at any distance x is $-Py$, the minus sign being used because it can be shown that the second derivative in this case is negative. Therefore

$$EI\frac{d^2y}{dx^2} = -Py$$

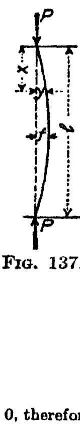


Fig. 137*A*.

Multiplying each side of the equation by dy, this may be integrated, and gives,

$$EI\left(\frac{dy}{dx}\right)^2 = -Py^2 + C$$

If the maximum deflection at the middle of the column is called f, it may be seen that when $y = f$, $\frac{dy}{dx} = 0$, and hence $C = Pf^2$.

Then

$$\frac{dy}{dx} = \left(\frac{EI}{P}\right)^{1/2}(f^2 - y^2)^{-1/2}$$

Integrating this gives

$$x = \left(\frac{EI}{P}\right)^{1/2}\sin^{-1}\frac{y}{f} + C_1$$

When $y = 0$, $x = 0$, and therefore $C_1 = 0$

Writing the equation in another form,

$$y = f\sin\left(\frac{P}{EI}\right)^{1/2}x \tag{1}$$

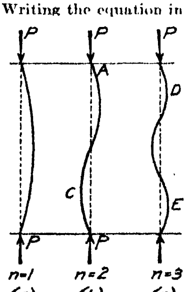


Fig. 137*B*.

This equation must satisfy the condition that when $x = l$, $y = 0$, therefore,

$$\left(\frac{P}{EI}\right)^{1/2}l = \pi\text{, or a multiple of }\pi$$

This is Euler's formula for long columns with round ends, and since $I = Ar^2$, it may take the form

$$\frac{P}{A} = \frac{n^2\pi^2E}{\left(\frac{l}{r}\right)^2} \tag{2}$$

Inserting the value $\frac{n\pi}{l}$ for $\left(\frac{P}{EI}\right)^{1/2}$ in eq. (1) ($n\pi$ representing any multiple of π) gives

$$y = f\sin\frac{n\pi}{l}x \tag{3}$$

If values of x in terms of l are now substituted in eq. (3), using $n = 1$, a curve is obtained as shown in Fig. 137*B* (*a*). The curves obtained for $n = 2$ and $n = 3$ are shown in Figs. 137*B*(*b*) and 137*B*(*c*). It is evident that Fig. 137*B*(*a*) shows the weakest case, and therefore columns with round ends are designed on the basis of eq. (2), using $n = 1$.

For long columns with round ends, Euler's formula for the unit load is

$$\frac{P}{A} = \frac{\pi^2 E}{\left(\frac{l}{r}\right)^2}$$

where P = axial load, A = area of cross-section, l = length of the column, r = least radius of gyration, $\frac{l}{r}$ = slenderness ratio and E = modulus of elasticity.

For long columns with fixed ends, the formula is

$$\frac{P}{A} = \frac{4n^2 E}{\left(\frac{l}{r}\right)^2}$$

It should be noted again that the value of P given by Euler's formula is the maximum load which the column can carry. It is the load under which the column will remain in a deflected position. It should also be noted that the unit stress which exists under this load is the elastic limit of the material, and that the formula does not enable the unit stress to be calculated. $\frac{P}{A}$ as obtained from the formula does not mean unit stress, but merely the value, which when multiplied by the area, will give the total load which the column can carry.

80. Ideal Column, Eccentrically Loaded.—In a manner similar to that used in deriving Euler's formula, it can be shown that an ideal column, carrying a load having an eccentricity, e, will have a maximum stress.

$$f = \frac{P}{A}\left(1 + \frac{ec}{r^2} \sec \frac{l}{2}\sqrt{\frac{P}{EI}}\right)$$

Here c is the distance to the extreme fiber from the neutral axis about which bending takes place, the other notation being the same as before.

This formula makes clear that there is a perfectly definite relation in this case between unit stress, f, and unit load P/A, which was not the case for the Euler formula. When the columns are short the term $\sec\frac{l}{r}\sqrt{\frac{P}{EI}}$ becomes practically equal to unity and the unit stress is then equal to $P/A\left(1 + \frac{ec}{r^2}\right)$ which is the ordinary formula for direct stress combined with bending.

81. Limitations of Euler's Formula.—Figure 138 shows the Euler formula for round ends plotted with unit load as ordinate and slenderness ratio as abscissa.

A column with one end round and one end fixed is approximately represented by the part of the column from A to C in Fig. 137$B(b)$. Substituting in eq. (2), $n = 2$, and replacing l by ⅔ l gives

$$\frac{P}{A} = 2\tfrac{1}{4}\cdot\frac{\pi^2 E}{\left(\frac{l}{r}\right)^2} \qquad (4)$$

which is the formula for the unit load for long columns fixed at one end and round at the other.

In the same way the column from C to D in Fig. 137$B(c)$ may represent a column fixed at both ends Substituting in eq. (2) gives

$$\frac{P}{A} = \frac{4\pi^2 E}{\left(\frac{l}{r}\right)^2} \qquad (5)$$

The case of a column with one end entirely free and the other end fixed is represented by the upper half of Fig. 137$B(a)$ and the formula for this case is

$$\frac{P}{A} = \frac{\pi^2 E}{4\left(\frac{l}{r}\right)^2} \qquad (6)$$

A value of 30,000,000 lb. per sq. in. was used for modulus of elasticity, and the curve therefore represents steel columns. If structural steel is being used an ultimate strength of about 60,000 lb. per sq. in. and a yield point of about 40,000 lb. per sq. in. may be expected. The curve shows that Euler's formula gives absurdly high values for unit load unless the column has a slenderness ratio of 175 or greater. It will be shown later that the results from Euler's formula should not be used unless they are lower than one-third of the ultimate strength.

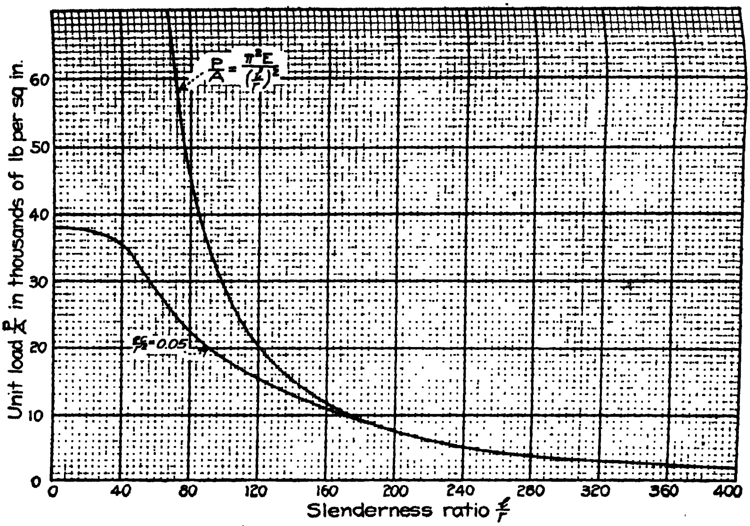


FIG. 138.

Figure 138 also shows the case of an eccentrically loaded steel column, the material being loaded up to its yield point of 40,000 lb. per sq. in. and the eccentricity being such that $\frac{ec}{r^2} = 0.05$. The curve makes very clear the effect of eccentricity of load in reducing column strength. Comparing this curve with the Euler curve it will be noted that the effect of eccentricity is very great for low values of $\frac{l}{r}$, but that for large values of $\frac{l}{r}$ the two curves practically coincide.

82. Columns of Practical Length.—Thus far only ideal columns have been discussed and it has been shown that Euler's formula cannot be used for columns of practical lengths. Furthermore, practical columns are used under quite different conditions than have been assumed for the ideal column. Practical columns are not perfectly straight, are not likely to have exactly axial loads, are not homogeneous, are likely to have initial stresses, and the various parts of built-up columns are not likely to act as a unit because of imperfect connections. These facts, together with the limitations of Euler's formula have lead to the development of a number of formulas which have in them constants obtained from tests on columns. Most of these formulas have a semi-theoretical basis.

83. The Rankine Formula.—The column formula developed by Rankine is based upon the assumption that columns fail by a combination of direct stress and bending. The direct stress is $f_1 = \frac{P}{A}$ and the stress due to bending is $f_2 = \frac{Mc}{I} = \frac{Pfc}{I}$, in which P is the load on the column and f is the maximum deflection of the column. The maximum stress is

$$f = \frac{P}{A} + \frac{Pfc}{I}$$

The assumption is now made that the deflection, f, is proportional to $\frac{l^2}{c}$, just as in the case for beams. Then,

$$f = \frac{P}{A} + k\frac{Pl^2}{I}$$

or, since $I = Ar^2$,

$$f = \frac{P}{A}\left[1 + k\left(\frac{l}{r}\right)^2\right]$$

or

$$\frac{P}{A} = \frac{f}{1 + k\left(\frac{l}{r}\right)^2}$$

In practice f as well as k are empirical constants which are given such values as to make the formula fit the results of column tests, f being dependent upon the kind of material, and k being dependent upon the kind of material and upon the end conditions of the column. The Cambria Steel Company's handbook uses the following constants for steel:

	Soft steel			Medium steel		
	Square ends	Pin and square ends	Pin ends	Square ends	Pin and square ends	Pin ends
f	45,000	45,000	45,000	50,000	50,000	50,000
k	$\frac{1}{36,000}$	$\frac{1}{24,000}$	$\frac{1}{18,000}$	$\frac{1}{36,000}$	$\frac{1}{24,000}$	$\frac{1}{18,000}$

Values of these constants for cast-iron columns are given in the chapter on Cast-iron Columns in Sec. 2.

84. The T. H. Johnson Straight-line Formula.—In 1886 T. H. Johnson[1] proposed a simple type of column formula which has been widely adopted. He found that a straight line tangent to Euler's curve could be made to fit quite well the results from column tests available at that time. The formula takes the form

$$\frac{P}{A} = f - k\frac{l}{r}$$

[1] *Trans. Am. Soc. C. E.*, vol. 15, 1886, p. 517.

In this formula, also, f and k are empirical constants which are chosen so as to make the formula fit the results of column tests.

The point of tangency of the straight line with the Euler curve is the limiting value of $\frac{l}{r}$ for which the straight line formula should be used. For larger values of $\frac{l}{r}$ than this limiting value the Euler formula should be employed. It can be shown that the point of tangency with Euler's curve occurs at a point at which the ordinate is $\frac{f}{3}$. It can also be shown that the constant f is dependent upon the kind of material and that the constant k is dependent upon the kind of material and upon the end conditions of the column.

The following table gives the values of f and k recommended by Johnson for structural steel, and also the limiting values of $\frac{l}{r}$.

STRUCTURAL STEEL

End conditions	f	k	Limit of l/r
Flat ends	52,500	179	195
Hinged ends	52,500	220	159
Round ends	52,500	284	123

Values of these constants for cast-iron and timber columns are given in Secs. 2 and 4.

85. The Parabolic Formula.—At a value of $\frac{l}{r}$ equal to zero the value of $\frac{P}{A}$ from the straight line formula is likely to be too high, and J. B. Johnson suggested that a parabola tangent to Euler's curve would fit better the results of column tests. The formula takes the form

$$\frac{P}{A} = f - k\left(\frac{l}{r}\right)^2$$

The following constants are recommended by Johnson for mild steel:

MILD STEEL

End conditions	f	k	Limit of l/r
Pin ends	42,000	0.97	150
Flat ends	42,000	0.62	190

86. Variation of Straight Line Formula.— A type of formula which has come into wide use is one which uses a horizontal line for low values of $\frac{l}{r}$ and an inclined straight line for the larger values. The American Railway Engineering Association formula for working loads is

$$\frac{P}{A} = 16{,}000 - 70\frac{l}{r}$$

This is limited to a value of $\frac{P}{A}$ which must not be higher than 14,000 lb. per sq. in., which means that for values of $\frac{l}{r}$ from zero to about 30 the curve used is a horizontal line.

The American Bridge Company uses a similar formula for the working loads for buildings.

$$\frac{P}{A} = 19{,}000 - 100\frac{l}{r}$$

This is limited to a maximum value of $\frac{P}{A}$ of 13,000 lb. per sq. in., and therefore for values of $\frac{l}{r}$ from zero to 60 the curve used is a horizontal line.

BENDING AND DIRECT STRESS

BY GEORGE A. HOOL AND W. S. KINNE

87. General Nature of the Problem.—In many cases, structural members which form a part of a composite structure are called upon to perform the combined duty of tension (or compression) members and beams. Such members are said to be subjected to bending and direct stress.

A few examples will now be given of what is meant by bending and direct stress. Figure 139*a* shows a simple beam carrying inclined loads. A section of

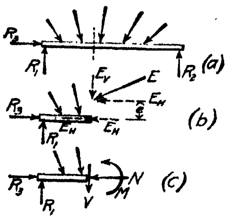


FIG. 139.

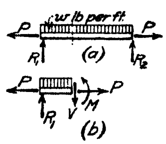


FIG. 140.

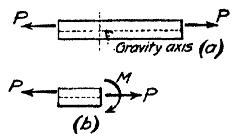


FIG. 141.

this beam at any point is shown in Fig. 139*b*. The force E represents the equilibrant of all forces to the left of the given section. This force has been resolved into its vertical and horizontal components, as shown in Fig. 139*b*. By the principles of statics it can be shown (see also Art. **88**) that the force E may be replaced by the system of forces shown at the cut section of Fig. 139*c*. At the section in question the beam is acted upon by a direct thrust T and a bending moment M. The transverse force V represents the shear on the section. Figure 139 represents a case typical of bending and direct stress conditions.

Figures 140 to 146, inclusive, show conditions resulting in structural members subjected to bending and direct stress. The conditions shown in Fig. 140 are typical of a truss member which also acts as a beam to carry either a roof or floor load. It may also represent the case of a tension or compression member subjected to bending due to its own weight. Figure 141 is typical of cases in which eccentric connections are used for tension or compression members. The

case of a post or column subjected to transverse loading is shown in Fig. 142. As shown in Fig. 142*b*, bending and direct stress occur at interior sections. Figure 143*a* shows a post or column which supports a load whose line of action does not coincide with the gravity axis of the member; Fig. 143*b* shows the conditions at any section of the member. Fig. 144*a* shows the top chord of a roof truss where conditions made it necessary to support the roof at points between joints of the truss. Fig. 144*b* shows the top chord member removed with the applied loads in position. This is a special case of Fig. 139. In Fig. 145*a*, the column supporting the end of a roof truss also supports an applied load W, due to a crane or

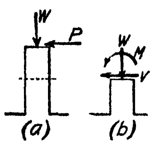


FIG. 142.

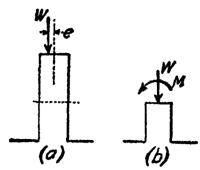


FIG. 143.

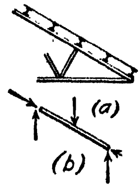


FIG. 144.

machinery load. The column is subjected to the forces shown in Fig. 145*b*. In Fig. 146, the column carries the load from a knee-brace, thus forming a member in bending and compression. Many other illustrations of bending and direct stress could also be given, but such cases would be combinations of those shown in Figs. 139 to 146, inclusive.

Problems in the determination of fiber stresses due to bending and direct stress may be divided into two main classes: one in which the deformation under the applied forces may be neglected; the other in which these deformations may *not* be neglected. Let Fig. 147*a* show a post carrying a load P at a distance e from the gravity axis of the post. For the conditions shown, the moment at

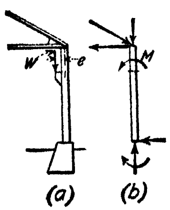


FIG. 145.

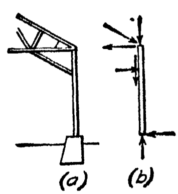


FIG. 146.

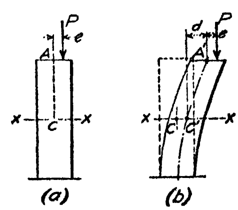


FIG. 147.

section x-x is $M = Pe$. This moment is calculated on the assumption that the post is not deformed by the action of the applied load. Now assume that the applied load does deform the post, and assume that the full lines of Fig. 147*b* show the deformed member. Point C, the center of gravity of the section, has moved to the position C', and the moment of the load P about C' is $M_o = P(d + e)$, where d is the horizontal distance between points A' and C'. Note that the moment for Fig. 147*b* is greater than that for Fig. 147*a*, and the difference in moment is due entirely to the deformation of the member. Similar conditions can be shown to exist in the cases shown in Figs. 139 to 146, inclusive.

Deformations of the nature described above take place in all elastic structures. However, there are certain classes of construction in which the members are so large that the deformation corresponding to d of Fig. 147b is so small that it may be neglected without appreciable error. Members composing a concrete or reinforced concrete structure are generally of this nature. In steel structures, and to a certain extent in timber structures, the members are generally comparatively slender, and the deformation corresponding to d of Fig. 147b is often so large that if it is neglected, the stresses determined with deformation disregarded would be considerably in error.

88. Determination of Total Fiber Stress, Deformation Neglected.

88a. Homogeneous Materials.—Let Fig. 148 represent any section of a beam and let E be the equilibrant of all forces to the left of the section (conditions similar to Fig. 139). Assume that E cuts the plane of the section at a point D which is located on one of the principal axes of the section. Resolve E into its vertical and horizontal components E_V and E_H, and assume these to be applied at point D, as shown in Fig. 148a. At C, the center of gravity of the section, apply two equal and opposite forces each equal to E_H. These additional forces will not disturb the equilibrium of the system. From statics, the equal and opposite parallel forces E_H (lettered 1 and 2 in Fig. 148) form a couple whose moment is

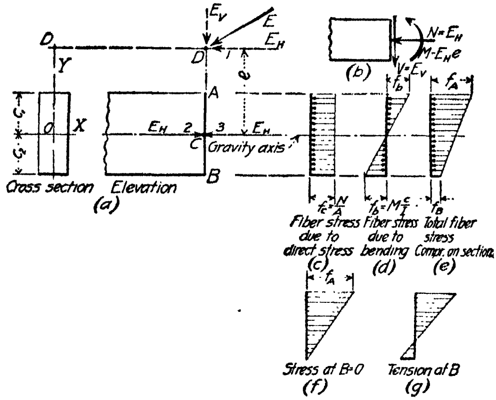


FIG. 148.

$$M = E_He \tag{1}$$

The remaining force E_H (lettered 3 in Fig. 148) acts as a thrust or compression on the section. If this thrust is represented by N, we have

$$N = E_H \tag{2}$$

The vertical component E_V which acts across the section, represents the *shear* on the section. Since it is generally assumed that the fiber stresses under consideration are not affected by shear, no further consideration will be paid to this force. Figure 148b shows the section with the several forces in position.

The fiber stress at points A and B of the beam section of Fig. 148 is generally calculated on the assumption that fiber stress due to the combined action of bending and direct stress is equal to the sum of the fiber stresses for bending and for direct stress considered as acting separately. The fiber stress due to the thrust N is uniform over the section and its value is

$$f_c = \frac{N}{A} \tag{3}$$

where f_c = stress intensity due to N, and A = area of section. Figure 148c

shows the variation in stress across the section. The stress due to bending is given by Art. **50b**, p. 22, as

$$f_b = \pm \frac{Mc}{I} \tag{4}$$

where f_b = extreme fiber stress due to bending; M = moment at section; c = distance from gravity axis to extreme fiber; and I = moment of inertia of section about gravity axis. For the conditions shown in Fig. 148 (positive moment), the stress on the top fiber A is compression and that on the bottom fiber B is tension. Figure 148*d* shows the variation in stress across the section.

As stated above, the total fiber stress is equal to the sum of the stresses due to bending and to direct stress. At the top fiber A, the total stress is

$$f_A = \left(\frac{N}{A} + \frac{Mc_1}{I}\right) \tag{5}$$

These stresses are to be added since both are compression for the conditions shown in Fig. 148. At the bottom fiber B, the total stress is

$$f_B = \left(\frac{N}{A} - \frac{Mc_2}{I}\right) \tag{6}$$

Since f_c is compression and f_b is tension, these stresses must be subtracted.

Equations (5) and (6) are sometimes reduced to a simpler form by the following substitutions. Letting f and c represent general values for f_A, f_B, c_1 and c_2 and using a plus or minus sign, these equations may be expressed by the single equation

$$f = \left(\frac{N}{A} \pm \frac{Mc}{I}\right)$$

From Fig. 148*a*, it can be seen that $M = E_Hc = Ne$. From Art. **18**, p. 2, $I = Ar^2$, where r = radius of gyration of the section. Placing these values in the above equation, we may write

$$f = \frac{N}{A}\left(1 \pm \frac{ec}{r^2}\right) \tag{7}$$

Equation (7) gives the same results as obtained by substitution in eqs. (5) and (6).

From eq. (6) it can be seen that the character of the fiber stress at B, Fig. 148*a*, depends upon the relative values of the terms $\frac{N}{A}$ and $\frac{Mc_2}{I}$. If $\frac{N}{A}$ is greater than $\frac{Mc_2}{I}$ the fiber stress is of the same character over the entire section (compression for the conditions shown in Fig. 148). Figure 148*e* shows the fiber stress variation for this case. When $\frac{N}{A} = \frac{Mc_2}{I}$ the fiber stress at B is zero. Figure 148*f* shows the fiber stress variation. If $\frac{Mc_2}{I}$ is greater than $\frac{N}{A}$, the fiber stress at B will be tension, and Fig. 148*g* shows the fiber stress variation across the section.

If in any case the equilibrant E of Fig. 148*a* acts in the opposite direction, the final fiber stresses given by eqs. (5) and (6) will have the same form. The character of stress, however, will be opposite to that indicated for Figs. 148*e*, *f*, and *g*. Equations (5) and (6) are general in nature and may be applied to any given case. The character of stress is best determined by inspection.

Special Values for Rectangular Sections.—For a rectangular section of width b and depth d, eqs. (5), (6), and (7) reduce to simple and useful forms. Since the gravity axis is at the center of the section, $c_1 = c_2 = \frac{d}{2}$. The moment of inertia of a rectangle of width b and depth d is $I = \frac{bd^3}{12}$. Also, the radius of gyration of this rectangle is given by the expression $r^2 = \frac{I}{A} = \frac{bd^3}{12} / bd = \frac{d^2}{12}$. Placing these values in eqs. (5) and (6), we have

$$\left.\begin{aligned} f_A &= \left(\frac{N}{A} + \frac{6M}{bd^2}\right) \\ f_B &= \left(\frac{N}{A} - \frac{6M}{bd^2}\right) \end{aligned}\right\} \tag{8}$$

and from eq. (7)

$$f = \frac{N}{A}\left(1 \pm \frac{6e}{d}\right) \tag{9}$$

For the special case shown in Fig. 148f, where the fiber stress at B is zero, the relation between e, the eccentricity of application of E_H, and d, the depth of section, may be determined from eq. (9) by placing $f = 0$ in this equation, and using a minus sign. Solving the resulting equation for e, we have

$$e = \tfrac{1}{6}d \tag{10}$$

That is, when the distance from the center of the rectangular section to the point of application of the force E_H of Fig. 148a is equal to one-sixth of the depth of the section, the fiber stress on the more remote extreme fiber is zero. Since a similar relation holds when the force E_H is applied at a point on the opposite side of the center, it can be seen from Figs. 148e, f, and g that when E_H is applied anywhere inside the *middle third* of the section, the fiber stress on the section is wholly compression or tension, depending upon the direction of E_H. This is the well-known middle third rule which is used in the design of masonry structures where no tension is allowed on the extreme fibers. From Fig. 147 and Fig. 148g, tension will exist on the extreme fiber when $\frac{6e}{d} > 1$, or when E_H is applied outside the middle third of the section.

A few problems will now be worked out by means of the equations given in this article. Since the effect of deformation is neglected in the derivation of the equations used in solving these problems, it will be interesting and instructive to solve the same problems by means of the more exact solutions given in later articles. The reader may then judge for himself as to the proper method of procedure in any given case.

Illustrative Problem.—A 2 × 12-in. steel eye bar, hinged at the ends and 30 ft. long, is subjected to a pull of 240,000 lb. Find the fiber stress at the center of the member due to direct stress and the bending due to the weight of the bar.

For the conditions stated, N = 240,000 lb.; A = 24 sq. in.; b = 2 in.; and d = 12 in. The weight of a 2 × 12-in. steel bar is 81.6 lb. per ft. Since the ends are hinged, the moment may be calculated as for simple beam conditions. Hence

$$M = \tfrac{1}{8}wl^2 = (\tfrac{1}{8})(81.6)(30)^2(12) = 110{,}160 \text{ in.-lb.}$$

From eq. (8)

$$f = \left(\frac{N}{A} \pm \frac{6M}{bd^2}\right) = \left[\frac{240{,}000}{24} \pm \frac{(6)(110{,}160)}{(2)(12)^2}\right]$$

Therefore

$$f_{\text{top fiber}} = 10{,}000 - 2{,}295 = 7{,}705 \text{ lb. per sq. in. (tensile)}$$
$$f_{\text{bottom fiber}} = 10{,}000 + 2{,}295 = 12{,}295 \text{ lb. per sq. in. (tensile)}$$

Illustrative Problem.—A portion of a top chord member of a roof truss is illustrated in Fig. 149. Find the extreme fiber stresses at the center and at the ends of the member for the loads indicated

For the conditions shown, $N = 30{,}000$ lb.; $A = (2)(2.09) = 4.18$ sq. in.; $c_{\text{top}} = 1.26$ in.; $c_{\text{bottom}} = 4.00 - 1.26 = 2.74$ in.; and $I = (2)(3.38) = 6.76$ in.^{3}

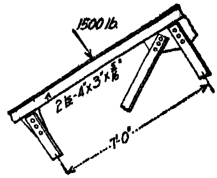


FIG. 149.

From the chapter on Restrained and Continuous Beams we note that, assuming fixed ends, the positive moment at the center is equal to $+\frac{1}{8}Wl$ and the negative moment at the ends is equal to $-\frac{1}{8}Wl$, where W is the centrally applied load in pounds and l is the span. Therefore $M = \frac{1}{8}Wl = (\frac{1}{8})(1{,}500)(7)(12) = 15{,}750$ in.-lb.

To find the extreme fiber stresses, eqs. (5) or (6) should be used.

At center of member:

$$f_{\text{top fiber}} = \frac{N}{A} + \frac{Mc}{I} = \frac{30{,}000}{4.18} + \frac{(15{,}750)(1.26)}{6.76} = 10{,}110 \text{ lb. per sq. in. (compressive)}$$

$$f_{\text{bottom fiber}} = \frac{N}{A} - \frac{Mc}{I} = \frac{30{,}000}{4.18} - \frac{(1\ ,750)(2.74)}{6.76} = 790 \text{ lb. per sq. in. (compressive)}$$

At end of member:

$$f_{\text{top fiber}} = \frac{N}{A} - \frac{Mc}{I} = \frac{30{,}000}{4.18} - \frac{(15{,}750)(1.26)}{6.76} = 5{,}240 \text{ lb. per sq. in. (compressive)}$$

$$f_{\text{bottom fiber}} = \frac{N}{A} + \frac{Mc}{I} = \frac{30{,}000}{4.18} + \frac{(15{,}750)(2.74)}{6.76} = 13{,}560 \text{ lb. per sq. in. (compressive)}$$

Illustrative Problem.—Figure 150 shows a building column which is subjected to bending stress under wind loads, due to the thrust of the knee brace. Find the extreme fiber stress.

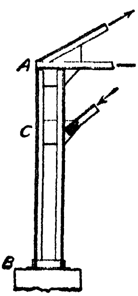


FIG. 150.

For the given conditions, $N = 60{,}000$ lb., $M = 1{,}200{,}000$ in.-lb., $A = 26.00$ sq. in., and with the angles placed $14\frac{1}{2}$ in. back to back, the moment of inertia of the column section, $I = 884.3$ in.^{4}

The extreme fiber stress may be found by using eqs. (5) or (6)

$$f = \frac{N}{A} \pm \frac{Mc}{I} = \frac{60{,}000}{26.00} + \frac{(1{,}200{,}000)(7.25)}{884.3} = 2{,}310 \pm 9{,}850$$

Therefore the extreme fiber stress on the side of the column adjacent to the knee brace is

$$f = 2{,}310 + 9{,}850 = 12{,}160 \text{ lb. per sq. in. (compressive)}$$

On the opposite side of the column, the extreme fiber stress

$$f = 2{,}310 - 9{,}850 = 7{,}540 \text{ lb. per sq. in. (tensile)}$$

88b. Non-homogeneous Materials.—Reinforced concrete members form typical examples of structural units composed of non-homogeneous materials. Methods for the analysis of such members subjected to bending and direct stress are given in the chapter on "Members Subject to Direct Compression and Bending" in Sec. 6.

89. Determination of Total Fiber Stress, Deformation Considered.—As stated in Art. 87, the moment at any point in a member subjected to bending and direct stress is influenced to some extent by the effect of the deflection of the member due to transverse loading. Two methods of analysis may be used in determining the total fiber stress in such members. One method, which is approximate in nature, assumes that the elastic curve of the deflected member is similar in form to the curve for a similar member under the action of transverse loads.

The moment due to deflection is estimated on this assumption and combined with the moment due to transverse loads. The other method, which is exact in nature, makes use of the differential equation of the elastic curve, derived in Appendix C. In the discussion which follows, special attention will be given to the approximate method of solution.

89a. Approximate Solution.—As stated above, the approximate solution is based on the assumption that the elastic curve for the member with the direct load removed is similar in form to the curve when the direct load is in place. It is also assumed that the deflection and moment under the combined loading are proportional to the deflection and moment for a similar member subjected only to transverse loading.

Figure 151*a* shows a beam freely supported or hinged at the ends and subjected to a transverse uniform load of w lb. per ft. and a direct axial tension of N lb. At point C, the center of this member, the combined moment is

$$M_c = \frac{wl^2}{8} - N\Delta = M_c' - N\Delta \qquad (a)$$

where M_c' represents the center moment in a simple beam under a uniform load, and M_c represents the center moment due to the combined action of direct and transverse loads.

It has been shown that for a simple beam uniformly loaded, the center moment is $M = \frac{1}{8}wl^2$ and the center deflection is $y_{max} = \frac{5}{384}\frac{wl^4}{EI}$. This center deflection may be expressed in terms of center moment as follows

$$y_{max} = \frac{5}{384}\frac{wl^4}{EI} = \frac{5}{48}\frac{l^2}{EI}(\tfrac{1}{8}wl^2) = \frac{5M_c'l^2}{48\ EI}$$

Assuming deflection and moment due to combined loading to be proportional to corresponding values for transverse loading only, we may write

$$M_c' : M_c :: y_{max} : \Delta$$

from which

$$\Delta = \frac{M_c}{M_c'}y_{max} = \frac{5}{48}\frac{M_cl^2}{EI}$$

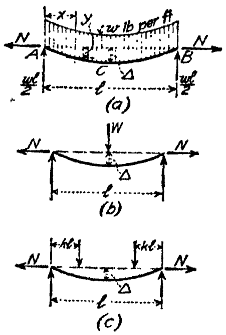


Fig. 151.

Substituting this value of Δ in eq. (*a*), replacing $\frac{5}{48}$ by the closely approximate value of $\frac{1}{10}$, and solving for M_c we derive,

$$M_c = \frac{M_c'}{1 + \dfrac{Nl^2}{10EI}} \qquad (11)$$

In Fig. 151*a*, N is shown as a tensile stress.

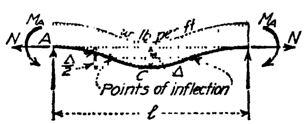


Fig. 152.

If N is compressive, a minus sign is to be used in the denominator of eq. (11).

When the ends of the member are restrained, as shown in Fig. 152, the effect of deflection on the moment may be estimated by assuming the points of inflection to be located as shown. Then

$$M_c = M_c' - N\frac{\Delta}{2}$$

and

$$M_A = M_A' - N\frac{\Delta}{2}$$

where M_c and M_A are the center and end moments due to combined loading and M_c' and M_A are corresponding values for a restrained beam when the direct stress N is removed. It has been shown that $M_c' = +\frac{1}{24}wl^2$, $M_A' = -\frac{1}{12}wl^2$ and y_c = deflection at $C = \frac{wl_4}{384EI}$. We may then write $\Delta = \frac{M_cl^2}{16EI} = \frac{1}{32}\frac{M_Al^2}{EI}$. Substituting this value of Δ in the above equations, we derive

$$M_c = \frac{M_c'}{1 + \frac{N}{32}\frac{l^2}{EI}} \tag{12}$$

and

$$M_A = \frac{M_A'}{1 + \frac{N}{64}\frac{l^2}{EI}} \tag{13}$$

A general equation applicable to the usual loadings and conditions of end supports may be written in the form

$$M = \frac{M'}{1 \pm C\frac{Nl^2}{EI}} \tag{14}$$

where M = moment due to combined loading, M' = moment in member when direct load N is removed, l = span length, EI = product of modulus of elasticity and moment of inertia of section, and C = a constant depending for its value upon the loading and end conditions for the member in question. When N is tension, use a + sign in the denominator of eq. (14) and when N is compression, use a minus sign.

Values of the constant C are as follows:

Hinged ends, uniform load (Fig. 151a) $C = \frac{1}{10}$

Hinged ends, single concentrated load (Fig. 151b) $C = \frac{1}{12}$

Hinged ends, two concentrated loads (Fig. 151c) $C = \frac{1}{24}(3 - 4k)$

Both ends fixed, uniform load (Fig. 152)

$C = \frac{1}{32}$ for center moment

$C = \frac{1}{64}$ for end moment

Both ends fixed, single concentrated load

$C = \frac{1}{48}$ for center and end moments

One end fixed, other end free, uniform load

$C = \frac{1}{17}$ for center moment

$C = \frac{1}{51}$ for end moment

One end fixed, other end free, single concentrated load

$C = \frac{1}{26}$ for center moment

$C = \frac{1}{42}$ for end moment

Since absolute fixity of the ends of a restrained beam is seldom realized in practice, most designers use a value $C = \frac{1}{10}$ for all cases.

The fiber stress due to bending and direct stress may be determined from eqs. (5), (6), or (7) by substituting in these equations the value of M determined from the above analysis.

Illustrative Problem.—A 2 × 12-in. steel eye bar, hinged at the ends, and 30 ft. long, is subjected to a pull of 240,000 lb. Find the fiber stresses at the center of the bar due to the combined effect of direct stress and weight of the member.

Use eq. (14) with a + sign in the denominator and $C = \frac{1}{10}$. For the given conditions, $N = 240{,}000$, $l = 30$ ft. $= 360$ in., $E = 30{,}000{,}000$ and $I = \frac{bd^3}{12} = \frac{(2)(12)^3}{(12)} = 288$.

Hence

$$\frac{1}{10}\frac{Nl^2}{EI} = \frac{(240{,}000)(360)^2}{(10)(30{,}000{,}000)(288)} = 0.360$$

A 2 × 12-in. steel bar weighs 81.6 lb. per ft. Hence $M' = \frac{1}{8}wl^2 = (\frac{1}{8})(81.6)(30)^2(12) =$ 110,160 in.-lb. From eq. (14)

$$M = \frac{M'}{1 + \frac{1}{10}\frac{Nl^2}{EI}} = \frac{110{,}160}{1 + 0.360} = 81{,}200 \text{ in.-lb.}$$

Then from eq. (8),

$$f = \left(\frac{N}{A} \pm \frac{6M}{bd^2}\right) = \frac{240{,}000}{24} \pm \frac{(6)(81{,}200)}{(2)(12)^2}$$

and

$$f_{\text{top fiber}} = 10{,}000 - 1{,}690 = 8{,}310 \text{ lb. per sq. in.}$$

$$f_{\text{bottom fiber}} = 10{,}000 + 1{,}690 = 11{,}690 \text{ lb. per sq. in.}$$

These are tensile stresses. On comparing these values with those given on p. 140, it can be seen the effect of the deflection is such as to cause a considerable decrease in the extreme fiber stresses. We therefore conclude that the effect of deflection should be considered when calculating combined stresses in long tension members.

Illustrative Problem.—Solve the Problem of Fig. 149 on p. 141, using eq. (14). Assume fixed ends.

For the assumed end and loading conditions, $C = \frac{1}{48}$ in eq. (14). With $N = 30{,}000$ $l = 7$ ft. $= 84$ in., $E = 30{,}000{,}000$ and $I = 6.76$ in.^{3}, we have

$$\frac{1}{48}\cdot\frac{Nl^2}{EI} = \frac{(30{,}000)(84)^2}{(48)(30{,}000{,}000)(6.76)} = 0.0217$$

For a central load of 1,500 lb.

$$M = \frac{1}{8}Wl = (\tfrac{1}{8})(1{,}500)(7)(12) = 15{,}750 \text{ in.-lb.}$$

Then from eq. (14)

$$M = \frac{15{,}750}{1 - 0.0217} = 16{,}100 \text{ in.-lb.}$$

From eq. (7), using values given on p. 141 and $M = 16{,}100$ in.-lb., the combined fiber stresses are found to be as follows:

At center of member:

$$f_{\text{top fiber}} = \frac{N}{A} + \frac{Mc}{I} = \frac{30{,}000}{4.18} + \frac{(16{,}100)(1.26)}{(6.76)} = 10{,}190 \text{ lb. per sq. in.}$$

$$f_{\text{bottom fiber}} = \frac{N}{A} - \frac{Mc}{I} = \frac{30{,}000}{4.18} - \frac{(16{,}100)(2.74)}{(6.76)} \quad 640 \text{ lb. per sq. in.}$$

At end of member:

$$f_{\text{top fiber}} = \frac{N}{A} - \frac{Mc}{I} = \frac{30{,}000}{4.18} - \frac{(16{,}100)(1.26)}{(6.76)} = 4{,}170 \text{ lb. per sq. in.}$$

$$f_{\text{bottom fiber}} = \frac{N}{A} + \frac{Mc}{I} = \frac{30{,}000}{4.18} + \frac{(16{,}100)(2.74)}{(6.76)} \quad 13{,}720 \text{ lb. per sq. in.}$$

All fiber stresses are compressive.

On comparing these values with those given on p. 141, it can be seen that they are practically identical. We therefore conclude that for compression members which are reasonably rigid, it is not necessary to take into account the effect of deflection in calculating combined fiber stresses.

89b. Exact Solution.—As previously stated, an exact solution for moment and fiber stress due to combined bending and direct stress may be made by means of the general differential equation of the elastic curve given in Art. 1, Appendix *C*. To illustrate the general methods, an exact solution will be made for a beam hinged at the ends and acted upon by a uniform load of w lb. per ft. and a direct tension N, as shown in Fig. 151*a*.

At any section distance x from the left end of this beam, the moment for the conditions shown in Fig. 151*a* is

$$M_x = \frac{wl}{2}x - \frac{wx^2}{2} - Ny \tag{15}$$

Placing this moment in the general equation for the elastic curve, we have

$$EI\frac{d^2y}{dx^2} = -M_x = Ny - \frac{wlx}{2} + \frac{wx^2}{2}$$

Let $\frac{N}{EI} = c^2$. Substituting this term in the above equation and integrating, we have

$$y = C_1e^{cx} + C_2e^{-cx} + \frac{wlx}{2N} - \frac{wx^2}{2N} - \frac{w}{c^2N} \tag{16}$$

To determine the constants of integration note from Fig. 151*a* that $y = 0$ when $x = 0$, and that $\frac{dy}{dx} = 0$ when $x = \frac{l}{2}$. The constants of integration are found to be

$$C_1 = \frac{w}{c^2N}\left(\frac{1}{1 + e^{cl}}\right)$$

and

$$C_2 = C_1e^{cl}$$

It is evident from Fig. 151*a* that the maximum combined moment occurs at the center of the beam. Let M_c represent this maximum moment. To determine M_c substitute $x = \frac{l}{2}$ in eq. (15). The value of y in eq. (15) is obtained from eq. (16) by substituting $x = \frac{l}{2}$ and values of C_1 and C_2 as given above. Performing the operations indicated we have finally

$$M_c = \frac{w}{c^2}\left[1 - \sec\frac{cl}{2}\right] = \frac{wl^2}{8}\left[\frac{8}{c^2l^2}\left(1 - \sec\frac{cl}{2}\right)\right] \tag{17}$$

As in the preceding article, the combined fiber stress may be determined from eqs. (5), (6), or (7) by substituting the value of M given by eq. (17).

Illustrative Problem.—Solve the problem on p. 140, using the exact method as given by eq. (17).

For the given conditions

$$c^2l^2 = \frac{Nl^2}{EI} = \frac{(240{,}000)(360)^2}{(30{,}000{,}000)(288)} = 3.60$$

and

$$\frac{cl}{2} = \frac{1}{2}\sqrt{3.60} = 0.948$$

From a table of hyperbolic functions,

$$\text{Sech}\frac{cl}{2} = \text{se}\ \ 0.948 = 0.673$$

From p. 140,

$$\tfrac{1}{8} wl^2 = 110{,}160 \text{ in.-lb.}$$

Then

$$M_o = \frac{wl^2}{8}\left[\frac{8}{c^2l^2}\left(1 - \operatorname{sech}\frac{cl}{2}\right)\right] = (110{,}160)\left(\frac{8}{3.6}\right)(1 - 0.673)$$
$$= 80{,}050 \text{ in.-lb.}$$

The exact value of the moment here calculated differs from the approximate value given on p. 144 by about 1.4 per cent. This difference between approximate and exact values is so small that we conclude that the approximate methods of Art. 89*a* are accurate enough for all cases encountered in practice. Similar conclusions hold also for fixed end members whether the direct stress is tension or compression.

General formulas for moment in members with free and restrained ends subjected to a uniform load and with a direct stress N taken as tension or compression are given below. In these formulas w = uniform load per unit of length, l = length of member, and $c = \frac{N}{EI}$, where N = direct stress and EI = product of modulus of elasticity and moment of inertia of section.

N is Tension

Beam hinged at ends:

$$\text{Center moment} = \frac{w}{c^2}\left(1 - \operatorname{sech}\frac{cl}{2}\right)$$

Beam fixed at ends:

$$\text{Center moment} = \frac{w}{c^2}\left(1 - \frac{cl}{2}\operatorname{cosech}\frac{cl}{2}\right)$$

$$\text{End moment} = \frac{w}{c^2}\left(\frac{cl}{2}\coth\frac{cl}{2} - 1\right)$$

N is Compression

Beam hinged at ends:

$$\text{Center moment} = \frac{w}{c^2}\left(\sec\frac{cl}{2} - 1\right)$$

Beam fixed at ends:

$$\text{Center moment} = \frac{w}{c^2}\left(\frac{cl}{2}\operatorname{cosec}\frac{cl}{2} - 1\right)$$

$$\text{End moment} = \frac{w}{c^2}\left(1 - \frac{cl}{2}\cot\frac{cl}{2}\right)$$

90. Members Subjected to Unsymmetrical Bending.—In the preceding analysis it has been assumed that the plane of bending coincides with the plane of a principal axis of the section. When the plane of bending does not coincide with a principal axis of the section, the fiber stress due to bending must be determined by the methods of unsymmetrical bending given in the chapter which follows. This fiber stress may then be combined with the fiber stress due to direct loading. The general formula for fiber stress may then be written

$$f = \frac{N}{A} \pm M\left(\frac{I_y y_A \sin\theta + I_x x_A \cos\theta}{I_x I_y}\right) \tag{18}$$

In this equation, the notation is the same as given on p. 151.

Illustrative Problem.—A $5 \times 3\tfrac{1}{2} \times \tfrac{1}{2}$-in. angle is subjected to a compressive stress of 15,000 lb., applied through a gusset plate connection, as illustrated in Fig. 153*a*. Find the maximum fiber stress.

The S-polygon shown in Fig. 153*b* is constructed and the neutral axis located by methods similar to those given in the following chapter on "Unsymmetrical Bending." OX and OY are the principal inertia axes of the section. By measurement it is found that point A is

farthest from the neutral axis, hence this will be the point of maximum fiber stress. The coördinates of point A with respect to the principal axes of inertia are $x_A = +1.63$, and $y = +2.61$. $I_x = 2.25$ in.^{4}, $I_y = 11.79$ in.^{4}, $A = 4.00$ sq. in., and $e = 2.21$ in. Substituting in eq. (11)

$$f = \frac{N}{A} + M\left(\frac{I_y y_A \sin\theta + I_x x_A \cos\theta}{I_x I_y}\right)$$

$$= \frac{N}{A} + N\left[\frac{(2.25)(2.61)(0.997) + (11.79)(1.63)(0.081)}{(2.25)(11.79)}\right]$$

$$= \frac{15{,}000}{4.00} + \frac{(15{,}000)(2.21)(7.40)}{26.53} = 3{,}750 + 9{,}250 = 13{,}000 \text{ lb. per sq. in. (compressive)}$$

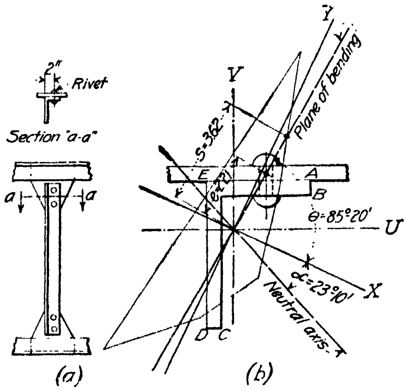


FIG. 153.

The fiber stress due to bending may also be determined from the S-polygon of Fig. 153b instead of by eq. (18).

From Fig. 153b, $S_A = 3.62$.

Hence,

$$f_A = \frac{M}{S_A} = \frac{(15{,}000)(2.21)}{3.62} = 9{,}170$$

This is within 1 per cent of the value given in the second part of the equation above.

91. The Kern of a Section.—In Art. **88** it was shown that for a rectangular section, the fiber stress over the entire section was of the same character, compression or tension, when the resultant of external forces was applied at a point on the principal axis which is inside the middle third of the section. The effect of bending in planes other than those of the principal axes will now be investigated by the methods used in the following chapter on "Unsymmetrical Bending."

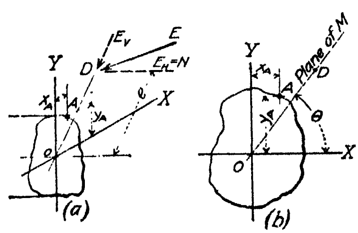


FIG. 154.

Let Fig. 154a show a section of a beam acted upon by a force E which is the equilibrant of all forces to the left of the given section. As shown in Fig. 154a, the plane containing the force E cuts the plane of a right section of the beam in the line OD, which passes through the center of gravity of the section. Figure 154b shows a projection of the plane of the

right section of the beam. As in Art. 88, the force E may be resolved into a thrust T acting at O perpendicular to the section and a moment M acting in the plane of OD at an angle θ with the principal axis OX.

The effect of the thrust N is a uniform fiber stress over the entire section equal to

$$f_c = \frac{N}{A}$$

and the effect of the unsymmetrical bending on any fiber, as A, Fig. 154b as given by eq. (2), p. 151, is

$$f_b = M\left(\frac{I_y y_A \sin\theta + I_x x_A \cos\theta}{I_x I_y}\right)$$

where I_x and I_y are respectively the moments of inertia of the section with respect to the OX and OY axes. Values of other terms are as indicated in Fig. 154.

The total stress on fiber A of Fig. 154 is then

$$f_A = f_c + f_b = \frac{N}{A} + M\left(\frac{I_y y_A \sin\theta + I_x x_A \cos\theta}{I_x I_y}\right)$$

From Fig. 154, $M = E_H e = Ne$; and also $I_y = Ar_y^2$ and $I_x = Ar_x^2$, where A is the area of the section and r_y and r_x are the radii of gyration of the section with respect to the Y and X axes respectively. Substituting these values in the above equation, we may write

$$f_A = \frac{N}{A}\left[1 + \frac{e(r_y^2 y_A \sin\theta + r_x^2 x_A \cos\theta)}{r_x^2 r_y^2}\right] \tag{19}$$

Placing $f_A = 0$ in eq. (19) and solving for e, we have

$$e = -\frac{r_x^2 r_y^2}{r_y y_A \sin\theta + r_x^2 x_A \cos\theta} \tag{20}$$

Equation (20) gives the locus of all points at which the resultant thrust $E_H = N$ of Fig. 154 must be applied in order to produce zero stress on fiber A.

On comparing the value of e given by eq. (20) and the value of S given by eq. (5), p. 152, it can be seen that the two equations are of the same form, but that eq. (20) has a minus sign while eq. (5) has a plus sign, and also that eq. (20) has terms containing the radius of gyration of the section where eq. (5) has terms containing the moment of inertia of the section. Hence, if we divide eq. (5) by $-A$, we obtain eq. (20).

Since eqs. (20) and (5) are identical in form, it is evident that, with certain modifications, the discussion of the chapter which follows on S-lines and S-polygons may also be applied to eq. (20).

Thus, by a line of reasoning similar to that given in the next chapter, it can be shown that for each extreme point of any section, eq. (20) represents a straight line. Any resultant force $N = E_H$ applied to the section between its center of gravity, O, Fig. 154, and the line represented by eq. (20) will cause a fiber stress over the entire section which is of the same character as fiber stress on the extreme point in question. If this is repeated for each extreme point of the section, a set of lines may be plotted which correspond to the S-lines of the chapter on "Unsymmetrical Bending."

If the several lines plotted from eq. (20) are produced so that lines from adjacent extreme points intersect, a closed area will be formed which is known as the

kern of the section. A resultant thrust T applied at any point in the kern of a section will cause fiber stresses of the same character at all points of the section. The kern of the section corresponds to the S-polygon of the next chapter.

The coordinates of the points of intersection of the lines forming the kern of a section may be determined by the methods given in the chapter on "Unsymmetrical Bending" for corresponding points on the S-polygon. However, since eq. (20) may be obtained from eq. (5), p. 152, by dividing the latter by $-A$, where A is the area of cross-section, it is evident that the coördinates of the apices of the kern of the section may be obtained from the coördinates of the apices of the S-polygon for the same section by dividing these values by $-A$.

The kern of a section forms a convenient graphical method of determining where loads may be placed on a given section without causing changes in the character of the fiber stress on the extreme points of the section. It can readily be seen that the middle third rule given in Art. **88***a* is a special case of the analysis of this article.

Figure 155 shows the kern of the section for a few standard sections.

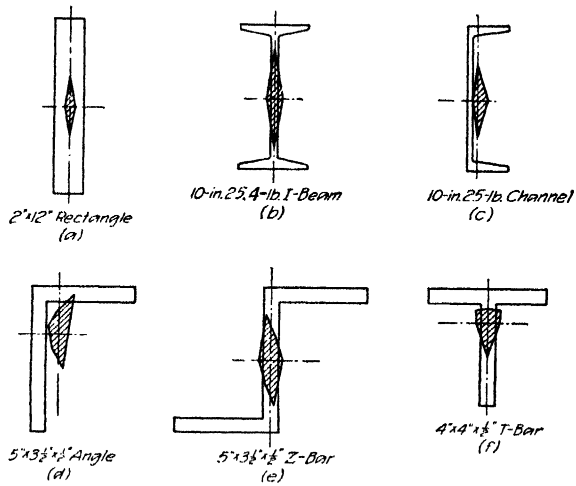


FIG. 155.

UNSYMMETRICAL BENDING

BY W. S. KINNE

In certain types of construction it is found necessary to place beam sections with their axes of symmetry at an angle to the plane of loading, as shown in Fig. 156. For the conditions shown, the principal axes of the section and the plane of loading do not coincide, as assumed in the cases considered in the preceding chapters. Bending of the nature shown in Fig. 156 is known as *unsymmetrical bending*. The brief treatment of the subject given in this chapter is confined to cases of pure bending only.

92. General Formulas for Fiber Stress and Position of Neutral Axis for Unsymmetrical Bending.—The full line rectangle of Fig. 157 shows a right section

of a straight beam of uniform cross-section subjected to a bending moment M acting in a plane which passes through the longitudinal axis of the beam, making an angle θ with OX, one of the principal axes of the section. In the work to follow, point O will be taken as the origin of coördinates, and the principal axes of the section, OX and OY of Fig. 157, will be taken as the coördinate axes. As the formulas are greatly simplified thereby, the properties of the section will be referred to the principal axes. These quantities are given directly or are easily calculated from data given in any of the structural steel handbooks.

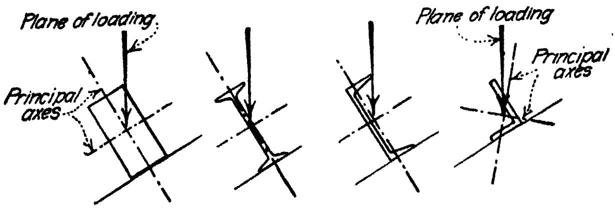


FIG. 156.

Let n-n of Fig. 157a represent the position of the neutral axis of the assumed section for the given plane of loading, and let α be the angle which the neutral axis makes with OX. Angle α and also angle θ are to be considered as positive when measured in a counter-clockwise direction. Figure 157b shows the fiber stress conditions on a line at right angles to the neutral axis, assuming linear distribution of stress

Let P, Fig. 157a, be any fiber of infinitely small area a at a distance v from the neutral axis. Assuming positive (clockwise) moment, the intensity of fiber stress at P is $f = -f_1v$, where f_1 is the fiber stress intensity at unit distance from the neutral axis. The minus sign indicates compression, for, as shown in Fig. 157, the fiber under consideration is above the neutral axis.

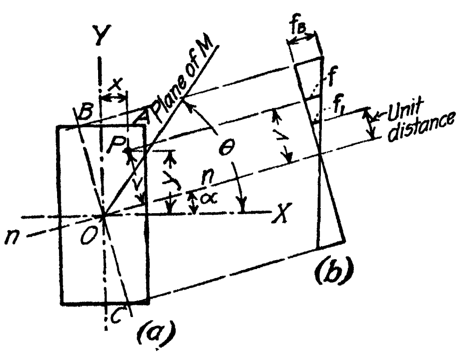


FIG. 157.

The moment of resistance of the section, which is equal to the stress on each fiber multiplied by its distance from the neutral axis is $M_R = \Sigma f_1 av^2$, where Σ represents the summation for the entire rectangle. But Σav^2 is the moment of inertia of the section about the neutral axis (see Art. 15), which will be denoted by I_n. With this notation, $M_R = f_1I_n$. Substituting for f_1 its value $-\frac{f}{v}$, we have

$$M_R = -\frac{f}{v}I_n$$

Since the beam is in equilibrium, the moments of internal and external forces at any section must be equal. Taking the neutral axis as the axis of moments, the external moment in a plane perpendicular to the neutral axis is $M \sin (\theta - \alpha)$. The moment of internal forces is the resisting moment of the section, which is given above as $M_R = -\frac{f}{v} I_n$. Equating these two expressions

$$f = -M \frac{v \sin (\theta - \alpha)}{I_n}$$

This expression can be placed in a more convenient form by referring both v and I_n to the principal axes of the section. From Fig. 157*a*, $v = y \cos \alpha - x \sin \alpha$. Values of x and y are positive when measured upward and to the right. In treatises on Mechanics it is shown that in terms of the principal moments of inertia of the section, I_x and I_y, the moment of inertia about the neutral axis is $I_n = I_x \cos^2 \alpha + I_y \sin^2 \alpha$. Substituting these values in the general equation given above

$$f = -M \frac{(y \cos \alpha - x \sin \alpha) \sin (\theta - \alpha)}{(I_x \cos^2 \alpha + I_y \sin^2 \alpha)}$$

To determine the relation between the angles α and θ, a summation of external moments about any two axes will yield two independent equations from which the desired relation can be obtained. Two convenient axes are OX and OY, the principal axes of the section.

For axis OX, using the value of v given above,

$$M \sin \theta = \Sigma f_1 avy = \Sigma f_1 (y^2 \cos \alpha - xy \sin \alpha)\, a$$

But Σay^2 is the moment of inertia of the section about the axis OX, which is denoted by I_x, and Σaxy is the product of inertia of the section, which is zero for principal axes. Then,

$$M \sin \theta = f_1 I_x \cos \alpha$$

In the same way, for axis OY,

$$M \cos \theta = -f_1 I_y \sin \alpha$$

Solving these equations for α, we have

$$\tan \alpha = -\frac{I_x}{I_y} \cot \theta \qquad (1)$$

which is the general equation for direction of the neutral axis for bending in any given direction.

Substituting the value of α, as given by eq. (1), in the above expression for f, we have

$$f = -M \left(\frac{I_y y \sin \theta + I_x x \cos \theta}{I_x I_y} \right)$$

which is the general expression for fiber stress at any point in a section of a beam due to a moment M acting in a plane at an angle θ to the axis OX. This equation can be made to apply to any particular point, as A, Fig. 157*a*, an extreme point of the section, by substituting for x and y the coördinates of the point in question. Let these coördinates be x_A and y_A, and let f_A be the resulting fiber stress. Then

$$f_A = -M \left(\frac{I_y y_A \sin \theta + I_x x_A \cos \theta}{I_x I_y} \right) \qquad (2)$$

Since in eqs. (1) and (2), x_A, y_A, I_x, and I_y are constants for any given point in a given section, it follows that the direction of the neutral axis and the intensity of the stress are dependent upon the value of θ. For θ = 90 deg., eq. (2) becomes $f_A = My_A/I_x$, and eq. (1) becomes, $\tan \alpha = 0$, or, $\alpha = 0$ deg. Again, for θ = 0 deg., eq. (2) becomes, $f_A = -Mx_A/I_y$, and eq. (1) becomes, $\tan \alpha$ = infinite, or, α = 90 deg.

It will be noted that these special values of fiber stress are of the form given in Art. **50***b*, p. 23, that is, $f = M\ (c/I)$, where I/c is known as the *section modulus* of the section. Also, the neutral axis in each case is perpendicular to the plane of loading. This condition holds true only when the plane of loading coincides with one of the principal axes of the section, at which time the other principal axis is the neutral axis, a fact which can be verified by a study of the values of α given above.

Equation (2) can also be written in the form

$$f_A = -\left[(M \sin \theta)\frac{y_A}{I_x} + (M \cos \theta)\frac{x_A}{I_y}\right] \tag{3}$$

As shown by the substitutions made above, this expression is the sum of two quantities obtained by resolving the bending moment into its components parallel to the principal axes of the section. Then by adding the fiber stresses due to these component moments, there is obtained an expression identical to eq. (3), and on transformation, to eq. (2). This offers a simple and easily remembered method for the calculation of fiber stresses due to unsymmetrical bending.

93. Flexural Modulus.—In Art. **50***b*, p. 23, it is shown that for bending in the plane of a principal axis, the fiber stress in a beam is given by an expression of the form

$$f = M(c/I) = \frac{M}{I/c}$$

where for any given section I/c is a constant quantity known as the *section modulus*.

In eq. (2), the reciprocal of the expression in parenthesis is seen to be a quantity of the same dimensions as the section modulus, but more general in nature, as it involves planes of loading other than the principal axes. Let S denote this quantity. Then

$$f = M/S \tag{4}$$

where

$$S = \frac{I_xI_y}{I_yy_A \sin \theta + I_xx_A \cos \theta} \tag{5}$$

The expression of eq. (5) is known as the *flexural modulus* of the section. For any given direction of loading and for any given point in a section, S is a constant. Having given the value of S for any given conditions, the resulting fiber stress is obtained by substitution in eq. (4).

94. The S-line.—For any point in a given section, the values of S as given by eq. (5), gives a measure of the strength of the section for bending in any direction.

From analytical geometry it can be shown that eq. (5) is in the form of the polar equation of a straight line. A convenient graphical representation of the

variation in flexural modulus for various planes of bending is thus readily obtained. In Fig. 158, the line C-D shows the variation in flexural modulus for point A, one of the corners of a rectangular section. This is known as an S-line of the section. The vector OE shows the value of S_A for bending moment at an angle θ to OX, one of the principal axes of the section.

It will be found convenient to express the equation of the S-line in terms of rectangular coördinates. If $y = S \sin \theta$ and $x = S \cos \theta$ be placed in eq. (5), we have

$$y = -\frac{I_x x_A}{I_y y_A} \cdot x + \frac{I_x}{y_A} \qquad (6)$$

which is the slope form of the equation of the S-line for point A, Fig. 158.

95. S-polygons.—Every extreme point or corner of a section is liable to become, at some time, a point of maximum stress. In order to determine graphically which of several extreme points is the one having maximum stress, it is necessary to plot the S-lines for all such points. In this way the values of S for the several points can be compared.

In Fig. 158, the line F-G represents the S-line for point B. The equation for this line is similar to that for point A, and can be obtained from eq. (6) by substituting x_B and y_B, the coördinates of B, in place of the corresponding values for A. Thus the required equation is

$$y = -\frac{I_x x_B}{I_y y_B} x + \frac{I_x}{y_B} \qquad (7)$$

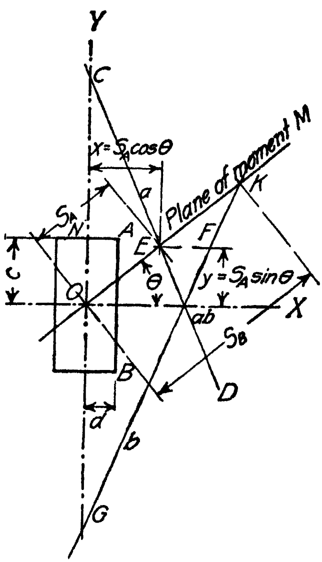


Fig. 158.

As before, the vector OK represents the value of S_B for bending at an angle θ to OX. Eq. (4) shows that the point of greatest stress is the one with the least S. Since vector OE is smaller than OK, fiber A has a greater stress than fiber B for the given plane of bending.

Equations similar to eqs. (6) and (7) can be made up for each extreme point of the section. If all these S-lines are plotted in Fig. 158, they will enclose a figure known as an S-polygon. Examples of S-polygons are given in Art. **96.**

S-polygons can be constructed by two different methods. One method of construction is carried out by plotting the S-lines, as given by equations similar to eqs. (6) and (7). The S-lines for adjacent points of the section are run to an intersection, and the resulting enclosed figure will form the desired S-polygon. Another and better method locates the coördinates of the points of intersection of adjacent S-lines by the methods of Analytical Geometry. This is done by solving simultaneously equations such as eqs. (6) and (7) for adjacent extreme points of the section. This process is repeated for each pair of adjacent points of the section. The resulting coördinates are plotted and connected up to form the complete S-polygon. This latter method, which is the one used in the work to follow, will now be explained in detail.

To determine the coördinates of the intersection of the S-lines for points A and B of Fig. 158, the equations for these lines, as given by eqs. (6) and (7), are to be solved simultaneously. Let x_{ab} and y_{ab} be the coördinates of the point of intersection—that is, the values of x and y common to the two equations. Then

$$x_{ab} = \frac{I_y(y_B - y_A)}{x_A y_B - x_B y_A} \tag{8}$$

$$y_{ab} = \frac{I_x(x_A - x_B)}{x_A y_B - x_B y_A} \tag{9}$$

Similar values for pairs of adjacent extreme points will differ only in the subscripts of x and y. The resulting values, when plotted and connected up, will form the desired S-polygon.

Equations (8) and (9) give general values for the coördinates of points of intersection of S-lines. Under certain conditions these equations take on a much simpler form. As shown in Fig. 158, extreme points A and B form an edge which is parallel to the axis OY, and $x_A = x_B = d$. If these values be placed in eqs. (8) and (9), the resulting equations are

$$x_{ab} = I_y/d \tag{10}$$

and

$$y_{ab} = 0 \tag{11}$$

For two adjacent points, as A and N of Fig. 158, which form a side parallel to the OX axis, $y_A = y_N = c$, and eqs. (8) and (9) become

$$x_{an} = 0 \tag{12}$$

and

$$y_{an} = I_x/c \tag{13}$$

In cases where S-polygons are to be determined for sections which are irregular in outline, as shown in Fig. 159, where some of the sides of the section are not parallel to the principal axes, OX and OY, eqs. (8) and (9) must be used in the determination of the coördinates of the S-polygon. It is possible, however, to make use of certain short cuts which will greatly simplify the calculations. This is done by revolving the axes of reference for coördinates of extreme points through such an angle that the side in question and the axes of reference will be parallel.

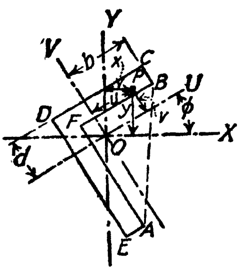


FIG. 159.

Suppose that the coördinates of the intersection points of the S-lines for adjacent points B and C of Fig. 159 are required. Choose a set of coördinate axes OU and OV, such that OV is parallel to the side C-B. Let ϕ be the angle which OU makes with OX, a principal axis of the section. This angle is to be considered as positive when measured counterclockwise. If x and y be the coördinates of any point P with respect to the OX and OY axes, and u and v be the coördinates of the same point with respect to the OU and OV axes, it can be shown from Fig. 159 that

$$y = v \cos \phi + u \sin \phi$$

and

$$x = u \cos \phi - v \sin \phi$$

In these equations u and v are considered positive when measured upward and to the right with respect to the axes OU and OV.

Substituting in eqs. (8) and (9) values of x and y as given by the above equations, using subscripts to correspond to the point in question, we have

$$x_{bc} = \frac{I_y[(u_B - u_C)\sin\phi + (v_B - v_C)\cos\phi]}{(u_C v_B - u_B v_C)}$$

and

$$y_{bc} = \frac{I_x[(v_B - v_C)\sin\phi + (u_C - u_B)\cos\phi]}{(u_C v_B - u_B v_C)}$$

Since the angle ϕ was so chosen that OV is parallel to side B-C, we have u_B $u_C = b$, as shown in Fig. 159. Substituting these values in the above equations, we have

$$\left.\begin{aligned} x_{bc} &= +\frac{I_y \cos\phi}{b} \\ y_{bc} &= +\frac{I_x \sin\phi}{b} \end{aligned}\right\} \tag{14}$$

In using eqs. (14) it is to be noted that the coördinates x_{bc} and y_{bc} are referred to the principal axes of the section, for in deriving the equations given above, only the coördinates of the extreme points of the section were referred to the axes OU and OV.

In a like manner, the coördinates of the intersection point of the S-lines for points D and C of the edge D-C, Fig. 159, parallel to the OU axis, are

$$\left.\begin{aligned} x_{dc} &= -\frac{I_y \sin\phi}{d} \\ y_{dc} &= +\frac{I_x \cos\phi}{d} \end{aligned}\right\} \tag{15}$$

where

$$d = v_D = v_C.$$

In this discussion it has been assumed that C-B and C-D are perpendicular sides. If they are not perpendicular, it will be necessary to determine the proper value of ϕ for each side in order to obtain the desired results.

When a section has a re-entrant corner, such as F, Fig. 159, it is quite evident that for any given plane of bending the fiber stress at F is less than at D. This is due to the fact that F is nearer the neutral axis for the plane of bending than is D. Hence the S-line for point D lies inside that for point F, whose S-line will be located entirely outside the S-polygon for the section. It is therefore necessary to draw S-lines only for the outside points of the section, as these points will be farthest from the successive positions of the neutral axis, and therefore have the least values of flexural modulus.

A simple and definite test for the determination of the points for which S-lines need be drawn is given by rolling a right line around the perimeter of the section for which the S-polygon is to be drawn. Since the successive positions of this rolling line are parallel to successive positions of the neutral axis as the plane of bending varies through all possible angles, it is evident that the points touched by this rolling line are those farthest removed from the neutral axis, and that they are points of possible maximum stress. It is to be noted that in rolling around the section, the right line will not cut across the section, which at once eliminates re-entrant corners.

For the section of Fig. 159, a line rolling as described above will touch points A, B, C, D, and E. The polygon formed by connecting these points is known as the *circumscribing polygon* of the section.

96. Construction of S-polygons.—The S-polygons for a few of the standard sections used as beams will now be calculated and constructed in order to illustrate the principles set forth in the preceding articles.

96*a*. S-polygon for a Rectangle.—The S-polygon for a 2×12-in. rectangle will be computed and constructed. Figure 160 shows the section with the principal axes OX and OY in position. The principal moments of inertia are $I_x = 288$ in.^{4}, and $I_y = 8$ in.^{4}; and the coördinates of the extreme points of the section, which in this case are also apices of the circumscribing polygon, are, $x_A = +1$, $y_A = +6$; $x_B = +1$, $y_B = -6$; $x_C = -1$, $y_C = -6$; and, $x_D = -1$, $y_D = +6$.

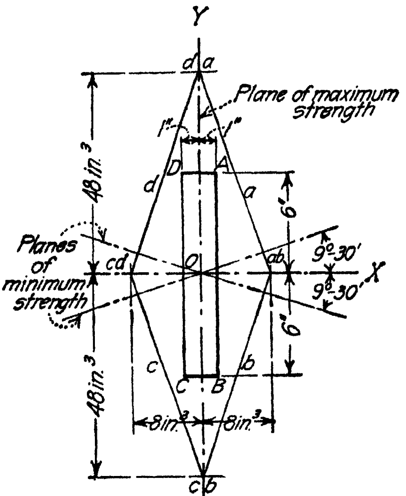


FIG. 160.—S-polygon for 2×12-in. rectangle.

Since the sides of the rectangle are all parallel to the principal axes of the section, the coördinates of the apices of the S-polygon are given by eqs. (10) to (13). For sides A-B and C-D, which are parallel to the OY axis, eqs. (10) and (11) are to be used. With $I_y = 8$ in.^{4}, and $d = x_A = x_B = +1$, eq. (10) gives, $x_{ab} = +\frac{8}{1} = +8$ in.^{3}; and eq. (11) gives, $y_{ab} = 0$. This apex of the S-polygon is located on the OX axis, as shown in Fig. 160. For side D-C the substitutions are similar to those for A-B, differing only in the signs of the coördinates of the extreme points. It will be found from eqs. (10) and (11) that $x_{cd} = -8$ in.^{3}, and $y_{cd} = 0$.

Sides A-D and C-B, which are parallel to the OX axis, require the use of eqs. (12) and (13). For side A-D, with $I_x = 288$ in.^{4} and $c = y_A = y_D = +6$ in., eq. (12) gives $x_{ad} = 0$, and eq. (13) gives $y_{ad} = +\frac{288}{6} = +48$ in.^{3} From the same equations we find for C-B, $x_{cb} = 0$, and $y_{cb} = -48$ in.^{3} These apices of the S-polygon are located on the OY axis, one above and the other below the OX axis, as shown in Fig. 160.

The complete S-polygon is obtained by plotting the points determined above, and connecting by straight lines the points which have a common letter, as, for example, points da and ab are connected by a line denoted by a in Fig. 160; likewise, points ab and bc are connected by a line denoted by b. Following this procedure for all points, the complete S-polygon is obtained, as shown in Fig. 160.

It will be noted that the coördinates of the apices of the S-polygon, as y_{da}, x_{ab}, etc., are equal to the section moduli of the rectangle for axes OX and OY respectively. This offers a convenient method for constructing this polygon without the use of eqs. (10) to (13). The section moduli can be calculated or

taken from the steel handbooks, plotted on the principal axes of the section, and the polygon drawn as described above.

96b. S-polygon for a 10-in. 25-lb. I-beam.—Fig. 161 shows the S-polygon for a 10-in. 25-lb. I-beam. As the circumscribing polygon for the I-beam is a rectangle, the methods of calculation are exactly the same as given above for the rectangular section. The detail calculations will not be given here. All data are shown on Fig. 161.

96c. S-polygon for a 10-in. 25-lb. Channel.—The circumscribing polygon for a channel is also a rectangle, but as the axis OY is not an axis of symmetry, the resulting S-polygon will not be symmetrical about the OY axis, as in the case of the rectangle and I-beam.

For a 10-in. 25-lb. channel, $I_x = 91.0$ in.^{4}, $I_y = 3.4$ in.^{4}; $x_A = +2.27$, $y_A = +5.0$; $x_B = +2.27$, $y_B = -5.0$; $x_C = -0.62$, $y_C = -5.0$; and, $x_D = -0.62$, $y_D = +5.0$. (All coördinates in inches.)

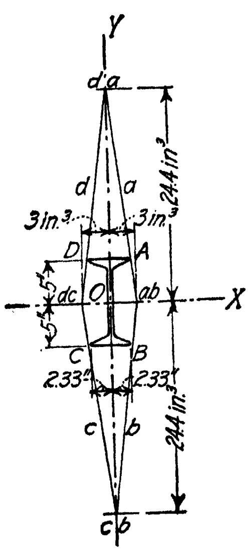


FIG. 161.—S-polygon for a 10-in. 25.4-lb. I.

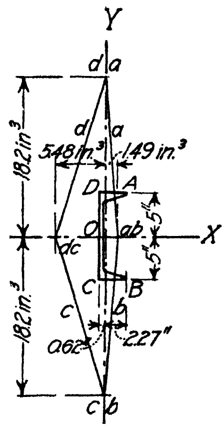


FIG. 162.—S-polygon for a 10-in. 25-lb channel.

Substituting these values in eqs. (10) to (13), the coördinates of the apices of the S-polygon are found to be

$$
\begin{aligned}
x_{ab} &= +3.4/2.27 = +1.49 \text{ in.}^3 \\
y_{ab} &= 0 \\
x_{bc} &= 0 \\
y_{bc} &= -91.0/5.0 = -18.2 \text{ in.}^3 \\
x_{cd} &= -3.4/0.62 = -5.48 \text{ in.}^3 \\
y_{cd} &= 0 \\
x_{da} &= 0 \\
y_{da} &= +91.0/5.0 = +18.2 \text{ in.}^3
\end{aligned}
$$

These values when plotted give the S-polygon of Fig. 162, on which all data are shown.

96a. S-polygon for an Angle Section.—The S-polygon for a $5 \times 3\frac{1}{2} \times \frac{1}{2}$-in. angle will be computed and constructed. In the case of angle sections, the steel handbooks do not give directly the principal moments of inertia of the section. The moments of inertia given are those for the gravity axes of the section (OU and OV of Fig. 163). By the application of a few well-known principles, the location of the principal axes and the values of the principal moments of inertia are readily determined.

Figure 163 shows the angle section with the gravity axes OU and OV in position. The moments of inertia for these axes are $I_u = 9.99$ in.^{4} and $I_v = 4.05$ in.^{4} Moments of inertia for principal axes are not given directly. However, the minimum radius of gyration of the section is given; this is a property of the minor principal axis of the section. From Art. 18, $I = Ar^2$, where A = area of section, and r = radius of gyration. For the section in question, $A = 4.0$ sq. in., and $r_y = 0.75$ in. Then, $I_y = (4.0)(0.75)^2 = 2.25$ in.^{4}

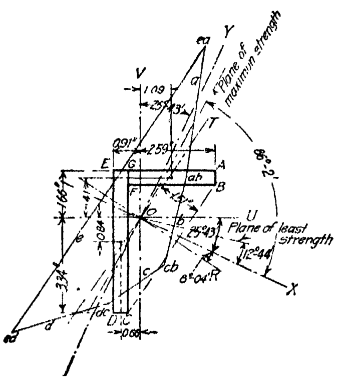


FIG. 163.—S-polygon for a $5 \times 3\frac{1}{2} \times \frac{1}{2}$-in. angle.

The value of I_x, the moment of inertia for OX, the major principal axis of the section, can be determined from the well-known relation connecting the moments of inertia for principal and other axes, which is: $I_x + I_y = I_u + I_v$. As I_x is the only unknown, we have: $I_x = I_u + I_v - I_y = 9.99 + 4.05 - 2.25 = 11.79$ in.^{4} These values may also be calculated from eqs. (11) and (12), p. 579, Appendix B.

The value of the angle between the principal and gravity axes, angle XOU of Fig. 163, may be determined from eq. (15) p. 580. If ϕ denote this angle, we have

$$\tan 2\phi = \frac{2J_{xy}}{I_y - I_x}$$

which may also be written

$$\tan \phi = \frac{2J_{xy}}{(I_y - I_x) \pm \sqrt{(I_y - I_x)^2 + 4J_{xy}}}$$

In these equations, J_{xy} = product of inertia of the section, as defined in Art. **6**, Appendix B. To determine J_{xy} for the angle section of Fig. 163, divide the figure into two rectangles $ABFG$ and $EGDC$. The coördinates of the centers of gravity of these rectangles with respect to the origin at O are given on Fig. 163. Then

$$J_{xy} = (3.0)(0.5)(1.41)(1.09) + (5.0)(0.5)(0.66)(0.84) = 3.69 \text{ in.}^4$$

Substituting in the first of the above equations,

$$\tan 2\phi = \frac{2J_{xy}}{I_y - I_x} = \frac{7.38}{4.05 - 9.99} = -1.24.$$

From the second equation $2\phi = 51° - 12'$ and $\phi = 25° \ 36'$

$$\tan \phi = \frac{7.38}{-5.94 \pm \sqrt{(5.94)^2 + (4)(3.69)^2}} = -0.479$$

$$\phi = 25° \ 36'$$

The gravity and principal axes are shown in their relative positions in Fig. 163. Values of tan ϕ for angle sections are given in the Cambria rolling mill handbook.

As shown in Fig. 163, the sides of the circumscribing polygon, *ABCDE*, are not parallel to either of the principal axes of the section. The coördinates of the apices of the *S*-polygon are to be calculated by eqs. (8) or (9); or, by rotating the axes of reference as explained by Fig. 159, eqs. (14) and (15) can be used. As the latter method is the simpler, it will be used here.

Axes *OU* and *OV* are parallel to sides *A-B*, *C-D*, *D-E*, and *E-A* of the circumscribing polygon, and will be used as the new axes of reference. The angle ϕ is seen from Fig. 163 to be 25 deg. 36 min.

For side *A-B*, which is parallel to the *OV* axis, eq. (14) is to be used. With ϕ = 25 deg. 36 min., $I_v = 2.25$ in.^{4}, and $u_A = u_B = 2.59$ in., we have,

$$x_{ab} = \frac{(+2.25)(0.902)}{2.59} = +0.784 \text{ in.}^3$$

$$y_{ab} = \frac{(+11.79)(0.432)}{2.59} = +1.97 \text{ in.}$$

In plotting these points it must be remembered that x_{ab} and y_{ab} are referred to axes *OX* and *OY*, the rotation of axes of reference having been made only with respect to the extreme points of the section.

Side *D-E* is also parallel to the *OV* axis, and eq. (14) is to be used, which gives

$$x_{de} = \frac{(+2.25)(0.902)}{-0.91} = -2.23 \text{ in.}^3$$

$$y_{de} = \frac{(+11.79)(0.432)}{-0.91} = -5.60 \text{ in.}^3$$

Sides *A-E* and *D-C* are parallel to axis *OU*. Substitution in eq. (15) gives

$$x_{ae} = \frac{(-2.25)(0.432)}{1.66} = -0.586 \text{ in.}^3$$

$$y_{ae} = \frac{(+11.79)(0.902)}{1.66} = +6.41 \text{ in.}^3$$

and

$$x_{dc} = \frac{(-2.25)(0.432)}{-3.34} = +0.291 \text{ in.}^3$$

$$y_{dc} = \frac{(+11.79)(0.902)}{-3.34} = -3.18 \text{ in.}^3$$

The side *B-C* of the circumscribing polygon is parallel to a pair of rectangular axes shown by *OR* and *OT* in Fig. 163. These axes make an angle of 33 deg. 40 min. with the gravity axes, or 8 deg. 4 min. with the principal axes of the section, as shown in Fig. 163. This angle can be calculated, or scaled with a protractor from a large layout of the section. Since the axis *OR* is in the fourth quadrant with respect to the axes *OX* and *OY*,

$$\phi = (360° - 8°4') = 351 \text{ deg. } 56 \text{ min.}$$

Using eq. (14), with ϕ as above and $b = 1.51$ in., as shown on Fig. 163, we have

$$x_{bc} = \frac{(+2.25)(0.990)}{1.51} = +1.48 \text{ in.}^3$$

$$y_{bc} = \frac{(+11.79)(-0.140)}{1.51} = -1.09 \text{ in.}^3$$

Plotting these points with respect to the *OX* and *OY* axes, and connecting the proper points, the complete *S*-polygon is obtained as shown in Fig. 163.

96c. S-polygons for Z-bars and T-bars.—Two rolled sections which are used occasionally as beam sections are the Z- and T-bars. *S*-polygons for these sections are shown in Fig. 164. The detail work of calculating these polygons will not be given, as the methods are similar to those used above.

Figure 164*a* shows the *S*-polygon for a 5 × 3¼ × ½-in. Z-bar. The coördinates of the apices of the *S*-polygon, referred to the principal axes of the section are:

$x_{ab} = -0.600$ in.^{3}, $y_{ab} = +8.56$ in.^{3}; $x_{bc} = +0.848$ in.^{3}, $y_{bc} = +4.38$ in.^{3}
$x_{cd} = +1.89$ in.^{3}, $y = 0$; $x_{af} = -1.89$ in.^{3}, $y_{af} = 0$;
$x_{ef} = -0.848$ in.^{3}, $y_{ef} = -4.38$ in.^{3}; $x_{de} = +0.600$ in.^{3}, $y_{de} = -8.56$ in.

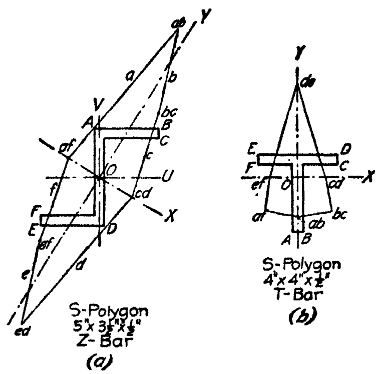


Fig. 164.

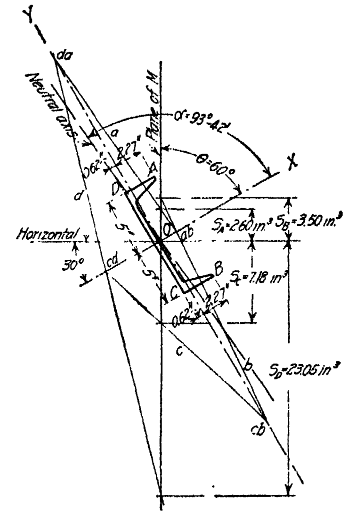


Fig. 165.

Figure 164*b* shows the *S*-polygon for a 4 × 4 × ½-in. T-bar, for which the coördinates of the *S*-polygon are:

$x_{ab} = 0$, $y_{ab} = -2.02$ in.^{3}; $x_{de} = 0$, $y_{de} = +4.83$ in.^{3};
$x_{cd} = +1.40$ in.^{3}, $y_{cd} = 0$; $x_{ef} = -1.40$ in.^{3}, $y_{ef} = 0$;
$x_{bc} = +1.69$ in.^{3}, $y_{bc} = -1.71$ in.^{3}; $x_{af} = -1.69$ in.^{3}, $y_{af} = -1.71$ in.^{3}

97. Solution of Problems in Unsymmetrical Bending.—Problems in unsymmetrical bending can be solved algebraically by the use of eqs. (1) and (2), or by semi-graphical methods involving the use of *S*-polygons. A few simple problems will be worked out to show the general methods employed.

In problems involving the determination of fiber stress in a given beam section under bending in any direction, the desired result is generally the maximum fiber stress and the fiber on which it occurs. A complete solution of this problem can be obtained by two methods. In the first method, the stresses are computed for all extreme fibers of the section. On comparing these values, the maximum can readily be determined. By the second, and better method, the neutral

axis of the section is located on a large scale layout of the section. From this sketch the fiber most remote from the neutral axis is determined by inspection, or by scaling if necessary, and a fiber stress calculation made only for this fiber, thus giving the required maximum stress intensity.

Illustrative Problem.—A 10-in. 25-lb. channel section is used as a beam to support a moment M acting in a vertical plane. Figure 165 shows the position of the channel and the direction of the plane of bending with respect to OX and OY, the principal axes of the section. The solution will be carried out for both of the general methods outlined above.

Algebraic Solution.—The moments of inertia of the section, as given by the steel handbooks, are: $I_x = 91.0$ in.^{4}, and $I_y = 3.4$ in.^{4} The coördinates of the extreme points of the section are: $x_A = +2.27$, $y_A = +5.0$; $x_B = +2.27$, $y_B = -5.0$; $x_C = -0.62$, $y_C = -5.0$; and, $x_D = -0.62$, $y_D = +5.0$. (All coördinates in inches.)

From eq. (2), with $\theta = 60$ deg., as shown in Fig. 165, and with the coördinates given above, we find for point A,

$$f_A = -M\left[\frac{(+3.4)(5.0)(0.866) + (91.0)(2.27)(0.50)}{(91)(3.4)}\right] = -\frac{+14.72 + 103.8}{309.5}M$$

$$f_A = -0.3835M$$

The minus sign indicates that the fiber stress is compressive.

For fiber B, substitution in eq. (2) involves the same quantities as for A, except that y_B is negative. The first term in the numerator of the above expression then becomes negative. Using the same form as given above, we have

$$f_B = -\frac{-14.72 + 103.8}{309.5}M = -0.2875M$$

In the same way, we have for points C and D

$$f_C = -M\left[\frac{(+3.4)(-5.0)(0.866) + (91.0)(-0.62)(0.5)}{(91)(3.4)}\right]$$

$$f_C = +\frac{+14.72 + 28.20}{309.5} = +0.1386M$$

and

$$f_D = +\frac{-14.72 + 28.20}{309.5} = +0.04355M$$

The plus signs indicate tensile stresses.

On comparing the calculated values, it will be found that fiber A has the maximum fiber stress, and that the stress intensity is $0.3835M$ lb. per sq. in., compression.

Proceeding with the second method of solution outlined above, we find from eq. (1) that the angle between the axis OX and the neutral axis for the given plane of bending is

$$\tan \alpha = \frac{(-91.0)(\cot 60°)}{3.4} = \frac{(-91.0)(0.5774)}{3.4} = -15.46$$

from which, $\alpha = 93$ deg. 42 min. In Fig. 165 the neutral axis, as located by this angle, is shown in position. It is evident by inspection that fiber A is most remote from the neutral axis. A single substitution in eq. (2) for fiber A gives the desired result. The calculations are as given above for point A; they will not be repeated.

Solution by Means of an S-polygon.—On Fig. 165 there is given a solution of this problem by means of an S-polygon. The S-polygon is constructed from the calculations made in Art. **96** and shown on Fig. 162.

From eq. (4) of Art. **93,** the fiber stress at any point is $f = M/S$, where S is the flexural modulus of the section. As explained in Art. **94,** the value of S for any point in the section is the distance measured along the plane of bending from the origin to the intersection of the plane of bending and the S-line for the given point. These intercepts are shown on Fig. 165, each with a subscript corresponding to the point for which the value of S is given. Then from eq. (4), the fiber stresses are: $f_A = M/2.60 = 0.385M$, $f_B = M/3.50 = 0.286M$, $f_C = M/7.18 = 0.139M$, and $f_D = M/23.05 = 0.0435M$.

The character of fiber stress is not given directly by the S-polygon. To determine the character of the fiber stress, locate the position of the neutral axis, as shown in Fig. 165. For positive moment, all points below the neutral axis will be under tensile stress, and points above the neutral axis will be under compression. Thus in the case under consideration, points A and B are above the neutral axis and are under compression, while C and D are below the neutral axis and are under tension. These results are checked by the algebraic solution given above.

Illustrative Problem.—A $5 \times 3\frac{1}{2} \times \frac{1}{2}$-in. angle with the longer leg vertical carries a moment M acting in a vertical plane, as shown in Fig. 166. Required the intensity of the maximum fiber stress and the fiber on which it occurs.

This is the angle section for which the S-polygon is calculated in Art. **96** and shown on Fig. 163. The principal moments of inertia of the section are: $I_x = 11.79$ in.^{4}, and $I_y = 2.25$ in.^{4} In Fig. 166 the principal axes OX and OY are shown in position.

Algebraic Solution.—The fiber of maximum stress intensity will be determined by plotting the position of the neutral axis on the angle section. From eq. (1), with $\theta = 115$ deg. 36 min., as shown on Fig. 166, we have

$$\tan \alpha = \frac{(-11.79)(\cot 115° 36')}{2.25} = +2.51, \text{ or, } \alpha = 68 \text{ deg. } 17 \text{ min.}$$

The position of the neutral axis is shown on Fig. 166. It will be found that fiber C is most remote from the neutral axis, and is therefore the fiber of maximum stress intensity.

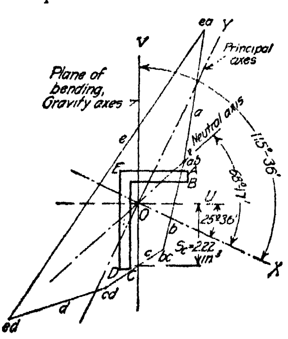


FIG. 166.

The coördinates of point C must be referred to the principal axes of the section, OX and OY, in substituting in eq. (2). This information is not given in the steel handbooks. It can be obtained by scaling from a large scale drawing of the section, or it can be calculated by means of the formulas for rotation of the axes of reference given for the conditions shown in Fig. 159 of Art. **95.** The values of u and v to be used in the formulas of Art. **95** can be found in the steel handbooks, for OU and OV are the gravity axes of the section. Then for $u_C = -0.41$, $v_C = -3.34$, and $\phi = 25$ deg. 36 min., we have, $y_C = (-3.34)(0.902) - (0.410)(0.432) = -3.19$, and, $x_C = (-0.410)(0.902) + (3.34)(0.432) = +1.07$, both values in inches. Calculated and scaled values were found to check.

Substituting in eq. (2) the values of x_C and y_C given above, and $\theta = 115$ deg. 36 min., the fiber stress at C is found to be

$$f_C = -M\left[\frac{(2.25)(-3.19)(\sin 115° 36') + (11.79)(1.07)(\cos 115° 36')}{(11.79)(2.25)}\right]$$

$$f_C = +0.450M$$

Fiber C is under tensile stress, as indicated by the positive sign of the result.

In calculating the tables of safe loads on angle sections given in the steel handbooks, it is usually assumed that the neutral axis is horizontal for all planes of bending. If the neutral axis be assumed to be parallel to the shorter leg of the angle of Fig. 166, the fiber stress at C is found to be: $f_C = Mc/I = 3.34 M/9.99 = 0.334M$, a result only about 75 per cent of the true stress given above.

Solution by S-polygon.—The S-polygon solution of the preceding illustrative problem is shown on Fig. 166. This polygon is constructed from data calculated in Art. **96** and shown on Fig. 163. From an inspection of Fig. 166, it can be seen that for the given plane of bending, fiber C has the least S, and is therefore the desired fiber of maximum stress. By scale from Fig. 166 we find $S_C = 2.22$ in.³ Therefore, $f_C = M/2.22 = 0.450M$, which checks the result obtained by the algebraic method. As fiber C is located below the neutral axis, the fiber stress is tensile.

The design of beams subjected to unsymmetrical bending is greatly simplified by the use of S-polygons. Where several possible loading conditions are involved, the algebraic calculations are long and tedious, while the semi-graphical S-polygon offers a comparatively simple and easily understood method of solution.

In designing by the S-polygon method, the process consists in comparing graphically the flexural modulus required for any plane of bending with that furnished by the assumed section. From eq. (4), Art. **93**, $S = M/f$. Having given the bending moment to be carried and the allowable working stress, the required flexural modulus is readily determined.

The required S is plotted to scale on a set of coördinate axes placed in the proper position in space. The S-polygons of the trial sections are then plotted to scale on the same set of axes. In order to answer the requirements of the design, the S furnished by the trial section must be equal to, or greater than, the required value.

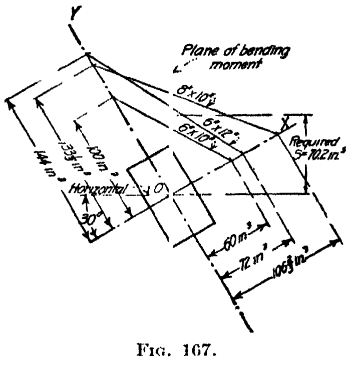


Fig. 167.

Illustrative Problem.—Design a wooden beam set with its faces at an angle of 30 deg. with the vertical, and subjected to an unsymmetrical bending moment acting in a vertical plane. The span of the beam is 12 ft., and the allowable working stress in the timber is 1,000 lb. per sq. in. Determine the beam section required to support a net uniform load of 300 lb. per ft.

As the weight of the beam section is not known to begin with, it will be assumed to be 25 lb. per ft. The total load to be carried is then 325 lb. per ft.; the bending moment in a vertical plane is $M = \frac{1}{8}wl^2 = \frac{1}{8}(325)(12)^2(12) = 70{,}200$ in.-lb.; and the required flexural modulus is $S = M/f = 70{,}200/1{,}000 = 70.2$ in. This is shown to scale in the proper position in Fig. 167.

From the S-polygon of a rectangle shown in Fig. 160, Art. **69**, it can be seen that for bending at an angle of 60 deg. with the axis OX, fibers A and C have values of S which are equal and smaller than those for D and B. It is evident, then, that it is necessary to draw only the S-line for point A in order to determine the proper section.

In Fig. 167 the S-lines for several rectangular sections are shown. The 6 × 10-in. section is too small, for the S furnished by the section is not equal to that required by the moment. The 6 × 12-in. section is a little too large, but as beams usually come in even inch sizes, it will be adopted.

Before this section is finally adopted, the assumed weight must be checked up. At 4 lb. per ft. board measure, a 6 × 12-in. section will weigh (12 × 6⁄12)4 = 24 lb. per ft. As the weight assumed in the calculations was 25 lb. per ft. a revision is not necessary.

98. Investigation of Beams.—An important problem in the investigation of the relative value of the various rolled sections when used as beams is their moment carrying capacity. By means of the S-polygons of the sections, a direct comparison can be made. Thus, if it be required to determine the relative moment carrying capacity of an I-beam and a channel of the same depth and weight per foot—as for example, a 10-in. 25.4-lb. I-beam and channel—we can refer to the S-polygons for these sections. Figure 161 gives the S-polygon for a 10-in. 25.4-lb. I-beam, and Fig. 162 gives the S-polygon for a 10-in. 25-lb. channel.

These polygons are drawn to the same scale so that the relative strength of the two sections is proportional to their sizes. It can be seen at once that the advantage is in favor of the I-beam section. In the same way, any sections can be compared by this method.

Another problem of considerable importance is the determination of the planes of greatest and least strength for any given section. In this way it is possible to place a section in such a position that its plane of greatest resisting moment coincides with the plane of the bending moment, and the section is used to its greatest advantage. It is also possible to avoid loading a beam in the plane of its least resisting moment.

From eq. (4) of Art. **93**, it can be seen that the fiber stress varies inversely as the value of S. Therefore the plane of greatest strength is the one with the largest S, and the plane of least strength is the one with the smallest S. The values are measured as shown by the vector OE of Fig. 158.

The plane of greatest strength in bending of the rectangle, I-beam, and channel sections, as shown by their S-polygons, (see Figs. 160, 161, and 162) is in the plane of the OY axis. By an inspection of the S-polygons, it can be seen that the plane of least strength is perpendicular to the S-lines, for on these planes the values of S are a minimum. There will be four such planes for the rectangle and I-beam sections, one for each S-line. For the channel section there are two planes of least strength, one perpendicular to the S-line a and another perpendicular to S-line b.

The angles which these planes make with the axis OX can be determined from a large scale drawing of the section by means of a protractor. The angles can also be determined by means of a proposition of Analytical Geometry which states that when a line is perpendicular to a given line, the slope of the perpendicular is the negative reciprocal of that of the given line. Thus from the equation of the S-line for fiber A, as given by eq. (6), Art. **94**, the slope of the perpendicular is $+\frac{I_y}{I_x}\frac{y_A}{x_A}$. For the rectangle of Fig. 160, we find from the data given in Art.

96*a*, that the angle between the OX axis and the plane of least strength, as determined from the above equation, is

$$\text{tan of slope} = +\tfrac{8}{288} \times \tfrac{6}{1} = +0.167, \text{ or slope angle} = 9 \text{ deg. } 30 \text{ min.}$$

This plane is shown in position on Fig. 160.

The determination of the planes of greatest and least strength of the angle section, for which the S-polygon is shown in Fig. 163, is not as simple a matter as for sections of rectangular form due to the unsymmetrical form of the S-polygon. From an inspection of the S-polygon of Fig. 163, it is evident that the angle section has its greatest strength as a beam for the plane of loading for which the fiber stresses, and hence the values of S, for fibers A and D are equal. This plane can be located by trial by means of a straight edge and a pair of dividers. It can also be located by means of eq. (5) of Art. **93**. If values of S, as given by eq. (5) for fibers A and D, be equated and the resulting expression be solved for θ, the result will be the desired plane of greatest strength. Performing the operation indicated above, we have

$$\tan\theta = -\frac{I_x}{I_y} \cdot \frac{x_A + x_D}{y_A + y_D}$$

For the angle section whose S-polygon is shown in Fig. 163, $x_A = +1.61$, $y_A = +2.60$; $x_D = +0.59$, $y_D = -3.40$; $I_x = 11.79$, and $I_y = 2.25$. From the above equation

$$\tan\theta = -\frac{11.79}{2.25} \cdot \frac{1.61 + 0.59}{2.60 - 3.40} = +14.40$$

or, $\theta = 86$ deg. 2 min. This plane of loading is shown in position on Fig. 163. The plane of least strength is determined by methods similar to those used for the rectangle. It is shown on Fig. 163.

In the above discussion the planes of greatest strength have been located and are shown in position on a few of the sections in general use as beams. To secure the best results, it is evident that the section should be so placed that the plane of bending and the plane of greatest strength coincide. It is not possible, however, to realize these ideal conditions in all cases. This is due to the fact that the methods of attaching the beam section to its supports determines the position of the beam. Thus beams supported on a sloping surface must usually be set with their faces perpendicular to the supporting surface.

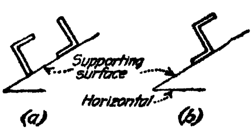


Fig. 168.

When an angle section is used as a beam, it should be placed as shown in Fig. 168*a*, for as shown by the S-polygon, this position is very close to its position for greatest strength for bending in a plane which is vertical or nearly so. At the same time, attachment to the supporting structure is readily made.

Z-bars are seldom used as beam sections, as it is difficult to obtain them except in large quantities. From the S-polygon for this section, Fig. 164*a*, it can be seen that for the position shown in Fig. 168*b*, the section is advantageously placed for bending in a vertical plane.

The T-bar, as shown by its S-polygon, Fig. 164*b*, does not form an ideal beam section, due to the fact that the fiber stresses on the extreme fiber of the stem are much greater than those on the flange. In any case it is desirable that

the section be placed with the stem down. The upper, and wider face, is then in compression, which increases the lateral stiffness of the section.

In some types of roof covering, T-bars closely spaced, are used to support tile or short span slabs carried directly on the T-bars. The stem of the T is placed up, the bottom flange forming a support for the title. From the discussion given above, it can be seen that the T-bar is not well placed in this type of construction, for the narrow stem of the T is in compression, and is liable to fail due to insufficient lateral support, unless low working stresses are maintained. The material is then not used to as great advantage as in the other sections considered.

99. Tables of Fiber Stress Coefficients for Beams.—The variety of conditions encountered in problems in unsymmetrical bending renders it impractical to attempt any very extensive tabulation of fiber stresses in beams. Each case must be worked out by means of the general equations or the S-polygon methods given in the preceding articles. Where S-polygon methods are to be used to any great extent, it will save time if the S-polygons of standard sections be plotted on tracing cloth, or some transparent material. The required S can be plotted on a sheet of paper, as explained in the illustrative problem, p. 163. By laying the plotted S-polygons over the required S, and shifting to different sections, the desired section can readily be determined.

There is, however, one very important and frequently encountered condition of unsymmetrical loading for which tabulations of fiber stress can be made. The case referred to is that of loading in a vertical plane on sections inclined at an angle to the vertical.

Table 1 gives coefficients for I-beams; Table 2 gives values for channels; and Table 3 gives values for angles. The fiber stress in any case is obtained by multiplying the moment, M, by the coefficient given in the tables. The sketch shows the conditions for which the values are given. These tables were taken from articles by R. Fleming, which appeared in the *Eng. Rec.*, March 3, 1917, and in the *Eng. News-Rec.*, Feb. 27, 1919.

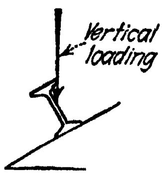


TABLE 1.—FIBER STRESS COEFFICIENTS, BENDING MOMENT DUE TO VERTICAL LOADING ON I-BEAMS

I-beam section	Pitch of roof in inches per foot								
	0	1	2	3	4	5	6	7	8
6-in. 12.5-lb.	0.138	0.212	0.284	0.352	0.415	0.473	0.526	0.573	0.614
7-in. 15.3-lb.	0.097	0.153	0.208	0.260	0.308	0.353	0.393	0.430	0.461
8-in. 18.4-lb.	0.070	0.114	0.157	0.196	0.234	0.268	0.300	0.328	0.352
9-in. 21.8-lb.	0.053	0.088	0.121	0.153	0.183	0.210	0.235	0.257	0.277
10-in. 25.4-lb.	0.041	0.069	0.096	0.122	0.146	0.168	0.188	0.206	0.222
12-in. 31.8-lb.	0.028	0.050	0.071	0.091	0.110	0.127	0.143	0.157	0.170

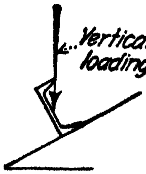


TABLE 2.—FIBER STRESS COEFFICIENTS, BENDING MOMENT DUE TO VERTICAL LOADING ON CHANNELS

Channel section	Pitch of roof in inches per foot								
	0	1	2	3	4	5	6	7	8
6-in. 8.2 -lb.	0.231	0.396	0.557	0.709	0.851	0.982	1.101	1.207	1.301
7-in. 9.8 -lb.	0.166	0.296	0.422	0.542	0.655	0.758	0.852	0.935	1.010
8-in. 11.5-lb.	0.124	0.228	0.330	0.427	0.517	0.600	0.676	0.743	0.804
9-in. 13.4-lb.	0.095	0.180	0.263	0.342	0.415	0.483	0.545	0.600	0.650
10-in. 15.3-lb.	0.075	0.145	0.214	0.279	0.340	0.397	0.448	0.494	0.535
12-in. 20.7-lb.	0.047	0.094	0.141	0.184	0.225	0.263	0.298	0.329	0.357

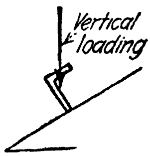


TABLE 3.—FIBER STRESS COEFFICIENTS, BENDING MOMENT DUE TO VERTICAL LOADING ON ANGLES

Angle section, inches	Pitch of roof in inches per foot								
	0	1	2	3	4	5	6	7	8
2½ × 2 × ¼	3.49	3.30	3.11	2.88	2.68	2.46	2.30	2.14	2.01
2½ × 2 × 5/16	2.91	2.76	2.61	2.41	2.22	2.04	1.90	1.78	1.67
3 × 2½ × ¼	2.33	2.22	2.10	1.98	1.85	1.71	1.60	1.49	1.38
3 × 2½ × 5/16	1.89	1.83	1.73	1.63	1.51	1.41	1.30	1.24	1.15
3½ × 2½ × ¼	1.80	1.69	1.60	1.46	1.35	1.22	1.15	1.06	1.12
3½ × 2½ × 5/16	1.47	1.39	1.31	1.22	1.14	1.02	0.96	0.89	0.93
4 × 3 × 5/16	1.06	1.00	0.94	0.88	0.81	0.75	0.69	0.65	0.66
4 × 3 × 3/8	0.92	0.87	0.81	0.75	0.70	0.63	0.59	0.55	0.52
5 × 3½ × 5/16	0.68	0.65	0.61	0.56	0.51	0.47	0.43	0.41	0.48
5 × 3½ × 3/8	0.60	0.57	0.53	0.48	0.43	0.40	0.37	0.35	0.41
6 × 4 × 3/8	0.41	0.38	0.35	0.32	0.29	0.27	0.25	0.27	0.30
6 × 4 × 7/16	0.35	0.33	0.31	0.28	0.25	0.23	0.22	0.23	0.26

100. Variation in Fiber Stress Due to Changes in Position of the Plane of Bending.—The S-polygon shows in a striking manner that small changes in the position of the plane of loading cause relatively large changes in the fiber stress on a given point in the section. This variation in position of the plane of loading may be due to a variety of causes. The deflection of the beam under loading may tend to twist the section about its longitudinal axis, thus changing the position of the plane of bending from that assumed in the design. In the case of wooden beams, warping of the timber may have a similar effect. To counteract these effects, the beam should be held rigidly in line by some form of lateral support.

Bridging in wooden floor construction is one method of providing this lateral support.

The effect of a small change in the position of the plane of loading will now be shown graphically by means of an S-polygon. Figure 169 shows a portion of the S-polygon of a 10-in. 25.4-lb. I-beam, data for which are given in Art. **96b**. A comparison will be made of fiber stresses for bending in the plane of the OY axis, and for bending in another plane 1 deg. away from the first plane; that is, for $\theta = 90$ deg. and 89 deg. respectively. By scale from Fig. 169, we have $S_1 = 24.4$ in.^{3} for $\theta = 90$ deg., and $S_2 = 21.3$ in.^{3} for $\theta = 89$ deg. The resulting fiber stresses are: $f_1 = 0.04099M$, and $f_2 = 0.04795M$. These values differ by 14.6 per cent of f_1. Values of S are also indicated for bending planes at 5 and 6 deg. from the axis OY. At this place the stresses differ by about 7.5 per cent.

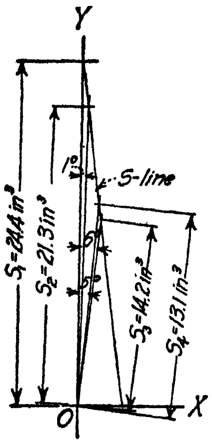


FIG. 169.

It can be seen by comparing the calculated values given above, and also by inspection from Fig. 169, that this percentage is a maximum for planes of loading near the OY axis.

An exact measure of the change in fiber stress can be determined by differentiation of eq. (2) of Art. **92**. Thus

$$df = -M\left(\frac{I_y y_A \cos\theta - I_x x_A \sin\theta}{I_x I_y}\right)d\theta$$

In this expression df is the change in fiber stress on a fiber A, coördinates x_A and y_A, due to a very small change $d\theta$ in the position of the plane of bending. This angular change $d\theta$ is to be measured in radians (a radian is 57.3 deg.).

The percentage change in fiber stress is given by dividing df by the value of f for the given θ. Then

$$\text{Percentage change in fiber stress} = \frac{df}{f}$$

$$= \frac{I_y y_A \cos\theta - I_x x_A \sin\theta}{I_y y_A \sin\theta + I_x x_A \cos\theta}\,d\theta$$

As stated above, this rate of stress change is a maximum for a loading plane at the OY axis. With $\theta = 90$ deg., the above equation becomes, when θ is expressed in degrees,

$$\frac{df}{f} = \left(\frac{I_x x_A}{I_y y_A}\right)\left(\frac{d\theta}{57.3}\right)$$

As an application of this equation, consider the case solved graphically in Fig. 169. For a 10-in. 25-lb. I-beam, $I_x = 122.1$ in.^{4}, $I_y = 6.9$ in.^{4}, $x_A = 2.33$ in. and $y_A = 5.0$ in. Then

$$\text{Percentage change} = df/f = \left(\frac{1}{57.3}\right)\left(\frac{122.1}{6.9}\right)\left(\frac{2.33}{5.00}\right)$$

$$= 14.4 \text{ per cent}$$

It will be noted in the above equation that the predominating factor is the ratio of principal moments of inertia, I_x/I_y. For sections in which this ratio is large—that is, in narrow deep sections the fiber stress increase is large for a relatively small

change in the direction of the plane of loading. To avoid this effect, beam sections should be chosen from rolled shapes or rectangular sections which have considerable lateral rigidity. If narrow sections must be used, they should be thoroughly braced to prevent overturning.

It is also interesting to note the change in position of the neutral axis due to changes in the plane of bending. This effect is best studied by means of eq. (1), Art. 92. For the beam section considered above, suppose, as before, that the plane of bending is 1 deg. from the axis OY, or $\theta = 89$ deg. in eq. (1). Then

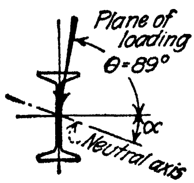
$$\tan \alpha = -(I_x/I_y) \cot \theta = \frac{(-122.1)(0.01746)}{6.9}$$

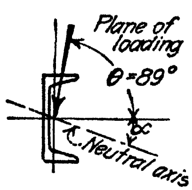
$$\tan \alpha = -0.309, \text{ or, } \alpha = 180° - 17° 10'$$

It will be noted that a 1-deg. change in the position of the plane of bending causes a 17-deg. change in the position of the neutral axis.

Table 4 gives the percentage change in fiber stress and the corresponding change in the position of the neutral axis due to a 1-deg. change in the direction of the plane of bending from the OY axis of standard I-beam and channel sections. These values were calculated by the methods given above.

Table 4.—Percentage Increase in Fiber Stress and Change in Position of Neutral Axis for a One-Degree Change in Direction of Plane of Bending





Section	$\frac{I_x}{I_y}$	Increase in fiber stress (per cent)	Change in slope of neutral axis α (degrees)
20-in. 65.4-lb. I-beam	41.8	22.8	36° 10′
18-in. 54.7-lb. I-beam	37.5	21.8	33° 15′
15-in. 42.9-lb. I-beam	30.2	19.3	27° 50′
12-in. 31.8-lb. I-beam	22.7	16.5	21° 35′
10-in. 25.4-lb. I-beam	17.7	14.4	17° 10′
9-in. 21.8-lb. I-beam	16.4	13.8	16° 0′
8-in. 18.4-lb. I-beam	15.0	13.1	14° 40′
7-in. 15.3-lb. I-beam	13.5	12.3	13° 20′
6-in. 12.5-lb. I-beam	11.8	11.5	11° 40′
15-in. 33.9-lb. channel	38.1	23.2	33° 40′
12-in. 20.7-lb. channel	32.8	21.4	29° 50′
10-in. 15.3-lb. channel	29.1	19.9	27° 0′
9-in. 13.4-lb. channel	26.3	18.5	22° 25′
8-in. 11.5-lb. channel	24.8	18.2	20° 35′
7-in. 9.8-lb. channel	21.5	16.5	20° 35′
6-in. 8.2-lb. channel	18.6	15.2	18° 0′

101. Deflection of Beams Under Unsymmetrical Bending.—The amount and direction of the deflection of a beam subjected to unsymmetrical bending is often desired. To determine the desired deflection, the bending moment can be resolved into its components parallel to the principal axes of the section and the deflection determined for these component moments by means of the usual formulas for the case in question. The required resultant deflection is equal to the vector sum of the component deflections.

Suppose the rectangular section of Fig. 170 is subjected to bending in a plane at an angle θ to axis OX due to a uniform load of w lb. per foot. Required the amount and direction of the resulting deflection.

As the components of moment parallel to the axes OX and OY are proportional to the components of the applied load for these same axes, the deflection parallel to the axes can be written from the deflection formula for uniform loading, which

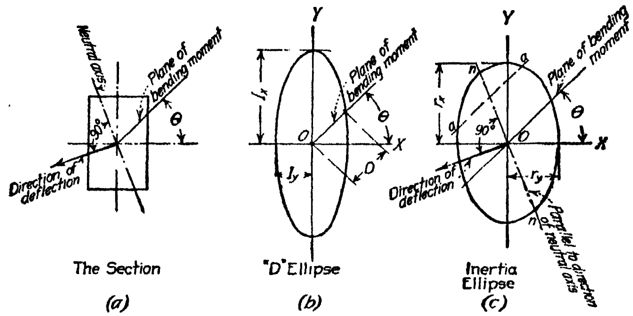


FIG. 170.

is, $d = \frac{5wl^4}{384EI}$ (see Sec. 1, Art. **63***b*). For the component of load parallel to the OX axis, we have from the above formula

$$d_x = \frac{5}{384} \cdot \frac{l^4}{E} \cdot \frac{w \cos \theta}{I_y}$$

and for the load parallel to the OY axis, we have

$$d_y = \frac{5}{384} \cdot \frac{l^4}{E} \cdot \frac{w \sin \theta}{I_x}$$

where d_x and d_y are the components of deflection for the OX and OY axes respectively.

The vector sum of these deflections is

$$d = (d_x^2 + d_y^2)^{1/2}$$

where d is the desired deflection. Substituting the above values of d_x and d_y, we have

$$d = \frac{5}{384} \cdot \frac{wl^4}{E} \left(\frac{I_x^2 \cos^2 \theta + I_y^2 \sin^2 \theta}{I_x^2 I_y^2} \right)^{1/2} \tag{16}$$

From Fig. 170*a* the angle which the resultant deflection makes with axis OX is

$$\tan \beta = \frac{d_y}{d_x} = \frac{I_y}{I_x} \tan \theta \tag{17}$$

As this expression is the negative reciprocal of that given in eq. (1), Art. **92,** it can be seen that the direction of deflection is perpendicular to the neutral axis for the given plane of bending.

If the loading conditions differ from those assumed in the above analysis, it is only necessary to change the value of the constant $\frac{5}{384} \frac{l^4}{E}$ of eq. (16) to meet the required conditions.

The amount and direction of deflection can also be determined by graphical methods which are based on certain properties of the ellipse. Equation (16) can be written in the form

$$d = \frac{5}{384} \cdot \frac{wl^4}{E} \times \frac{1}{D}, \qquad \text{where } D = \left(\frac{I_x^2 I_y^2}{I_x^2 \cos^2 \theta + I_y^2 \sin^2 \theta}\right)$$

This value of D can be shown to be the equation of an ellipse with major and minor axes I_x and I_y. Figure 170 *b* shows the D-ellipse for a rectangular section. The vector D, measured as shown in Fig. 170*b*, gives the denominator of the above equation for loading on the given plane.

As stated above, the direction of deflection is perpendicular to the neutral axis. The neutral axis can be located by means of the inertia ellipse of the section. A complete discussion of the inertia ellipse will be found in advanced works on Mechanics, to which the reader is referred.

Figure 170*c* shows the inertia ellipse for a rectangular section. It is constructed with major and minor axes equal to the radii of gyration of the section for the axes OX and OY. To locate the neutral axis, draw through point O a line parallel to the plane of bending. Draw *a-a*, any chord of the ellipse parallel to the plane of bending. Bisect this chord, and through its center point draw a line *n-n* which passes through the point O. This line is parallel to the direction of the neutral axis for the given direction of bending. This construction is based on the fact that eq. (1) expresses the relation which exists between the conjugate diameters of an ellipse.

A line perpendicular to *n-n* gives the direction of the desired deflection, as shown in Fig. 170*c*.

SECTION 2

DESIGN OF STEEL AND CAST-IRON MEMBERS

STEEL SHAPES AND PROPERTIES OF SECTIONS

1. Steel Shapes.—The steel used in structures is in the form of single pieces, or combinations of two or more pieces, to which the general term *shapes* is applied. The procedure in the manufacture of these shapes consists of the following operations: (1) smelting iron ore and producing pig iron; (2) converting the pig iron into rectangular prisms of steel, called *ingots;* and (3) rolling the ingots to the desired shapes. The shapes used in building construction are: Square and round rods or bars, flat bars or *flats*, plates, angles, channels, I-beams, H-sections, zees and tees. Flat members 6 to 7 in. wide and less are usually designated as *bars* or *flats;* over 6 to 7 in. wide are designated as *plates*. Zees and tees are not now used to any great extent. Zees have been used extensively for columns but are rapidly becoming obsolete. H-sections are designed for use as columns.

The process of rolling I-beams, channels, angles, etc. is in general as follows: The ingots are brought to a uniform temperature in the *soaking pit*, and then are taken out and passed several times through a set of rolls, called *blooming rolls*. These rolls give to a piece only the general shape (rectangular, flat, or square) of the finished product. The next step is to pass the steel through the *roughing rolls*, and then the piece is passed to the *finishing rolls* where the final shaping takes place. The pieces, still very hot, are then passed on by movable tables to circular saws where they are cut into required lengths.

The method of increasing sectional area of standard shapes is shown in Fig. 1. For example, suppose it is desired to roll channels or I-beams having the same depth, but different thicknesses of web. These sections are always rolled horizontally and the increase in thickness of web is accomplished by changing the distance between the rolls, the effect being to change the width of flange as well. Thus, two beams with the same height but different weights differ simply by a rectangle as shown. It will be seen, also, that for an angle with certain size of legs the effect of increasing weight is to change slightly the length of legs, and to increase the thickness.

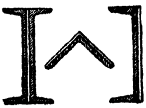


Fig. 1.

Bethlehem beam, girder and H-sections are shaped by four rolls instead of the two grooved rolls used for manufacturers' standard shapes. The use of so many rolls makes possible a variation of height as well as width, and both are increased with additional weight in H-sections.

Plates when rolled to exact width, the width being controlled by a pair of vertical rolls, are known as *universal mill* or *edged* plates. Plates rolled without the width being controlled have uneven edges and must be sheared to the correct width. Such plates are known as *sheared* plates.

The properties of the standard shapes manufactured by the different steel companies are the same. The *standard* shapes of the Assoc. of Am. Steel Mfrs., are rolled by all mills, but each company also has its own list of *special* shapes. These special shapes, which are different for the different mills, are not as likely to be in stock as the standard shapes.

Standard I-beams are rolled in depths from 3 to 24 in. and standard channels from 3 to 15 in. The different depths of standard I-beams are: 3 to 10 in. consecutively, then 12 in., 15 in., 18 in., 20 in., and 24 in. For channels, 3 to 10 in., consecutively, then 12 in. and 15 in. For each depth of I-beam and channel, there are several standard weights.

Minimum sizes of steel shapes are more likely to be found in stock and are the most efficient for resisting bending considering the weight of material used. The rolls are made especially for these sections and the heavier sections for a given depth of beam are obtained by spreading the rolls as explained above.

I-beams and channels, 15 in. and under, and angles 6 in. and under, take the *base price*. Heavier sections are charged for at a higher rate, usually 10 c. per 100 lb., above base price.

2. Properties of Steel Sections.—The fundamental properties of sections may be said to be: Sectional dimensions, location of the center of gravity, and the moments of inertia about the various axes. The distance from the center of gravity to the most stressed fiber c; the section modulus s; and the radius of gyration r, follow from these. The methods of finding the properties of sections are given in Section 1 and Appendix B.

To facilitate the work of the designer, properties of steel sections are published. The facility with which a designer can find and use these properties, which are given in manufacturers' handbooks and elsewhere, has much to do with the amount of work which he can accomplish.

Beams.—The steel manufacturers' handbooks give very complete tables of properties of steel beam section. Uniformly loading I-beams, channels, and angles should be selected from the tables of safe or allowable uniform loads. These tables can also be adapted for other loadings, such as for a load concentrated at the center, in which case a beam should be selected which will carry twice the load, uniformly distributed. For a number of load concentrations, approximately equal in amount and spacing, the load may be considered as uniform.

For irregular loadings on I-beams and channels the moment and shear should be computed and the tables used which give the allowable resisting moment and shear of the various shapes. If desired, however, the beams may be designed by computing the section modulus and selecting the proper size of beam from the tables of properties. Angles, tees and other miscellaneous shapes used as beams must usually be designed by use of the section modulus, as few tables of safe loads or resisting moments and shears are given for these shapes.

Bethlehem beams and girders differ from the manufacturers' standard sections rolled by other manufacturers. The beams have heavier flanges, and, where moment is the consideration, they are lighter for the same strength than other sections. Their webs are lighter than in standard sections. Bethlehem girder sections are, for their depths, the strongest sections rolled. They have nearly twice the carrying capacity of the manufacturers' standard section for the

same depth, but they are uneconomical where there is room for a deeper section. Tables of uniform loads for Bethlehem sections are given in Bethlehem Handbook. The common properties are also given.

Built-up steel beam properties usually have to be computed with the properties of the component parts as a basis. Some properties of the more common plate-girder sections are given in the principal steel handbooks.

Columns.—I-beams are occasionally used as columns. Their properties will be found as noted under beams. The only rolled steel column section in common use is the H-section. The Carnegie Co. rolls 4-, 5-, and 6-in. H-sections; and the Bethlehem Co. rolls 8-, 10-, 12- and 14-in. H-sections in a large range of weights. The properties of various built-up columns of pairs of channels, both latticed and with cover plates, and of plate and angle sections are given in the steel handbooks.

There are also patent columns such as Lally columns[1] and cast-iron columns for second-class construction or light loads, whose properties are given in books issued by the manufacturers.

Struts and Ties.—In the design of struts and ties, it is found convenient to have tables giving the values of the radius of gyration r, and also tables giving net areas deducting rivet holes. The principal steel handbooks give values of r for pairs of different angles back to back, and also the net areas for angles. It should be noted that the minimum r for a single angle is not about an axis parallel to either leg. This minimum r is given in the tables of the properties of angles.

STEEL BEAMS

By C. R. Young

3. Stress Conditions to be Met.—In order that a steel beam may be able to perform the service required of it satisfactorily, it must be secure against failure by bending, flange buckling, vertical or horizontal shear and web crippling. At the same time it must not deflect to such an extent as to endanger or damage dependent construction. By carefully selecting the type of section to be employed, large economies may often be effected in meeting each of the above tendencies to failure. Where bending moment and tendency to excessive deflection are of paramount importance, the selection of beams possessing heavy flanges and large depth in relation to area will be found in the interests of economy. If the beam be without lateral support for a long distance in relation to its width, the selection of a broad-flanged type is desirable. Where shear and the accompanying tendency to web crippling are large, it may be prudent to select relatively shallow beams with heavy webs which do not require stiffening or reinforcing. To meet any stress condition, it is most desirable to utilize, where possible, single rolled sections, of standard rather than special type, and so obviate the relatively expensive procedure of building up a section from several shapes and plates.

4. Proportioning for Moment.—In selecting a rolled section, such as an I-beam, channel, angle or tee for the resistance of a given bending moment, it is convenient to employ the flexure formula $f/c = M/I$ in the form $S = M/f$,

[1] The Lally Column Co., New York and Chicago.

where S, known as the section modulus, $= I/c$. Having found the required section modulus S by dividing the computed bending moment by the permissible fiber stress, all that is necessary in designing for moment is to find from a book of tables a section having a section modulus at least as great as that required. It should be remembered, however, that adequacy for moment is by no means a guarantee of entire safety.

Illustrative Problem.—If the permissible bending stress on a steel beam which is subjected to a bending moment of 30,000 ft.-lb. is 16,000 lb. per sq. in., suggest a suitable section.

Care must be taken to express the bending moment in the formula in inch-pounds, since the unit of length involved in the permissible fiber stress and in the tabulated section moduli is the inch.

$$S = \frac{(30,000)(12)}{16,000} = 22.5$$

An economical section would be the Carnegie supplementary 10-in., 22.4-lb. I-beam for which the section modulus = 22.7, or a Bethlehem 10-in., 23.5-lb. I-beam with a section modulus = 24.6. The standard 10-in., 25.4-lb. I-beam with a section modulus of 24.4 would also suffice. Relative pound prices and availability would govern the choice, provision being made of course for shear, web crippling and flange buckling (see Arts. **16, 17** and **15** respectively).

Illustrative Problem.—If a girder of 21-ft. span, consisting of one 18-in., 54.7 lb. I-beam, is to be loaded by two equal concentrated loads at the third points (including the weight of the girder), and the permissible flexural stress = 16,000 lb. per sq. in., find how great each of the concentrated loads may safely be, so far as bending is concerned.

If P be one of the concentrated loads, the maximum moment $= (P)(7)(12) = 84P$ in.-lb.

Moment of resistance of section, or its capacity to resist moment

$$M = Sf = (88.4)(16,000) = 1,415,000 \text{ in.-lb.}$$

Hence

$$P = \frac{1,415,000}{84} = 16,850 \text{ lb.}$$

Great saving in time in proportioning beams for moment may be effected by using the tables of flexural capacity for rolled sections in the handbooks. By using these tables it is possible to find very easily the necessary size of rolled shape to carry a given uniform load over a given span, or to find the carrying capacity of a given beam over a given span.

Tables of safe loads per foot of span given in the handbooks have the merit that the safe bending capacity for any practicable span may be found by dividing the safe load per foot, or the coefficient of strength as it is sometimes called, by the span. The correctness of this is evident from the formula for the total safe load on a uniformly loaded beam,

$$W = \frac{8Sf}{l}$$

where l = the span in inches and S and f are as previously defined. The capacity of spans of a fractional number of feet may be found in this way without interpolation.

Illustrative Problem.—What is the total safe load in pounds of a 7-in., 15.3-lb. I-beam of 9.37-ft. span at a permissible bending stress of 16,000 lb. per sq. in.?

From Cambria, the coefficient of strength, or allowable load for a span of 1 ft. = 110,410 lb.

Safe load for stipulated span,

$$W = \frac{110,410}{9.37} = 11,780 \text{ lb.}$$

In tables of capacity, the weight of the beam is included, so that if the superimposed, or net load is desired, the weight of the beam must be deducted from the quantity taken from the tables.

5. Economic Section for Flexure.—In most cases it is more economical for the resistance of moment, so far as the beam itself is concerned, to select a section of the minimum weight available for a given depth than to use an intermediate or a maximum weight. The addition of area to a section that comes from thickening the web and widening the flanges an equal amount, as is done in widening the rolls to produce the heavier weights of I-beams and channels, is much less effective in increasing the bending capacity than would be the addition of area by increasing the depth. For a rectangular section the section modulus increases as the square of the depth and as the first power of the width. A somewhat similar rule applies to variations in width and depth of rolled sections. From Fig. 2 it is seen for Carnegie I-beams: (1) that the section modulus per square inch of sectional area is in every case greater for the minimum weight than for the maximum weight beam of the same depth, and (2) that this quantity gradually increases with the depth of beams. It is, therefore, economical of steel, so far as moment is concerned, to utilize the deepest practicable sections available.

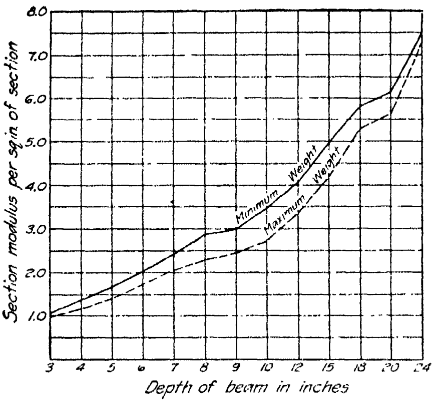


FIG. 2.—Comparison of flexural efficiency of Carnegie I-beams of minimum and maximum weights.

Illustrative Problem.—Compare the bending capacities per square inch of area of the following standard I-beams: 15-in., 45-lb.; 15-in., 55-lb.; 18-in., 48.2-lb.; and 18-in., 54.7-lb.

Beam	Section modulus (in.³)	Area (in.²)	Section modulus per sq. in. of area	Relative efficiencies
15-in., 45-lb.	60.5	13.12	4.61	1.09
15-in., 55-lb.	67.8	16.06	4 22	1.00
18-in., 48.2-lb.	81.9	14.09	5.82	1.38
18-in., 54.7-lb.	88.4	15.94	5.55	1 32

It is thus evident that by increasing the depth of beams from 15 to 18 in. with little or no change in weight, the bending capacity per unit of area or volume is increased at least 32 per cent and that the efficiency of an 18-in., 48.2-lb. I-beam although 12.4 per cent lighter than a 15-in., 55-lb. is 38 per cent greater.

Formerly, only the so-called "standard" sections of I-beams were to be had, but in 1902 in Germany and in 1908 in America the rolling of broad-flanged beams

on the Grey mill began. The Bethlehem series of beams brought out at the latter date was designed to have the same bending strength as standard sections then existing but with generally 10 per cent less weight. This result was attained by thinning the web and widening the flange, thus dispensing with what was generally unnecessary shear material and adding it in the flanges where it was useful for the major requirement, the bending moment. In recent years the rolling of Carnegie "supplementary" beams has made available a highly economical beam so far as flexure is concerned. Care must be taken in using these beams, as also in using Bethlehem beams, to see that the safe stresses in shear and web crippling are not exceeded (see Arts. **16** and **17**).

6. Relative Efficiencies of Beams and Channels.—In selecting a rolled section for a given duty, the respective advantages of the I-beam and the channel should be studied. The fact that I-beams have a larger percentage of their area in the flanges than have channels and a smaller percentage of their area in the web, would indicate the superiority of the I-beam for flexure and the channel for shear and web crippling. If then, the governing stress is a flexural one, the I-beam is best, but if it is shear or web crippling, the channel is best.

The truth of this will be evident from the comparative figures for representative beams and channels listed in Table 1. In the second column the amount of section modulus per square inch of each I-beam is seen to be greater than that for the channel with which it is most naturally compared. On the other hand, in the third column it is seen that the percentage of shear area is higher for channels than for the corresponding I-beams. As is explained in Art. **16,** it is assumed that the entire shear is resisted by the web, taking its area as the depth of the beam d multiplied by the web thickness.

TABLE 1.—RELATIVE FLEXURAL AND SHEARING EFFICIENCIES OF TYPICAL I-BEAMS AND CHANNELS

Section	Section modulus per square inch of total area, $\frac{S}{A}$	Shearing area per square inch of total area, $\frac{dt}{A}$
8-in. 11.5-lb. channel	2.41	0.523
8-in. 18.4-lb. I-beam	2.66	0.405
10-in. 15.3 lb. channel	3.00	0.538
10-in. 25.4-lb. I-beam	3.30	0.420
10-in. 35.0-lb. channel	2.24	0.798
10-in. 35.0-lb. I-beam	2.86	0.581
12-in. 20.7-lb. channel	3.55	0.557
12-in. 31.8-lb. I-beam	3.89	0.453
15-in. 33.9 lb. channel	4.22	0.607
15-in. 42.9-lb. I-beam	4.72	0.492
15-in. 55.0-lb. channel	3.55	0.757
15-in. 55.0-lb. I-beam	4.22	0 606

7. Location of Neutral Axis of Punched Beams.—When holes are punched or drilled through the flanges or web of a beam on the tension side of the neutral axis, some diminution of bending capacity is thereby brought about. It is

customary to assume that if holes on the compression side are filled with rivets, no reduction of strength is occasioned. Such could not be assumed, however, for holes filled with loosely fitting bolts.

Before the effect of holes on the tension side can be calculated accurately, it is necessary to determine the position of the neutral axis of the punched beam. If it be assumed that this axis is at the center of gravity of the right section passing through the rivet holes in question, the moment of inertia and section modulus of the net sect on must then be computed about that axis—a time consuming operation if many cases have to be considered.

There are reasons, however, for believing that the position of the neutral axis does not change greatly by reason of the insertion of a few holes on the tension side even though they be spaced at the minimum practicable distances apart. As the deformation of the beam, and hence the stress variation in it, depends upon

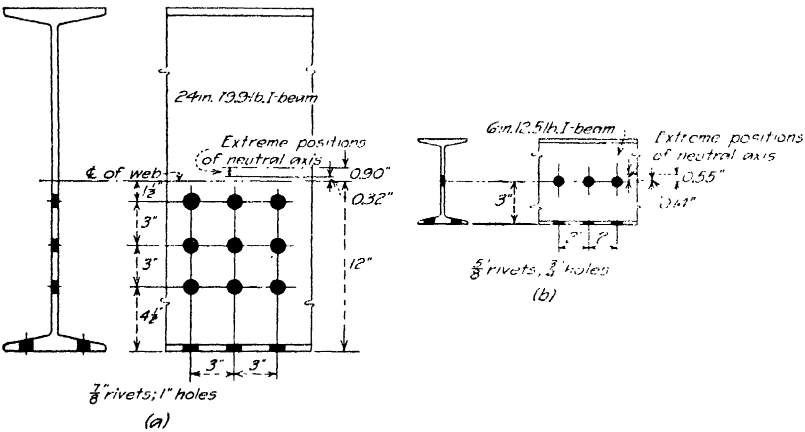


FIG. 3.—Effect of holes on tension side of beams on position of neutral axis.

the gross rather than on the net section, the position of the neutral axis is likely to be affected more by the gross section between the rivet holes than by the net section through them. In order to show the small movement of the neutral axis that results if the predominance of the gross section be given due weight, consider the two extreme cases shown in Fig. 3: One a 24-in., 79.9-lb. I-beam with three horizontal rows of 1-in. holes in the lower half of the web and two rows of holes of the same size in the tension flange, the web and flange holes being opposite; the other a 6-in., 12.5-lb. I beam with a line of 3/4-in. holes along the center line of the web and two lines of holes in the tension flange, the holes being opposite. Assume a longitudinal spacing of 3 in. in the first case, Fig. 3*a*, and of 2 in. in the second case, Fig. 3*b*.

That no important change in the position of the neutral axis is brought about, even in the case of such extreme punchings as those shown, seems probable when one considers the small diminution of volume brought by the insertion of the holes. The volume of the holes in a 3-in. longitudinal section of the 24-in. I is 2.56 cu. in. or 3.7 per cent of the gross volume of the section. If the moment of the volume

of the holes be taken about the neutral axis of the gross section and be divided by the net volume of the 3-in. section, the upward shift of the neutral axis is found to be 0.32 in. For the 6-in., 12.5-lb. I, the loss of volume of a 2-in. length of beam brought about by the punching shown in Fig. 3*b* is 6.1 per cent of the gross volume, and the movement of the neutral axis would be 0.14 in. If the neutral axis is assumed as located at the center of gravity of the net sectional area through a transverse line of holes, this eccentricity would be respectively 0.90 and 0.55 in. for the two beams of Fig. 3.

Just where the neutral axis actually lies at a reduced section it is impossible to say, but no doubt it is somewhere between the two extreme positions found, and rather nearer the neutral axis of the gross section than a consideration of net sectional area alone would indicate. Since in the examples the arrangement of rivets is an extreme one and the beams selected are of minimum-weight sections, in which the effect of punching is relatively important, the change in position of the neutral axis brought about by punching cannot be great in the average case.

In any accurate investigation of stress conditions at a cross section a knowledge of the extreme possible position of the neutral axis is of assistance. To show the alteration in position of the neutral axis brought about by an arrangement of rivet holes as severe as is likely to be commonly encountered in average practice, Table 2 has been computed. The figures therein contained are based on the extreme assumption that the neutral axis depends only on the extent and disposition of the net sectional area at the transverse section in question. Two opposite holes of a diameter equal to the size of the maximum permissible rivet or bolt plus $\frac{1}{8}$ in. are assumed to be located in one flange only. The sections are, in general, the minimum and maximum weights of Carnegie and Bethlehem I-beams and Bethlehem girder beams. From the right-hand column it is seen that the alteration in position of the neutral axis is, in relation to the depth of the beam, a maximum for the shallowest and lightest beams and a minimum for the deepest and heaviest ones. It ranges from 10.5 per cent for a 3-in., 5.7-lb. I to 2.2 per cent for a 30-in., 200-lb. Bethlehem girder beam. The shift is, in general, greater for the lighter weights of any depth of beam than for the heavier weights of the same depth. With the aid of this table it is easy to determine the shift of the neutral axis for any rolled beam with two maximum holes in one flange. If there be but one hole in one flange, the shift in position of the neutral axis ranges from about 45 to 48 per cent of that for two holes in one flange, the former figure applying to very shallow, light beams and the latter to deep and heavy ones. An average of 47 per cent may be safely taken.

8. Proportioning of Punched Beams.—If it be assumed that the neutral axis of a punched beam passes through the center of gravity of the area of the reduced section, the capacity of the beam, even considering rivet holes in the tension side only, may be calculated in the same manner as the capacity of an unpunched beam from the formula $M = Sf$. The section modulus is to be taken for the net section, being equal to the net moment of inertia divided by the distance of the extreme fiber from the neutral axis. For an assigned working stress, the capacity of the beam is consequently reduced in the same ratio as the reduction of section modulus.

This reduction is brought about by two causes. First, while the *depth* of the tensile portion of the section is increased, its area is reduced, since the diminution

TABLE 2.—MAXIMUM ALTERATION IN POSITION OF NEUTRAL AXIS OF ROLLED BEAMS AND GIRDERS CAUSED BY TWO OPPOSITE HOLES IN ONE FLANGE

NOTE: For one hole in one flange, take 47 per cent of the "shift" figures given

Section, Standard and Carnegie I-beams (in.) (lb.)	Dia. of holes = dia. of maximum rivet + ⅛ in.	Shift of neutral axis (in.)	Ratio of shift to depth of beam	Section, Bethlehem I-beams (in.) (lb.)	Dia. of holes = dia. of maximum rivet + ⅛ in.	Shift of neutral axis (in.)	Ratio of shift to depth of beam
3 × 5.7	0.50	0.32	0.105	10 × 23.5	0.875	0.56	0.056
3 × 7.5	0.50	0.23	0.075	10 × 28.5	0.875	0.46	0.046
4 × 7.7	0.625	0.39	0.099	12 × 28.5	0.875	0.58	0.048
4 × 10.5	0.625	0.27	0.068	12 × 36.0	0.875	0.59	0.049
5 × 10.0	0.625	0.45	0.090	15 × 38.0	1.0	0.75	0.050
5 × 14.75	0.625	0.28	0.057	15 × 71.0	1.0	0.69	0.046
6 × 12.5	0.75	0.52	0.086	18 × 48.5	1.0	0.75	0.042
6 × 17.25	0.75	0.35	0.059	18 × 59.0	1.0	0.60	0.034
7 × 15.3	0.75	0.48	0.069	20 × 59.0	1.0	0.75	0.038
7 × 20.0	0.75	0.35	0.050	20 × 82.0	1.0	0.64	0.032
8 × 17.5	0.875	0.56	0.070	24 × 73.0	1.0	0.80	0.033
8 × 18.4	0.875	0.63	0.078	24 × 84.0	1.0	0.75	0.031
8 × 25.5	0.875	0.50	0.063	26 × 90.0	1.125	0.90	0.034
9 × 21.8	0.875	0.69	0.076	28 × 105.0	1.125	0.90	0.032
9 × 35.0	0.875	0.40	0.045	30 × 120.0	1.125	0.89	0.030
10 × 22.4	0.875	0.54	0.054	Bethlehem			
10 × 25.4	0.875	0.64	0.064	girder beams			
10 × 40.0	0.875	0.39	0.039	8 × 32.5	1.0	0.38	0.048
12 × 27.9	0.875	0.60	0.050	9 × 38.0	1.0	0.39	0.043
12 × 31.8	0.875	0.68	0.056	10 × 44.0	1.0	0.42	0.042
12 × 55.0	0.875	0.50	0.042	12 × 55.0	1.125	0.52	0.043
15 × 37.3	0.875	0.55	0.036	12 × 70.0	1.125	0.50	0.042
15 × 42.9	0.875	0.70	0.046	15 × 73.0	1.125	0.56	0.037
15 × 75.0	0.875	0.54	0.036	15 × 140.0	1.125	0.38	0.025
18 × 48.2	1.0	0.67	0.037	18 × 92.0	1.125	0.60	0.033
18 × 54.7	1.0	0.90	0.050	20 × 112.0	1.125	0.61	0.031
18 × 90.0	1.0	0.70	0.039	20 × 140.0	1.125	0.63	0.031
20 × 65.4	1.0	0.82	0.041	24 × 120.0	1.125	0.71	0.029
20 × 100.0	1.0	0.70	0.035	24 × 140.0	1.125	0.65	0.027
21 × 60.4	1.0	0.69	0.033	26 × 150.0	1.125	0.77	0.030
24 × 74.2	1.0	0.72	0.030	26 × 160.0	1.125	0.67	0.026
24 × 79.9	1.0	0.94	0.039	28 × 165.0	1.125	0.79	0.028
24 × 115.0	1.0	0.82	0.034	30 × 180.0	1.125	0.71	0.024
27 × 90.0	1.0	0.79	0.029	30 × 200.0	1.125	0.66	0.022

of section caused by the rivet holes is greater than the increase of section due to the upward shifting of the neutral axis. In the second place, the compression area is lessened, as will be seen from Fig. 4, and the maximum stress developed on the extreme compressive fiber, if the maintenance of a plane section be assumed (as usual), will be only

$$f_c = f_t\left(\frac{c - e}{c + e}\right)$$

where f_t = extreme fiber stress on tensile side; c = distance from center of gravity of gross section to extreme tensile fibers, and e = shift of neutral axis towards compression side due to insertion of holes. In the case of, say, a 24-in., 79.9-lb. I, with two holes in the tension flange,

$$f_c = f_t\left(\frac{12 - 0.94}{12 + 0.94}\right) = 0.855f_t$$

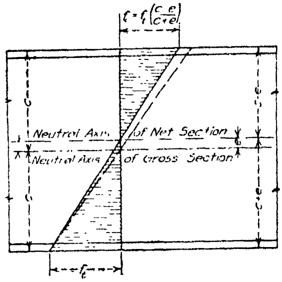


FIG. 4.—Reduction in working stress on extreme compressive fiber due to effect of holes on tension side.

The calculation of the moment of resistance of a typical section on the assumption that the neutral axis passes through the gravity axis of the net section will illustrate the procedure necessary for exact analysis.

Illustrative Problem.—Compute the safe moment of resistance of a 15-in., 73-lb. Bethlehem girder beam with two 1-in. rivets through one flange. f = 16,000 lb. per sq. in. Assume the neutral axis to pass through the center of gravity of the net section.

Statical moment of two $1\frac{1}{8}$-in. holes through $\frac{11}{16}$-in. metal of flange taken about axis through center of section = (2)(1.125)(0.688)(7.16) = 11.10 in.³

Shift of neutral axis = statical moment of holes ÷ net area of section =

$$\frac{11.10}{21.49 - (2)(1.125)(0.688)} = 0.56 \text{ in. above center of web.}$$

Moment of inertia of net section about center of gravity of net section is

$$I + Ae^2 - a\left(\frac{d - g}{2} + e\right)^2$$

where I = moment of inertia of gross section about its own gravity axis, A = gross area of section, e = shift of neutral axis; a = area of two holes; d = depth of beam; g = thickness of beam flanges at rivet lines, or grip. This is

$$883.4 + (21.49)(0.56)^2 - 1.55\left(\frac{15.0 - 0.688}{2} + 0.56\right)^2 = 797.7$$

Section modulus of reduced section = 797.7/(7.5 + 0.56) = 99.0, as compared with 117.8, the section modulus of the gross section, a reduction of 16 per cent.

Safe moment of resistance = Sf = 99.0(16,000) = 1,585,000 in.-lb.

9. Net Section Modulus.—Obviously the accurate computation of the section modulus and moment of resistance of a punched beam on the supposition that the neutral axis is at the center of gravity of the net section is too laborious a task to be often undertaken. To obviate this labor, Table 3 has been computed. This gives generally for the minimum and maximum weights of Carnegie and Bethlehem I-beams, Bethlehem girder beams and standard channels, the percentage of gross section modulus developed with holes in the flanges. The holes are $\frac{1}{8}$ in. larger than the maximum rivet or bolt specified for the section considered in the handbooks. In the case of beams, the punchings assumed are one hole in one

flange or in each flange, and two opposite holes in one flange or in each flange. For channels, one hole is assumed in one or in each flange.

In order to appreciate the relation of the reduction of section modulus on the above hypothesis to that obtained on the assumption that the neutral axis does not move, the percentage reductions for unsymmetrical punchings on the latter basis have been listed parallel with the others. The grip, or thickness of the flanges at the rivet lines has been taken at the fractional values given in the handbooks, and the moment of inertia of the holes about their own gravity axis has been neglected. Since the error involved in these assumptions is unimportant, and may be offset by the selection of somewhat different gages from those assumed, this procedure is justifiable, the more so because of the uncertainty as to just where the neutral axis actually does lie. The diameters of the holes have been taken as the diameter of the maximum rivet or bolt plus ⅛ in.

From a study of Table 3 it is evident that the percentage reduction of section modulus brought about by punching of the flanges is greatest for shallow, light beams and channels and least for deep, heavy ones, ranging, in the case of two holes in one flange, from 41.6 to 10.2 per cent. For beams of a given depth, the reduction is, in general, greatest for the minimum weight beam and least for the maximum weight. Where the punching consists of but one hole in one flange, the percentage reduction is very nearly one-half of that occasioned by two holes in one flange only.

Where the punching is symmetrical—that is, the same number of holes are taken out of each flange—the uncertainty respecting the position of the neutral axis disappears and the computation of net section modulus is thereby greatly simplified. From the table, the fact appears that the percentage loss in section modulus is only slightly more with one hole out of each flange than with one hole out of one flange only. Thus, the percentage of gross section modulus developed in a 9-in., 21.8-lb. I with one ⅞-in. hole out of one flange is 84, assuming the neutral axis at the gravity axis of the net section, whereas with one hole out of each flange it is 81.4, thus showing losses of 16 and 18.6 per cent respectively. Although the loss of area in the second case is double that in the first case, the loss in section modulus is only as 1.16 to 1. The same general principles apply if two holes are taken out of one flange and two out of each flange. The greater weakening accompanying unsymmetrical punching, despite the fact that there are only half as many holes, is due to the important effect of the shifting of the neutral axis.

The importance of the alteration in position of the neutral axis brought about by punching is further exhibited in the relation of the relative percentages of gross section modulus developed on the assumption (1) that the axis moves, (2) that it is stationary. In the latter case the reduction is everywhere less. Thus, for a Bethlehem 18-in., 48.5-lb. I, with two holes out of one flange, the relative percentages are 81.7 and 89.2. If the neutral axis were to take up a position based wholly on the net area through the punched section in question, the common assumption of stationary axis would be seriously in error. It leads to reductions in section modulus that are only from 55 to 65 per cent of what they should be if the assumption of a shifting axis holds. The first figure applies to the shallower and lighter sections and the second to deep and heavy ones. A close approximation to the net section modulus on the basis of the shift theory can be

TABLE 3.—UNCORRECTED PERCENTAGE OF GROSS SECTION MODULUS DEVELOPED BY BEAMS AND CHANNELS WITH HOLES IN FLANGES

Section (in.) (lb.)	Diameter of holes = diameter of maximum rivet + ⅛ in.	Percentage of gross section modulus developed					
		Neutral axis at center of gravity of net section		Neutral axis at center of gravity of gross section			
		One hole in one flange	Two holes in one flange	One hole in one flange	Two holes in one flange	One hole in each flange	Two holes in each flange
Standard and Carnegie I-beams							
3 × 5.7	0.50	78.8	58.4	89.0	78.0	78.0	56.0
3 × 7.5	0.50	85.2	67.8	90.1	80.2	80.2	60.4
4 × 7.7	0.625	80.6	61.4	89.0	78.0	78.0	56.0
4 × 10.5	0.625	86.0	70.3	90.5	81.0	81.0	62.0
5 × 10.0	0.625	82.8	64.4	89.6	79.2	79.2	58.4
5 × 14.75	0.625	86.6	73.0	91.7	83.4	83.4	66.8
6 × 12.5	0.75	82.2	64.4	89.8	79.6	79.6	59.2
6 × 17.25	0.75	86.1	72.3	91.5	83.0	83.0	66.0
7 × 15.3	0.75	85.0	70.1	91.5	83.0	83.0	66.0
7 × 20.0	0.75	88.0	76.2	92.6	85.2	85.2	70.4
8 × 17.5	0.875	85.8	71.2	91.9	83.8	83.8	67.6
8 × 18.4	0.875	83.9	67.3	90.4	80.8	80.8	61.6
8 × 25.5	0.875	85.4	70.7	91.0	82.0	82.0	64.0
9 × 21.8	0.875	81.0	67.6	90.7	81.4	81.4	62.8
9 × 35.0	0.875	88.8	77.5	92.9	85.8	85.8	71.6
10 × 22.4	0.875	88.7	76.8	93.3	86.6	86.6	73.2
10 × 25.4	0.875	86.2	72.4	91.9	83.8	83.8	67.6
10 × 40.0	0.875	90.2	80.0	93.7	87.4	87.4	74.8
12 × 27.9	0.875	89.4	78.1	93.6	87.2	87.2	74.4
12 × 31.8	0.875	87.4	74.7	92.5	85.0	85.0	70.0
12 × 55.0	0.875	89.8	79.3	93.5	87.0	87.0	74.0
15 × 37.3	0.875	91.7	83.2	95.0	90.0	90.0	80.0
15 × 42.9	0.875	89.4	78.5	93.8	87.6	87.6	75.2
15 × 75.0	0.875	91.0	82.1	94.4	88.8	88.8	77.6
18 × 48.2	1.0	91.5	82.6	94.8	89.6	89.6	79.2
18 × 54.7	1.0	88.5	76.8	93.0	86.0	86.0	72.0
18 × 90.0	1.0	90.7	81.4	94.2	88.4	88.4	76.8
20 × 65.4	1.0	90.5	80.5	94.1	88.2	88.2	76.4
20 × 100.0	1.0	91.2	82.5	94.5	89.0	89.0	78.0
21 × 60.4	1.0	92.0	84.5	95.2	90.4	90.4	80.8
24 × 74.2	1.0	92.8	85.4	95.6	91.2	91.2	82.4
24 × 79.9	1.0	90.8	81.7	94.4	88.8	88.8	77.6
24 × 115.0	1.0	92.0	83.6	95.0	90.0	90.0	80.0
27 × 90.0	1.0	93.0	85.8	95.6	91.2	91.2	82.4
Bethlehem I-beams							
10 × 23.5	0.875	88.7	77.5	93.8	87.6	87.6	75.2
10 × 28.5	0.875	90.5	80.4	94.4	88.8	88.8	77.6
12 × 28.5	0.875	89.8	79.5	94.1	88.2	88.2	76.4
12 × 36.0	0.875	89.5	79.0	94.0	88.0	88.0	76.0
15 × 38.0	1.0	89.2	78.4	93.7	87.4	87.4	74.8
15 × 71.0	1.0	90.0	80.0	94.2	88.4	88.4	76.8
18 × 48.5	1.0	90.8	81.7	94.6	89.2	89.2	78.4
18 × 59.0	1.0	92.0	84.1	95.2	90.4	90.4	80.8
20 × 59.0	1.0	91.6	83.1	95.0	90.0	90.0	80.0
20 × 82.0	1.0	92.5	85.2	95.6	91.2	91.2	82.4

TABLE 3 (*Continued*)

Section (in.) (lb.)	Diameter of holes = diameter of maximum rivet + ⅛ in.	Percentage of gross section modulus developed: Neutral axis at center of gravity of net section		Neutral axis at center of gravity of gross section			
		One hole in one flange	Two holes in one flange	One hole in one flange	Two holes in one flange	One hole in each flange	Two holes in each flange
Bethlehem I-beams							
24 × 73.0	1.0	92.5	81.9	95.5	91.0	91.0	82.0
24 × 84.0	1.0	92.8	85.7	95.7	91.4	91.4	82.8
26 × 90.0	1.125	92.0	84.1	95.3	90.6	90.6	81.2
28 × 105.0	1.125	92.6	85.3	95.6	91.2	91.2	82.4
30 × 120.0	1.125	93.2	86.4	95.9	91.8	91.8	83.6
Bethlehem girder beams							
8 × 32.5	1.0	90.3	80.5	91.5	89.0	89.0	78.0
9 × 38.0	1.0	91.0	82.1	95.0	90.0	90.0	80.0
10 × 44.0	1.0	91.4	82.6	95.1	90.2	90.2	80.4
12 × 55.0	1.125	91.0	82.2	95.0	90.0	90.0	80.0
12 × 70.0	1.125	91.5	82.5	95.0	90.0	90.0	80.0
15 × 73.0	1.125	92.0	81.0	95.5	91.0	91.0	82.0
15 × 140.0	1.125	94.0	88.5	96.7	93.1	93.4	86.8
18 × 92.0	1.125	92.8	85.5	95.9	91.8	91.8	83.6
20 × 112.0	1.125	93.3	86.3	96.1	92.2	92.2	84.4
20 × 140.0	1.125	93.5	86.4	96.1	92.2	92.2	84.4
24 × 120.0	1.125	93.5	86.8	96.2	92.1	92.1	84.8
24 × 140.0	1.125	94.1	88.0	96.5	93.0	93.0	86.0
26 × 150.0	1.125	93.5	86.6	96.2	92.4	92.4	84.8
26 × 160.0	1.125	94.3	88.6	96.7	93.4	93.4	86.8
28 × 165.0	1.125	93.7	87.4	96.3	92.6	92.6	85.2
30 × 180.0	1.125	94.5	89.0	96.8	93.6	93.6	87.2
30 × 200.0	1.125	95.0	89.8	97.1	91.2	91.2	88.4
Standard channels							
3 × 4.1	0.625	67.8		82.2		61.4	
3 × 6.0	0.625	77.8		86.0		72.0	
4 × 5.4	0.625	71.1		82.6		65.2	
4 × 7.25	0.625	74.8		83.5		67.0	
5 × 6.7	0.625	75.4		85.8		71.6	
5 × 11.5	0.625	85.6		89.7		79.4	
6 × 8.2	0.75	76.7		85.4		70.8	
6 × 15.5	0.75	82.5		88.4		76.8	
7 × 9.8	0.75	75.8		85.1		70.2	
7 × 19.75	0.75	81.4		89.2		78.4	
8 × 11.5	0.875	75.3		85.0		70.0	
8 × 21.25	0.875	82.5		88.5		77.0	
9 × 13.4	0.875	75.6		85.2		70.4	
9 × 25.0	0.875	82.8		88.8		77.6	
10 × 15.3	0.875	78.4		86.9		73.8	
10 × 35.0	0.875	87.5		91.4		82.8	
12 × 20.7	1.0	79.0		87.1		74.2	
12 × 40.0	1.0	84.5		89.7		79.4	
15 × 33.9	1.0	83.4		89.7		79.4	
15 × 55.0	1.0	87.5		91.8		83.6	

made by computing the percentage of gross section modulus developed on the basis of the stationary axis theory and then reducing it by multiplying it by a factor less than one. An examination of Table 3 shows that this factor should range from about 0.75 to 0.98 depending upon the size of the beam and whether there are one or two holes in each flange.

Where the diameter of the holes is less than the maximum permissible diameter, or the diameter assumed, the percentage loss of section modulus may, with sufficient accuracy be assumed as bearing the same ratio to those tabulated in Table 3 as the new diameter of hole does to the diameter assumed in the table.

Illustrative Problem.—Find the percentage of gross section modulus developed in a Bethlehem 18-in., 59-lb. I-beam with holes for two ¾-in. rivets opposite each other in the tension flange. Assume that the neutral axis is at the center of gravity of the net section.

Effective percentage of gross section modulus with two 1-in. holes (from Table 3) = 84.1, and hence loss of gross section modulus = 15.9 per cent.

Loss of section modulus with two ⅞-in. holes instead of two 1-in. holes = $\frac{0.875}{1.00}$ (15.9) = 13.9 per cent.

Percentage of section modulus effective = 86.1 per cent.

Since the moment of inertia of a portion of the section increases approximately as the square of the distance of its center of gravity from the neutral axis of the beam, and since at the same time holes through the flange remove a larger area of metal, flange holes have very much greater effect in reducing the bending capacity than web holes have. No material error is committed by neglecting the weakening effect of the latter so far as moment is concerned. An example will suffice to show this.

Illustrative Problem.—Determine the reduction of bending capacity, of a 9-in., 21.8-lb. I-beam with two ⅞-in. holes opposite each other in the tension flange and one ⅞-in. hole in the web 2 in. out from the neutral axis with its center on the right section through the flange holes. Assume the neutral axis to be at the center of the web.

Gross S of 9-in., 21.8-lb. I = 18.9

Area of two flange holes, the grip of beam being ½ in. = (2)(0.875)(0.5) = 0.875 sq. in

Distance of center of gravity of flange holes from neutral axis = 4.5 − 0.25 = 4.25 in. Moment of inertia of two holes about axis through center of web, neglecting their moment of inertia about their own gravity axes = $(0.875)(4.25)^2$ = 15.8.

Area of web hole = (0.875)(0.290) = 0.25 sq. in. Approximate moment of inertia of hole = $(0.25)(2)^2$ = 1.0.

Section moduli of 3 holes = (15.8 + 1.0)/4.5 = 3.7, hence net section modulus = 18.9 − 3.7 = 15.2, or 19.6 per cent less than the gross section modulus. This represents the reduction of bending capacity caused by the holes.

The relatively small weakening caused by the web hole indicates that but little error would be committed by neglecting web holes.

10. Correction of Approximate Section Modulus.—In making calculations respecting the bending strength of beams on the basis of the fixity of the neutral axis, it should be assumed that an originally plane cross-section does not remain a plane during flexure, but resolves itself into two planes intersecting at the neutral axis, as shown in Fig. 5*a*. This involves a greater extreme fiber stress at the tension edge than at the compression edge in order that the total tension, T, on the punched half of the beam may equal the total compression, C, on the compression half, as shown in Fig. 5*c*. Dividing the total moment of resistance into

two parts—that contributed by the tensile portion of the cross-section, M_t, and that contributed by the compression portion, M_c, we have

$$M = M_t + M_c = \frac{f_t}{c} \cdot I_t + \frac{f_c}{c} \cdot I_c \tag{1}$$

where f_t and f_c are the extreme fiber stresses on the tensile and the compressive faces respectively, and I_t and I_c are the moments of inertia of the corresponding halves. In computing the total moment of resistance on the basis of this theory, therefore, it is incorrect to assume, as is commonly done, that for beams of the same depth—that is, with f/c constant—it varies directly with the net moment of inertia. Change in the number and arrangement of holes not only affects the value of I_t, but also the value of the coefficient of I_c—that is, $\frac{f_c}{c}$. A reduction of 20 per cent in the moment of inertia of the tension half, therefore, brings

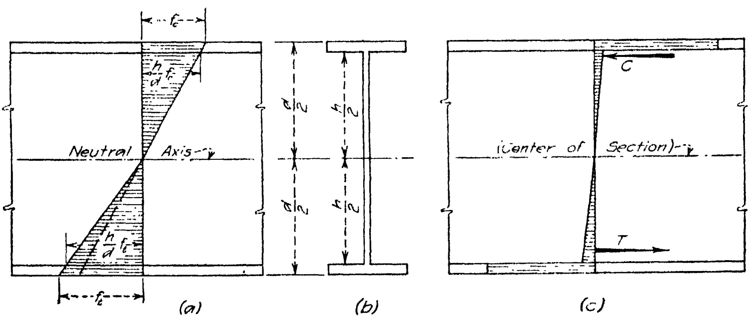


FIG. 5.—Reduction of flexural strength of I-beam due to holes in tension flange—Neutral axis at center of section.

about a reduction of total moment of resistance of *more* than 20 per cent. If there were no holes, the moment of resistance would be

$$M = M_t + M_c = \frac{f}{c} \cdot I_t + \frac{f}{c} \cdot I_c \tag{2}$$

but the punching not only reduces the first term of the right-hand member by 20 per cent, but the second term in the ratio of f_c/f.

The reduction of flexural strength, assuming the neutral axis to remain fixed in position, may be computed for the general case of an I-beam with holes in one flange by substituting an equivalent rectangle for each flange and calculating the necessary increase in extreme fiber stress on the tension face as compared with that on the compression face in order that compensation may be made for the effect of the holes. Let f_c = extreme fiber compressive stress and f_t = extreme fiber tensile stress. If the equivalent section of the beam be as shown in Fig. 5*b*, *h* being the clear distance between flanges and *d* the depth of the beam, the total compression may be written

$$A_f\left[\frac{f_c + \frac{h}{d}f_c}{2}\right] + \frac{A_w}{2}\left[\frac{\frac{h}{d} \cdot f_c}{2}\right] = \tfrac{1}{2} f_c\left[A_f\left(1 + \frac{h}{d}\right) + \tfrac{1}{2} A_w\frac{h}{d}\right]$$

where A_f = gross area of one flange and A_w = gross area of entire web.

The total tension may in like manner be written

$$A_f'\left[\frac{f_t + \frac{h}{d}\cdot f_t}{2}\right] + \frac{A_w}{2}\left[\frac{\frac{h}{d}\cdot f_t}{2}\right] = \tfrac{1}{2} f_t\left[A_f'\left(1 + \frac{h}{d}\right) + \tfrac{1}{2} A_w \cdot \frac{h}{d}\right]$$

where A_f' = net area of one flange.

Equating the total compression and the total tension, there results the relation

$$\frac{f_c}{f_t} = \frac{A_f'\left(1 + \frac{h}{d}\right) + \frac{1}{2} A_w \frac{h}{d}}{A_f\left(1 + \frac{h}{d}\right) + \frac{1}{2} A_w \frac{h}{d}} \tag{3}$$

By taking account of the reduced value of I_t compared with half the moment of inertia of the unpunched beam, or with I_c, and also of the lesser value of the ratio f_c/c as compared with f_t/c, as given by Formula (3), the net moment of resistance in rigid accordance with the arbitrary assumption of fixed neutral axis may be found.

Illustrative Problem.—Calculate the percentage reduction of the moment of resistance of an 18-in., 54.7-lb. I with two 1-in. holes in one flange, assuming the neutral axis as at the center of the web.

Moment of inertia of one-half the gross section = 397.8.

Moment of inertia of two 1-in. holes through ¾-in. flange material =

$$(2)(1)(0.75)(8.625)^2 = 112.0.$$

Net moment of inertia of tension half of beam, I_t = 397.8 − 112.0 = 285.8, or 0.72 I_c.

The gross area of one flange (taking its average thickness) = 4.15 sq. in.; the net area = 2.77 sq. in., the web area = 7.64 sq. in., the average flange thickness = 0.691 in., and the value of h/d = 0.924. Then, the ratio

$$\frac{f_c}{f_t} = \frac{(2.77)(1.924) \times (0.5)(7.64)(0.924)}{(4.15)(1.924) + (0.5)(7.64)(0.924)} = 0.77$$

Formula (1) becomes for the case in point

$$M = \frac{f_t}{c}.0.72I_c + 0.77\frac{f_t}{c}.I_c$$

or, since $I_c = \frac{1}{2} I$, or, half the gross moment of inertia of the entire beam,

$$M = \frac{f_tI}{c}(0.36 + 0.385) = 0.745\frac{f_tI}{c}$$

or 74.5 per cent of what the moment of resistance of the gross section would be with an extreme fiber stress on both upper and lower faces equal to the maximum permissible stress f_t.

Comparing this with the efficiency of the same beam, when computed on the assumption that the neutral axis is at the center of gravity of the net section, it is found by consulting Table 3 that in the latter case it is 76.8 per cent, or somewhat higher than is obtained in accordance with the arbitrary assumption that the neutral axis is fixed at the center of gravity of the gross section. The latter must, therefore, be in error on the side of severity.

To compare the efficiencies developed in accordance with the two common assumptions upon which Table 3 is based with that found by the method illustrated in the last problem, efficiencies have been listed for a number of typical beams and girder beams in Table 4. These beams are assumed to have two maximum holes in one flange.

The efficiencies tabulated in column A are the same as those in column 4 of Table 3 for the same beams and are based on the neutral axis being at the center of gravity of the net section. Those in column B are based on the neutral axis

being at the center of gravity of the gross section, and take into account in accordance with the method illustrated in the last problem, the fact that f_c must be less than f_t. The efficiencies in column C are calculated on the assumption that the neutral axis is at the center of gravity of the gross section and that the fiber stress f_t exists at both compressive and tensile extreme fibers. The figures correspond to those in column 6 of Table 3 for the same beams.

TABLE 4.—CORRECTED FLEXURAL EFFICIENCY OF TYPICAL I-BEAMS WITH TWO MAXIMUM HOLES IN ONE FLANGE

Size of beam, (in.) (lb.)	Diam. of holes (in.)	Percentage of gross flexural capacity developed: Neutral axis at center of gravity of net section A $f_c < f_t$	Neutral axis at center of gravity of gross section B $f_c < f_t$	C $f_c = f_t$	Coefficient "K" by which method "C" results are to be multiplied	Method C results adjusted by coefficient K
Standard I-beams						
6 × 12.5	0.75	64.4	61.4	79.6	0.90	71.6
6 × 17.25	0.75	72.3	68.5	83.0	0.90	74.5
12 × 31.8	0.875	74.7	71.9	85.0	0.95	80.8
12 × 55.0	0.875	79.3	77.0	87.0	0.95	82.6
18 × 54.7	1.0	76.8	74.4	86.0	0.95	81.8
18 × 90.0	1.0	81.4	79.0	88.4	0.95	84.0
24 × 79.9	1.0	81.7	78.9	88.8	0.95	84.4
24 × 115.0	1.0	83.6	81.3	90.0	0.95	85.5
Bethlehem I-beams						
12 × 28.5	0.875	79.5	76.6	88.2	0.95	83.6
18 × 48.5	1.0	81.7	78.8	89.2	0.95	84.6
24 × 73.0	1.0	84.9	82.8	91.0	0.95	86.4
Bethlehem girder beams						
12 × 55.0	1.125	82.2	80.0	90.0	0.95	85.5
18 × 92.0	1.125	85.5	83.6	91.8	0.95	87.2
24 × 120.0	1.125	86.8	84.8	92.1	0.95	87.6
30 × 180.0	1.125	89.0	86.8	93.6	0.95	89.0

It will be seen from this table that if the resistance of the beam were developed in the manner assumed in calculations by method B, that the efficiency would be less than if the neutral axis were to shift to the center of gravity of the net section, which is in itself, according to the principle of least work, an indication that the axis does shift, and that the efficiency cannot be lower than given by method A. It probably does not shift to the center of gravity of the net section by reason of the influence of the gross section on either side of the holes, but it is reasonable to assume that it shifts to a mid-position, so that the actual efficiency developed

is neither so low as methods A or B would indicate nor so high as method C would indicate. Since the employment of the latter method greatly facilitates computation, it is recommended that in supplementing Table 3 it be followed and a correction applied to give results between those given by method A and by method C. If, for beams 6 in. deep and less, the efficiency found by method C be multiplied by the coefficient 0.90 and, for beams over 6 in. deep, including rolled girder beams, it be multiplied by 0.95, the efficiencies so obtained will probably be very close to those actually developed. In the last column of Table 4 the efficiencies found for certain typical beams by this procedure are given, affording a comparison with those obtained by methods A, B and C. For a greater refinement of design than is obtained by the usual arbitrary assumptions, it is recommended that for unsymmetrical punchings the quantities in the fifth and sixth columns of Table 3 be employed, multiplying them first by the appropriate coefficients K from Table 4.

The same correcting coefficient may be applied without material error to beams having only one hole in the tension flange or to beams having reinforcing plates riveted on the flanges.

Illustrative Problem.—A Bethlehem 28-in., 165-lb. girder beam has two $1\frac{1}{8}$-in. holes opposite each other in its tension flange. Estimate the probable efficiency.

Gross moment of inertia of girder beam = 6,562.7.

Moment of inertia of two holes, assuming fixed axis (method C) = $(2)(1.188)(1.125)(13.41)^2 = 480$. Net moment of inertia = 6,562.7 − 480 = 6,082.7.

If the correction factor be 0.95, as suggested in Table 4 for sections of this depth, the probable efficiency is

$$\frac{(0.95)(6{,}082.7)}{6{,}562.7} = 0.88 - 88 \text{ per cent.}$$

This result may be obtained readily from Table 3, if available, by multiplying the efficiency given for this beam in column 6 by the factor 0.95.

11. Limiting Longitudinal Position of Flange Holes.—Where flange holes are located in unreinforced rolled beams at points remote from the center, no attention need be paid to the weakening effect produced by them. Within a certain distance from the center of a beam symmetrically loaded, and from the point of maximum moment of a beam unsymmetrically loaded, the reduction of section modulus should be computed in accordance with one of the methods given above. The exact position of this critical point depends on the nature of the distribution of the loading and on the end conditions. For the common cases of simple beams uniformly loaded, loaded at the third-points, and loaded at the center, the diagram of Fig. 6, similar to one suggested by Henry Kercher in *Engineering News-Record*, May 12, 1921, p. 800, will be found helpful. By means of the curves there given it is possible to determine directly the point in the span length, with respect to the center, at which a given fraction of the section modulus provided at the point of maximum moment would be permissible. If it be desired, for example, to place two holes in the tension flange of a symmetrically-loaded beam at a stated distance from the center, the ratio of the net section modulus at the holes to the section modulus at the center may be found by the methods already described or by the use of tables, and a reference to the appropriate curve will show whether this reduced section modulus would suffice at the proposed location of the holes.

In cases where plates are riveted to the flanges of rolled beams for purposes of reinforcement, it is very convenient to be able to discover readily if the net section of the beam at the point of cut-off of the plates is adequate.

Illustrative Problem.—An 18-in., 48.2-lb. I-beam of 21-ft. span carries a uniformly distributed load. It is desired to drill two 1-in. holes in the bottom flange at a point 3 ft. from the center. Is this permissible? Assume the neutral axis at the center of the web and apply the adjusting factor 0.95 recommended in Table 4.

From Table 3 the percentage of the gross section modulus developed on the assumption made is 89.6 per cent. This corrected = (0.95)(0.896) = 0.852.

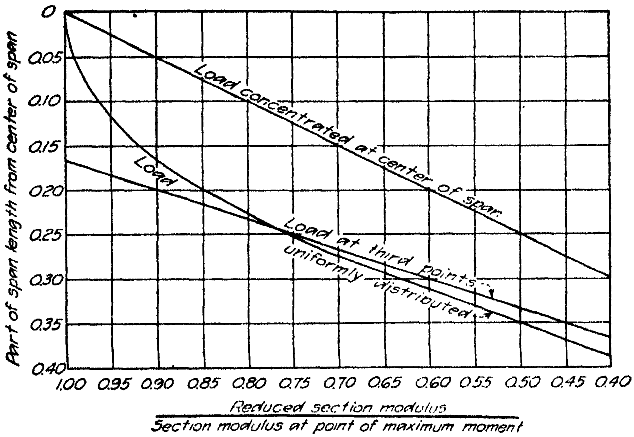


FIG. 6.—Proportion of maximum provided section modulus required at various points of span.

On the curve for uniformly distributed loading in Fig. 6 find the point where the vertical through 0.852 on the horizontal axis intersects the curve, and directly opposite to the left is found 0.20l. The holes may, therefore, not be located closer to the center than (0.20)(21) = 4.2 ft., and hence the proposed location is not permissible, without reinforcement of the section or use of a heavier beam.

12. Reinforcement of Beams for Bending.—Occasionally through the desire to utilize a rolled section that chances to be on hand for a bending moment in excess of its normal capacity, or in order to keep the construction especially shallow, or to strengthen an overloaded beam already in position, a beam is reinforced for flexure by the addition of a plate or plates to each flange. Such procedure is only practicable where the shear requirement does not also demand reinforcement. The bending capacity of the beam is thereby increased in direct ratio to its increase in section modulus. To determine this increase, the moment of inertia of the plates on the compression flange, considered as unpunched, plus the net moment of inertia of the plates on the tension flange is added to the gross moment of inertia of the original section less the moment of inertia of the holes through the tension flange. The total net moment of inertia divided by the distance to the extreme fiber of the reinforced beam will then give the net section modulus. In making this computation it is most convenient to consider the

neutral axis as at the center of gravity of the gross area and then apply whatever correction is deemed reasonable for the section under consideration.

Allowance for rivet holes on the tension side of the beam should be for the most unfavorable arrangement likely to be adopted. A saving in section may, however, be effected by staggering the rivets in the two gage lines of the flange, since for the large spacing obtaining at the point of maximum moment, one flange hole only need then be deducted. While web holes on this critical section should also in strict accuracy be considered, it is generally unnecessary to do so unless there be several of them. If for any reason flange holes are opposite as they will be at the ends of the plate, allowance should be made for two holes in computing the section modulus.

Illustrative Problem.—Calculate the moment of resistance of a 12-in., 31.8-lb. I (Fig. 7) with one 6 × 3/8-in. plate riveted to each flange by 3/4-in. rivets, staggered in two lines. Permissible flexural stress = 16,000 lb. per sq. in.

Gross moment of inertia of I-beam = 215.8.

Approximate moment of inertia of two unpunched plates = (2)(6)(0.375)(6.19)² = 172.4.

Total gross I = 388.2.

Moment of inertia of *one* hole through tension flange, including plate, the grip of the I-beam being 9/16 in., is

$$(1)(0.875)(0.94)(5.91)^2 = 28.8$$

Net moment of inertia of section = 388.2 − 28.8 = 359.4, and adjusted net section modulus (0.95)(359.4)/6.375 = 53.6, applying the correction factor 0.95 recommended in Art. **10** for sections over 6 in. in depth.

Moment of resistance = 53.6 (16,000) = 858,000 in.-lb.

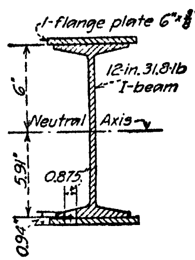


Fig. 7.—Cross-section of reinforced I-beam.

It should be remembered when calculating the section modulus of parts of a compound section that this quantity is the moment of inertia of the part under consideration about the neutral axis of the compound section divided by the distance from the neutral axis to the extreme fiber of the total assemblage of parts existing at the section considered. If, for example, a 12-in., 31.8-lb. I with section modulus of 36.0 is used in a reinforced beam or box girder with one 1/2-in. plate on each flange, the gross section modulus of the beam in the assemblage is

$$\frac{6.00}{6.50}(36) = 33.2.$$

13. Length of Reinforcing Plates.—Because of the lessening moment near the ends, it is unnecessary to carry the reinforcing plates the full length of the beam. They should end, theoretically, at the points nearest the center where the net section modulus of the unreinforced beam is sufficient for the moment requirement. If the total loading is uniformly distributed, the position of these points may be found readily by either graphical or analytical means.

Since for a beam carrying a uniformly distributed load, the moment varies as the ordinates to a parabola with vertex at the center of the span and axis vertical, the required section modulus must vary in precisely the same manner, given a constant working stress in flexure. The truth of this is evident from the formula

$$S = \frac{M}{f}$$

It is possible, therefore, to plot, as has been done in Fig. 8, a curve of required section modulus at each point of the half span and, to superimpose on this, a diagram of provided section modulus. Such diagram may be prepared for any system of loading, although that of Fig. 8 is for uniform loading. The maximum ordinate S represents the required net section modulus of the reinforced beam at the point of maximum moment, and the depths of the various strips required to cover the area beneath the curve of required section modulus represent the net section moduli of the punched beam and the successive flange plates with respect to the total cross-section existing at the point. The minimum length of these strips required to cover completely the area mentioned may be easily scaled from the diagram. The theoretical length, x, of any flange plate is twice the scaled distance from the center of the span to the point where the inner (lower) horizontal bounding line for the appropriate strip cuts the curve. To this length a certain addition is made, for reasons set forth below.

If it happens that by reason of staggered rivet spacing in the central portion of the reinforcing plates (or for any other cause), the net section modulus, s_3' of the primary beam is less near the ends of the flange plates (where the rivets will be opposite each other) than the net section modulus, s_3, near the center, the upper horizontal bounding lines for the successive strips should be jogged down inside the end of the first (outside) flange plate, as shown in Fig. 8. For approximate work the net section modulus of the beam may be taken as the same throughout—that is, at the least value—and the jogging of the horizontal bounding lines thereby obviated. In general, the strip representing the net section modulus of the pair of outside flange plates will be wider than is actually required to cover the remaining area of the diagram, but this does not affect the graphical determination of length of the outside flange plates.

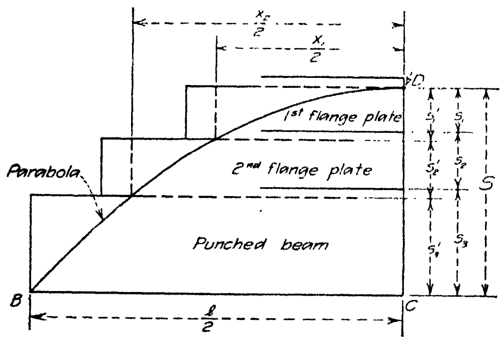


FIG. 8.—Determination of length of flange plates for reinforced steel beams.

To determine the theoretical lengths by analytical means, let $\frac{x_1}{2}$, $\frac{x_2}{2}$, etc., Fig. 8, be the required half lengths of the flange plates numbered consecutively from the outside. Let the actual effective section modulus of the punched beam at the center be s_3 with respect to the maximum section and at the ends of the flange plates be s_3' with respect to the beam section alone. Let s_2 be the actual effective section modulus at the center of the span for the two flange plates in immediate contact with the punched beam with respect to the maximum section, and s_2' be the corresponding quantity at the ends of these plates with respect to an assemblage consisting of the beam and these two plates. Let s_1 be the *required* net section modulus of the two outer (first) flange plates with respect to the maximum section, and s_1' be the required net section modulus of these plates at their

ends. Let S be the total required net section modulus at the center of the reinforced beam. The relation of these quantities will be clear from Fig. 8. Then, since the curve BD is a parabola

$$\frac{\left(\frac{x_1}{2}\right)^2}{\left(\frac{l}{2}\right)^2} = \frac{s_1'}{S}$$

or

$$x_1 = l\sqrt{\frac{s_1'}{S}}$$

In a similar manner the length of any plate may be derived; thus,

$$x_2 = l\sqrt{\frac{s_1' + s_2'}{S}}$$

$$x_n = l\sqrt{\frac{s_1' + s_2' + \ . \ . \ . \ s_n'}{S}}$$

To ensure that the reinforcing plates are able to take stress at the points where they are first needed, they should be carried past the points of theoretical ending a distance sufficient to accommodate at least two transverse rows of rivets. The addition of 9 in. at each end of the flange plates will make this possible, for the close spacing of rivets usually adopted at the ends of the reinforcing plates. Closer spacing at the ends of the plates than at the center is desirable since the increment of flange stress per lineal inch of girder is greater near the ends than near the center, and since it is well to transfer stress to the flange plates as near their ends as possible in order that they may be fully and uniformly stressed at the points where they are most needed.

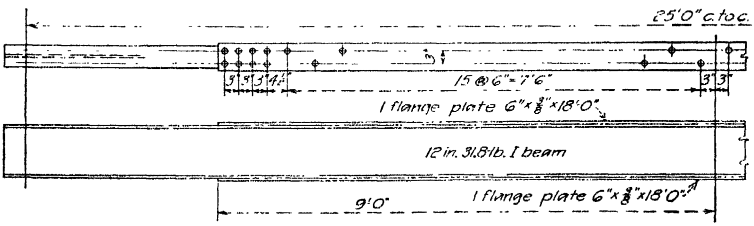


FIG. 9.—Details of flange reinforcement for I-beam.

Illustrative Problem.—Determine the theoretical and practical lengths of the reinforcing plates in the problem of Art. **12**, if the span is 25 ft.

Net section modulus required at center, with one ⅞-in. flange hole out = 53.6.

Effective section modulus of 12-in., 31.8-lb. I with *two* ⅞-in. holes out of tension flange, according to Table 4 = (0.808)(36) = 29.1.

Required net section modulus s_1', of two plates to satisfy requirements at the ends of these plates = 53.6 − 29.1 = 24.5.

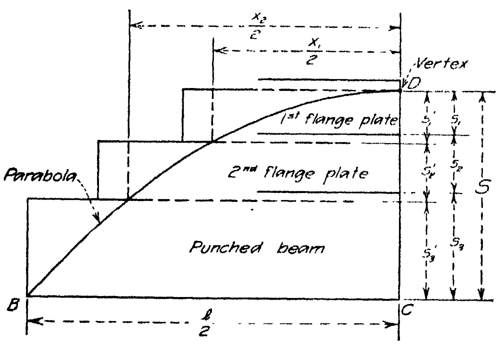
Theoretical length of flange plates required,

$$x = l\sqrt{\frac{s_1'}{S}} = 25\sqrt{\frac{24.5}{53.6}} = 16.9 \text{ ft.}$$

By making the plates 18 ft. long, or a little more than 6 in. longer at each end, two rows of rivets, 3 in. apart, may be driven outside the theoretical point of cut off and at the same time sufficient end distance allowed for the plate. The relation of the plates to the primary beam may be seen in Fig. 9.

It is possible, therefore, to plot, as has been done in Fig. 8, a curve of required section modulus at each point of the half span and, to superimpose on this, a diagram of provided section modulus. Such diagram may be prepared for any system of loading, although that of Fig. 8 is for uniform loading. The maximum ordinate S represents the required net section modulus of the reinforced beam at the point of maximum moment, and the depths of the various strips required to cover the area beneath the curve of required section modulus represent the net section moduli of the punched beam and the successive flange plates with respect to the total cross-section existing at the point. The minimum length of these strips required to cover completely the area mentioned may be easily scaled from the diagram. The theoretical length, x, of any flange plate is twice the scaled distance from the center of the span to the point where the inner (lower) horizontal bounding line for the appropriate strip cuts the curve. To this length a certain addition is made, for reasons set forth below.

If it happens that by reason of staggered rivet spacing in the central portion of the reinforcing plates (or for any other cause), the net section modulus, s_3' of the primary beam is less near the ends of the flange plates (where the rivets will be opposite each other) than the net section modulus, s_3, near the center, the upper horizontal bounding lines for the successive strips should be jogged down inside the end of the first (outside) flange plate, as shown in Fig. 8. For approximate work the net section modulus of the beam may be taken as the same throughout—that is, at the least value—and the jogging of the horizontal bounding lines thereby obviated. In general, the strip representing the net section modulus of the pair of outside flange plates will be wider than is actually required to cover the remaining area of the diagram, but this does not affect the graphical determination of length of the outside flange plates.



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$$\frac{\left(\frac{x_1}{2}\right)^2}{\left(\frac{l}{2}\right)^2} = \frac{s_1'}{S}$$

or

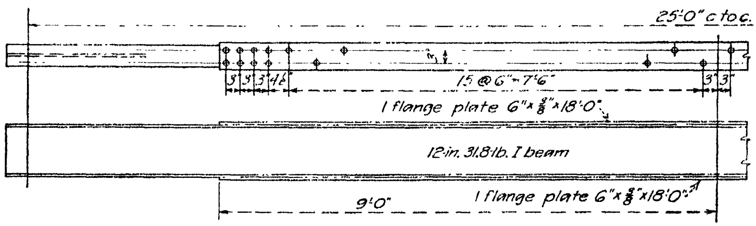
$$x_1 = l\sqrt{\frac{s_1'}{S}}$$

In a similar manner the length of any plate may be derived; thus,

$$x_2 = l\sqrt{\frac{s_1' + s_2'}{S}}$$

$$x_n = l\sqrt{\frac{s_1' + s_2' + \cdots s_n'}{S}}$$

To ensure that the reinforcing plates are able to take stress at the points where they are first needed, they should be carried past the points of theoretical ending a distance sufficient to accommodate at least two transverse rows of rivets. The addition of 9 in. at each end of the flange plates will make this possible, for the close spacing of rivets usually adopted at the ends of the reinforcing plates. Closer spacing at the ends of the plates than at the center is desirable since the increment of flange stress per lineal inch of girder is greater near the ends than near the center, and since it is well to transfer stress to the flange plates as near their ends as possible in order that they may be fully and uniformly stressed at the points where they are most needed.



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Required net section modulus s_1', of two plates to satisfy requirements at the ends of these plates = 53.6 − 29.1 = 24.5.

Theoretical length of flange plates required,

$$x = l\sqrt{\frac{s_1'}{S}} = 25\sqrt{\frac{24.5}{53.6}} = 16.9 \text{ ft.}$$

By making the plates 18 ft. long, or a little more than 6 in. longer at each end, two rows of rivets, 3 in. apart, may be driven outside the theoretical point of cut off and at the same time sufficient end distance allowed for the plate. The relation of the plates to the primary beam may be seen in Fig. 9.

14. Riveting of Reinforcing Plates.—For the adequate attachment of the plates to the primary beam, enough rivets should be employed to develop between the ends and the center of the span, the total stress that exists in the plates at that point. It is sufficiently accurate to assume that the extreme fiber stress obtains over the entire net cross-section of the plate at the span center, although at the fibers nearest the neutral axis, the intensity of stress is somewhat less than at the extreme fibers. The total force to be developed between the end and the center of a tension reinforcing plate will, therefore, be the net area multiplied by the stipulated extreme fiber stress. On the compression side the area of the plates will not be reduced by rivet holes and hence if the plates be the same as on the tension side, the average stress over the plate section will be lower, but the total force to be provided for will be the same. The number of rivets found for the tension side will, therefore, apply also to the compression side. So few are they in relation to the length of the plates generally needed that the actual spacing is usually dictated by practical requirements such, for example, as that rivet spacing in the line of stress must not exceed 16 times the thickness of the outside plate nor 6 in. in any case.

Illustrative Problem.—Determine the correct rivet spacing in the reinforcing plates of the preceding problem, if the safe shearing and bearing stresses on rivets are 10,000 and 20,000 lb. per sq. in. respectively, and the spacing in the line of stress must not exceed 16 times the thickness of the reinforcing plate, nor 6 in.

Net section of one 6 × 3/8-in. plate = [6 − 1(0.875)](0.375) = 1.92 sq. in. Hence safe tensile strength = (1.92)(16,000) = 30,700 lb.

Least value of 3/4-in. rivet bearing on 3/8-in. plate or 9/16-in. flange of beam is the single shear value = (0.44)(10,000) = 4,400 lb.

Number of rivets required in one plate from end to center to develop stress at center = 30,700/4,400 = 7. This would give an average spacing of about 1½ ft. even though the rivets are staggered, and hence the rule that the spacing must not exceed $16t$ = (16)(3/8) = 6 in. becomes operative. Beginning at a point 3 in. on each side of the center line, the rivets will be staggered 6 in. apart for 7 ft. 6 in. and then a closer spacing will be used for the remaining distance, as shown in Fig. 9. This closer spacing is desirable since the increment of flange stress per lineal inch is much greater near the ends of the girder than near the center.

15. Proportioning for Flange Buckling.—In proportioning beams for bending moment, it is unsafe to use without reduction, the permissible extreme fiber stress given in specifications if the compression flange is unsupported laterally for a distance exceeding about 10 or 15 times the width of flange. This source of weakness arises from the fact that the compression flange acts in some measure as a column and, like a column, tends to fail by buckling sidewise between points of lateral support. It is not free to do so, however, to the same extent as a column, by reason of the fact that both the web and the tension flange tend, with an effect that depends largely on the depth of the beam and the web thickness, to hold it in a straight line. If the web is very deep and thin, it obviously could afford relatively small support to the compression flange and could transmit but little support from the tension flange.

The compression flange differs from a column also in the manner of loading. A column receives its loading at one point, or a group of points, near the top. If the load be concentric, there is, then, neglecting the buckling stress, a uniformly distributed stress over the cross section from one end to the other. The com-

pression flange of a beam, however, receives its loading at innumerable points from one end to the dangerous section, or if it be a restrained beam, from the point of inflection to the dangerous section.

Early efforts to formulate rules for reducing the permissible flexural stress to compensate for the buckling tendency in compression flanges were based upon unsatisfactory experimental evidence, as was pointed out by R. Fleming in an excellent discussion of the subject in Engineering News, April 6, 1916.

The Pencoyd rule, adopted in the Bethlehem, Jones & Laughlin, Phoenix and several other handbooks was based more on judgment than on actual tests. As will be seen from Table 5, no reduction of bending strength is required by this rule where the ratio of unsupported length to width of compression flange does not exceed 20. A reduction of 10 per cent for each additional 10 flange-widths of length is recommended up to 70 flange-widths where the table stops. This rule is the most lenient in common use.

A basis for the Cambria formula, $p = \dfrac{18{,}000}{1 + \dfrac{l^2}{3{,}000\,b^2}}$, given in Table 5, was found by considering the top flange a strut of rectangular cross section of width b, sufficiently supported by the web and tension flange to warrant a fiber stress of 18,000 lb. per sq. in. for short lengths. Applying the Rankine formula for fixed ends, $p = \dfrac{18{,}000}{1 + \dfrac{l^2}{36{,}000\,r^2}}$, and replacing r^2 by $\dfrac{b^2}{12}$, the Cambria formula was derived. According to this formula, a fiber stress of 16,000 lb. per sq. in. would be attained at a value of $\dfrac{l}{b} = 19.37$ and therefore no reduction is necessary with ratios of less than approximately 20 flange-widths.

The Carnegie formula, recommended since 1913, $p = 19{,}000 - 300\dfrac{l}{b}$ is an approximate application of the column formula $p = 19{,}000 - 100\dfrac{l}{r}$ to the compression flange, considering it as a rectangle. Reduction begins at 10 flange-widths and no beam is allowed to have an unsupported length exceeding 40 flange-widths, a restriction of greater severity than had till that time been customarily imposed.

Two characteristic reduction formulas for railway bridge work are; the American Railway Engineering Association formula, $p = 14{,}200 - 200\dfrac{l}{b}$ and the Canadian Engineering Standards Association formula, $p = 16{,}000 - 200\dfrac{l}{b}$. No maximum allowable value of $\dfrac{l}{b}$ without reduction is specified in either case, but beams of greater length than 20 ft. without lateral support are not allowed. The first is included in Table 5 for purposes of comparison, although it is intended only for plate girder flanges. The second applies to either.

R. Fleming recommends in Engineering News-Record, Feb. 24, 1921, that the permissible flexural stress for beams over 10 in. deep, and for plate girders without cover plates, be $p = 19{,}000 - 250\dfrac{l}{b}$, while for beams 10 in. deep, and

TABLE 5.—PERMISSIBLE EXTREME FIBER STRESSES IN STEEL BEAMS WITH UNSUPPORTED COMPRESSION FLANGES

(lb. per sq. in.)

l = unsupported length in inches, b = breadth of flange in inches

Rule or formula	Maximum allowable $\frac{l}{b}$ without reduction	Ratio of unsupported length to flange width									Maximum allowable $\frac{l}{b}$ specified
		15	20	25	30	40	50	60	70	80	
Bethlehem, Jones & Laughlin, Phoenix	20.0	16,000	16,000	15,200	14,400	12,800	11,200	9,600	8,000	Table stops	None
Cambria: $p = \dfrac{18,000}{1 + \dfrac{l^2}{3,000b^2}}$	19.37	16,000	15,880	14,900	13,850	11,740	9,820	8,180	6,840	5,750	Table carried to 110
Carnegie: $p = 19,000 - 300\frac{l}{b}$	10.0	14,500	13,000	11,500	10,000	7,000	.....	.....	.....	.....	40
A.R.E.A.: $p = 14,200 - 200\frac{l}{b}$ for plate girders	.....	11,200	10,200	9,200	8,200	Max. unsupported length = 20 ft.					None
C.E.S.A.: $p = 16,000 - 200\frac{l}{b}$ for beams or girders	....	13,000	12,000	11,000	10,000	Max. unsupported length = 20 ft.					None
Fleming: $p = 19,000 - 250\frac{l}{b}$ for beams not over 10 in. and plate girders without cover plates	12.0	15,250	14,000	12,750	11,500	9,000	6,500	.....	.....	.....	50
Fleming: $p = 19,000 - 225\frac{l}{b}$ for beams 10 in. and under and plate girders with cover plates	13.3	15,630	14,500	13,440	12,250	10,000	7,750	.....	.....	.....	50
Young: $p = 16,000 - 120\frac{l}{b}$	.....	14,200	13,600	13,000	12,400	11,200	10,000	.....	.....	.....	50

under, and for plate girders with covers, $p = 19{,}000 - 225\frac{l}{b}$ is recommended. If the compression flange is rigidly held at the ends, the length to be taken is ¾ the span.

A valuable study of flange buckling made by H. F. Moore was reported in 1913 in Bulletin 68 of the University of Illinois Engineering Experiment Station. From theoretical considerations and from a study of all available tests, Mr. Moore concluded that the ultimate fiber stress for steel I-beams not restrained against sidewise buckling of the compression flange, is given by the formula $f_1 = 40{,}000 - 60\frac{ml}{r'}$, where f_1 = extreme fiber stress in lb. per sq. in. computed by the usual flexure formula, l = span of beam in inches, r' = radius of gyration of the I-section about a gravity axis parallel to the web, and m = a coefficient dependent upon the end conditions and method of loading, ml being an equivalent column length. Values of m vary from 1.0 for a cantilever beam with an end load to 0.25 for a fixed-ended beam with a mid-point load. For a simple beam with uniform load, it is 0.667, while for a simple beam with a single concentrated load at any point of the span, it is 0.50.

From the above formula for ultimate flexural stress, it is possible to derive a working formula with a fairly satisfactory basis in actual experimental results. If r' be considered as equal to 0.20 b, which is a very close average value, and the maximum listed value of m be taken, that is 1, then the formula for ultimate buckling strength becomes $f_1 = 40{,}000 - 300\frac{l}{b}$. Applying a factor of safety of 2½—a reasonable one in column design and sufficiently severe in the present case if m be taken at its extreme value of 1—there results the working formula $p = 16{,}000 - 120\frac{l}{b}$. The test results indicate that such a formula should be applied for all ratios of $\frac{l}{b}$ up to the arbitrary limit for the ratio set by good practice. It is recommended that the latter do not exceed 50.

In Table 5 the permissible stresses on compression flanges given by the formulas discussed above are listed for values of $\frac{l}{b}$ from 15 to the upper limit allowed. From this it is seen that for short unsupported lengths, disregarding the A.R.E.A. formula which applies to plate girders only, the C.E.S.A. formula is the most severe, while for long lengths the Carnegie formula is the most severe. In the case of the A.R.E.A. formula, considerable weight was given to recent experimental evidence of low strength of short length columns and to the fact that the top flanges of through plate girder spans may be indifferently braced. In view of Moore's investigations, the low values given by the Carnegie formula for high width ratios appear unwarranted. In the case of the recommended formula, the necessity for applying the reduction to beams without lateral support for all values of $\frac{l}{b}$ from zero to 50, might appear to be severe for beams with width ratios deour 10 or 15, for which some formulas permit the specified working stress withnut reduction. In practice, however, if a beam is supported at intervals of less than 10 or 15 times the flange width, it is likely to be supported continuously,

by flooring or other construction furnishing effective restraint against lateral buckling, and justifying the employment of the full working stress.

To apply the recommended formula when using tables of safe capacity of beams based on an extreme fiber stress of 16,000 lb. per sq. in., the tested loads may be multiplied by the following reduction factors:

$\frac{l}{b}$	Reduction factor	$\frac{l}{b}$	Reduction factor
0	1.000	30	0.774
10	0.924	35	0.737
15	0.887	40	0.700
20	0.850	45	0.662
25	0.812	50	0.625

Illustrative Problem.—A 20-in., 65.4 lb. girder of 21-ft. span carries a load of 20,000 lb. at each of the third points. If the concentrated loads be considered as including the weight of the girder and there be no lateral restraint between the two points of loading, express an opinion as to the safety of the girder if $p = 16{,}000 - 120\frac{l}{b}$.

Moment $= (20{,}000)(7)(12) = 1{,}680{,}000$ in.-lb.
Extreme fiber stress $= M/S = 1{,}680{,}000/116.9 = 14{,}400$ lb. per sq. in.
Permissible stress by formula

$$p = 16{,}000 - (120)(84)/6.25 = 14{,}390 \text{ lb. per sq. in.}$$

As this is almost exactly equal to the existing stress the beam is safe.

Illustrative Problem.—If the total capacity of an 18-in., 60-lb. I, 17 ft. in span, is listed in a handbook as 58,700 lb. with a fiber stress of 16,000 lb. per sq. in., find the safe capacity assuming that the beam is without lateral support between its bearings. $p = 16{,}000 - 120\frac{l}{b}$.

$$\text{Width ratio, } \frac{l}{b} = (17)(12)/6.087 = 33.5$$

Interpolating in the table of reduction factors, above, 74.8 per cent of the tabular load should be regarded as safe, or $(0.748)(58{,}700) = 43{,}900$ lb.

Wherever a beam must be employed with any considerable portion of its length without lateral support, it is advantageous in order to reduce the flange buckling stress to select a section with a relatively wide flange. By so doing a higher permissible flexural stress may be used than for beams with narrower flanges and an important economy effected.

Care must be taken not to assume a beam as supported against lateral buckling unless the lateral restraint is known to be effective. Separators between the webs of I-beams cannot be regarded as fully supporting the compression flange, particularly if they be of the gas-pipe type. Tie rods and sag rods have small value in preventing the compression flange from buckling under overload.

16. Proportioning for Shear.—When a rolled beam or channel is proportioned to be sufficiently strong in flexure, it will generally be adequate also for both vertical and horizontal shearing stresses, so that there is usually no necessity of investigating the shearing capacity of the beam. If, however, the span of the beam be short in relation to its depth and it is loaded to capacity in bending, the shearing stresses may be excessive. Such beams should consequently be investigated for shear.

Analysis of the internal stresses in a beam[1] shows that the intensity of either the vertical or the horizontal shearing stress at any point on any cross section may be expressed by the formula

$$v = \frac{QV}{It} \tag{1}$$

where Q = the statical moment of the area as one side of the point considered, about the neutral axis; V = total vertical shear; I = moment of inertia of the section; and t = thickness of section at the point. It has also been established that for an I-section, by far the greater part of the total vertical shear is resisted by the web and that no material error is committed by considering that the web resists all of it.

16*a*. Vertical Shearing Stress.—While for many purposes it is sufficiently accurate to assume the total vertical shear to be uniformly distributed over the web, considering the web area to be the extreme depth of the beam multiplied by the web thickness, the error involved, which is on the side of weakness, must be offset by the use of a low working stress in shear. In cases of close designing or investigation of seriously overloaded beams, the design for shear should be on the basis of the correct theory.

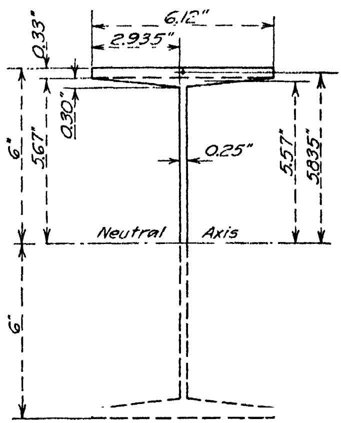


Fig. 10.—Calculation of true shearing stress at neutral axis of an I-beam.

Illustrative Problem.—Find the maximum intensity of the shearing stress on the web of a 12-in., 28.5-lb. Bethlehem I-beam of 12-ft. span carrying a total uniformly distributed load of 32,000 lb., and compare it with the average stress assuming the total shear to be uniformly distributed over the web.

As the maximum stress will be at the neutral axis, Q is to be taken for half the cross-sectional area.

Q for flange rectangle (Fig. 10) = (6.12)(0.33)(5.835)	=	11.80
Q for 2 flange triangles = (2.935)(0.30)(5.57)	=	4.90
Q for half web = (5.67)(0.25)(2.835)	=	4.02
Total statical moment	=	20.72
Moment of inertia, I of beam	=	216.2
End shear = 32,000/2	=	16,000 lb.

Shearing stress, horizontal or vertical, at neutral axis

$$v = \frac{(20.72)(16{,}000)}{(216.2)(0.25)} = 6{,}130 \text{ lb. per sq. in.}$$

Assuming the total shear as uniformly distributed over the web

$$v = \frac{16{,}000}{(12)(0.25)} = 5{,}333 \text{ lb. per sq. in.}$$

The actual stress is, therefore, 15.0 per cent in excess of the average.

To simplify the design of beams for shear by the accurate formula, Table 6 has been prepared. It contains for a wide range of rolled I-beams, girder beams and channels the following quantities:

[1] See Sec. 1, Art. 51.

Table 6.—Relation of True Maximum Shearing Stress to Average Shearing Stress for Rolled Beams, Girders and Channels

$$\text{Maximum stress, } v_m = \frac{Q}{It} \cdot V$$

$$\text{Average stress, } v_a = \frac{1}{dt} \cdot V$$

Section (in.) (lb.)	Statical moment of ½ gross area about neutral axis, Q	$\frac{Q}{It}$	$\frac{1}{dt}$	Excess of v_m over v_a (per cent)
Standard and Carnegie I-beams				
3 × 5.7	0.96	2.26	1.96	15.3
3 × 7.5	1.16	1.15	0.96	20.3
6 × 12.5	4.1	0.83	0.73	13.9
6 × 17.25	5.2	0.43	0.36	19.6
7 × 15.3	5.9	0.66	0.57	14.9
7 × 20.0	7.2	0.38	0.32	19.8
8 × 17.5	8.1	0.63	0.57	11.5
8 × 18.4	8.2	0.53	0.46	14.8
8 × 25.5	10.3	0.28	0.24	20.1
9 × 21.8	10.8	0.44	0.38	14.9
9 × 35.0	15.2	0.19	0.15	23.0
10 × 25.4	14.0	0.37	0.32	15.0
10 × 40.0	19.4	0.17	0.14	22.8
12 × 31.8	20.8	0.28	0.24	15.6
12 × 55.0	32.5	0.13	0.10	21.8
15 × 42.9	34.3	0.19	0.16	16.2
15 × 75.0	55.7	0.093	0.077	21.4
18 × 54.7	51.6	0.14	0.12	16.9
18 × 90.0	83.0	0.083	0.070	18.6
20 × 65.4	68.5	0.12	0.10	17.2
20 × 100.0	99.4	0.069	0.057	20.5
24 × 79.9	101.4	0.097	0.083	16.2
24 × 100.0	119.1	0.067	0.056	20 4
27 × 90.0	126.2	0.082	0.07	15.4
Bethlehem I-beams				
10 × 23.5	14.1	0.46	0.40	15.0
10 × 28.5	15.9	0.30	0.26	18.2
12 × 28.5	20.7	0.38	0.33	15.0
12 × 36.0	25.9	0.31	0.27	15.4
15 × 38.0	34.0	0.27	0.23	15.3
15 × 71.0	62.9	0.15	0.13	18.7
18 × 48.5	49.2	0.19	0.17	10.8
18 × 59.0	56.3	0.13	0.11	15.0
20 × 59.0	68.0	0.15	0.13	15.9
20 × 82.0	92.1	0.10	0.088	18.1
24 × 73.0	101.0	0.12	0.11	15.8
24 × 84.0	115.9	0.106	0.091	16.7
26 × 90.0	133.4	0.098	0.083	17.4
28 × 105.0	167.0	0.083	0.072	16.2
30 × 120.0	204.2	0.072	0.062	16.9
Bethlehem girder beams				
8 × 32.5	16.4	0.50	0.43	14.9
9 × 38.0	21.7	0.42	0.37	14.2
10 × 44.0	27.8	0.37	0.32	13.5
12 × 55.0	41.0	0.26	0.23	14.3
12 × 70.0	51.9	0.21	0.18	15.6
15 × 73.0	67.5	0.18	0.16	14.8
15 × 140.0	125.8	0.10	0.083	18.5
18 × 92.0	101.3	0.13	0.12	14.6
20 × 112.0	134.7	0.10	0.091	14.8
20 × 140.0	169.4	0.090	0 078	15.3
24 × 120.0	172.9	0.090	0.079	14.9
24 × 140.0	201.2	0.080	0.070	14.7
26 × 150.0	229.6	0.071	0.061	15.7
26 × 160.0	248.8	0.070	0.061	14.9
28 × 165.0	271.9	0.063	0.054	16.0
30 × 180.0	317.1	0.056	0.048	15.9
30 × 200.0	353.3	0.052	0.044	16.0
Standard channels				
3 × 4.1	0.65	2.40	1.96	22.1
3 × 6.0	0.86	1.15	0.93	24.5
6 × 8.2	2 6	0 99	0.83	18.5
6 × 15.5	4.2	0.38	0.30	28.6
7 × 9.8	3.6	0.80	0.68	18.0
7 × 19.75	6.1	0.30	0.23	29.8
8 × 11.5	4.8	0.67	0.57	18.4
8 × 21.25	7.6	0.28	0.22	28.7
9 × 13.4	6.2	0.57	0.48	18.3
9 × 25.0	10.1	0.23	0.18	28.4
10 × 15.3	7.9	0.49	0.42	18.2
10 × 35.0	15.2	0.16	0.12	31.3
12 × 20.7	12.7	0.35	0.29	21.0
12 × 40.0	21.2	0.14	0.11	29.0
15 × 33.9	25.2	0.20	0.17	21.0
15 × 55.0	36.9	0.11	0.082	28.9

(1) The statical moment, Q, of one-half of the gross area of the section about the neutral axis.

(2) The coefficient $\frac{Q}{It}$, by which the total vertical shear, V, is to be multiplied to give the maximum shearing stress, v_m, according to the exact theory.

(3) The coefficient $\frac{1}{dt}$, by which the total vertical shear, V, is to be multiplied to give the approximate, or average, shearing stress, v_a, assuming that the total shear, V, is uniformly distributed over the web area dt.

(4) The percentage by which the true maximum shearing stress exceeds the approximate or average shearing stress.

While in general only two places of decimals have been given for the coefficients $\frac{Q}{It}$ and $\frac{1}{dt}$, the percentages in the last column were calculated from coefficients with three places of decimals. The sections for which the quantities have been listed are, in most cases, the minimum and maximum weights rolled for each depth. The values for intermediate sections can be approximated by interpolating in the following manner:

Where sections are of the same section index or number, being produced by the same rolls through slightly varying their distances apart, the figures for sections of the same depth as those listed, but of intermediate weight, may be determined on the basis of relative weights. Thus, the widening of the rolls a distance Δt, increases the statical moment by

$$\Delta Q = \tfrac{1}{2}\, \Delta t \left(\frac{d}{2}\right)^2$$

where d is the depth of the beam, or decreases it if Δt be regarded as the narrowing of the rolls from the width required for an upper weight section. Having the figures in the table for any beam of the same section number as that being investigated, the percentage change in Q, or the value of Q for the section in hand, may be found. The value of the coefficient $\frac{Q}{It}$ is then easily determined by dividing Q by quantities found in the handbooks. As the coefficient $\frac{1}{dt}$ is readily found for any beam, the excess of the maximum over the average shearing stress is easily determined.

Table 6 may be used in any one of several ways. The true maximum shearing stress, v_m, may be obtained by multiplying the total shearing force, V, by the appropriate coefficient $\frac{Q}{It}$, interpolating as explained between the values given for intermediate weights of sections. The average shearing stress, v_a, may be obtained in similar manner by multiplying the shearing force, V, by the coefficient $\frac{1}{dt}$. From this approximate stress the true maximum stress may be obtained by increasing the former by the appropriate percentage in the last column, or by a percentage obtained by proper interpolation.

An examination of the table shows that the true maximum stresses exceed the average or approximate stresses by from about 10 to 30 per cent, and generally from 15 to 20 per cent. The error involved in using the approximate method of

design is greater for the maximum weights of sections than for the minimum weights. This is because the former section approaches the rectangle more nearly than the latter. Particularly large differences occur for the maximum weights of channels in each depth, reaching in one case over 31 per cent. By reason of the fact that there is less difference in weight between the minimum and maximum Bethlehem sections for a given depth, the percentage excess of true over approximate stresses is more nearly constant than for standard sections.

Illustrative Problem.—The maximum shear on a 9-in., 25-lb. I-beam is 35,000 lb. Compute the true maximum shearing stress on the cross-section.

Since for a 9-in., 25-lb. I the increment to the statical moment 10.8 (Table 6) for a 9-in., 21.8-lb. I is

$$\Delta Q = \tfrac{1}{2}\Delta t\left(\frac{d}{2}\right)^2 = (\tfrac{1}{2})(0.107)(4.5)^2 = 1.1$$

the statical moment for the heavier beam becomes 10.8 + 1.1 = 11.9.

The coefficient $\frac{Q}{It}$ is, therefore,

$$\frac{11.9}{(91.4)(0.397)} = 0.328$$

and the maximum shearing stress

$$v_m = (0.328)(V) = (0.328)(35,000) = 11,531 \text{ lb. per sq. in.}$$

16b. Horizontal Shearing Stress.—Since the horizontal and vertical shearing stresses at a point are equal, the maximum horizontal shearing stress occurs, as does the maximum vertical shearing stress, at the neutral axis of the beam. The calculation of its intensity at any point may be made in precisely the same manner as for the vertical shearing stress. Unlike timber, steel is able to take shearing stresses almost equally well in all directions—at least near enough so for purposes of ordinary design. The presence of possible horizontal lines of rivet holes along or near the neutral axis may render the beam weaker in horizontal shear than in vertical shear.

16c. Permissible Shearing Stress.—In proportioning steel beams for shear, it is commonly specified that the shearing stress on the gross section of the web considered as uniformly distributed shall not exceed 10,000 lb. per sq. in., or 62.5 per cent of the customary permissible tensile stress. Since the ultimate shearing strength of structural steel is about 75 per cent of the ultimate tensile strength, the shearing unit stress might appear too low. However, the shearing stress is really not uniformly distributed and since the presence of holes in the web somewhat reduces its strength, the unit is seen to be justifiable. More conservative specifications fix the safe shearing stress at 10,000 lb. per sq. in. on net area. In either case the gross area is taken as dt, where d = extreme depth of beam and t = thickness of web.

The effect of lines of holes in the plane of shear considered is, of course, to weaken the section to a degree proportionate to the number and size of the holes. If, for example, a vertical line of 1-in. holes 4 in. apart center to center lies on the section being investigated, the area has been reduced 25 per cent and the statical moment by an equal amount. The moment of inertia of the web, if deep, is reduced in approximately the same ratio, so that the shearing stresses would be increased approximately in the same ratio as the area is decreased. For a further discussion of this point see Art. 41.

To facilitate the design of beams for shear by the approximate or average method, tables of the safe shearing capacity of rolled beams are inserted in the handbooks. In the tables of safe bending loads the upper limit of loads beyond which excessive shearing stresses (or really web crippling stresses) would be produced are indicated, thus making it easy to avoid sections weak in shear. No provision for loss of section by holes is made except in the lowness of the prescribed working stress.

Illustrative Problem.—If the permissible shearing stress on the webs of beams is 10,000 lb. per sq. in., gross area considered as uniformly distributed, report on the safety in shear of a 15-in., 42.9-lb. I-beam supporting a total uniformly distributed load of 150,000 lb.

$$\text{Total end shear, } V = (\tfrac{1}{2})(150{,}000) = 75{,}000 \text{ lb.}$$
$$\text{Gross area of web} = (15)(0.410) = 6.15 \text{ sq. in.}$$

Average shearing stress on web = 75,000/6.15 = 12,200 lb. per sq. in. As this is greater than the prescribed shearing stress, the beam is unsafe in shear. A 15-in., 50-lb. I with a ½-in. web would be adequate.

17. Diagonal Buckling of Web.—Although the web of a beam may be safe so far as either vertical or horizontal shearing stresses are concerned, it may be unsafe for the resistance of the diagonal compression resulting from a combination of shearing and flexural stresses. As has been shown elsewhere in this volume, both the magnitude and the direction of the resultant compressive stress at a point depends on the relative intensities of these two kinds of stress. If f_m = the maximum compressive or tensile stress at any point on either side of the neutral axis; f = the flexural stress at the point; and v = the shearing stress; then, as has been shown,[1]

$$f_m = \frac{f}{2} \pm \sqrt{\tfrac{1}{4}f^2 + v^2}$$

The positive sign is to be used for compressive stresses above the neutral axis and for tensile stresses below it, the negative sign is for tensile stresses above and compressive stresses below the neutral axis.

At the center section of a uniformly loaded beam, the shear is zero and the resultant stress at any point on the section is horizontal. On the other hand, near the support, the flexural stress, f, approaches zero and the resultant diagonal stress makes an angle of nearly 45 deg. with the horizontal throughout the greater part of the depth of the beam. For beams carrying concentrated loads, the shear may be practically constant, and very near its maximum for a large portion, or perhaps all, of its length. In such cases, therefore, the highest diagonal stresses are likely to occur near the section of maximum moment. Such is also true for cantilever, restrained, and continuous beams.

The maximum diagonal compression existing in the web is particularly likely to arise at some point near the junction of the web with the flanges, where a large flexural stress is augmented by the shearing stress. At the ends of beams of ordinary length, the diagonal compressive stress may not be so large as exists at points near the center, but if the beam be short and the shearing stresses heavy, the critical region so far as web crippling is concerned is likely to be near a support. Near this point the diagonal compression throughout the depth of the beam may be regarded as equal in intensity to the vertical or hori-

[1] See Sec. 1, Art. 53.

zontal shearing stress and as making an angle of 45 deg. with the neutral axis relations that are exactly true at the neutral axis.

Because of its relative thinness, the web of a beam tends to buckle or cripple under the action of the diagonal compressive stresses, but such action cannot proceed in the same manner as if the strip of web under consideration were an isolated bar of metal. At right angles to the strip there are tensile stresses which at the neutral axis have a magnitude equal to the compressive stresses, tending to restrain the strip from buckling, and it is therefore in a more favorable condition than a true column.

A rational method of proportioning a beam so that the compressive stresses in the web will not cause failure through buckling, is that followed in the Cambria handbook. It is assumed that the safe stress on a strip of web making an angle of 45 deg. with the neutral axis is represented fairly by the Rankine formula for fixed-ended columns,

$$p = \frac{12{,}000}{1 + \dfrac{l^2}{36{,}000r^2}}$$

where p = safe compressive stress in lb. per sq. in.

l = length of diagonal strip between fillets.

r = radius of gyration of the web normal to its plane.

If h = the clear vertical distance between fillets, and t = thickness of web,

$$p = \frac{12{,}000}{1 + \dfrac{h^2}{1{,}500\,t^2}} \tag{1}$$

If the equality of shearing and diagonal compressive stresses which exists at the neutral axis is assumed to hold throughout the depth of the beam, the average shearing stress in the beam web should also incidentally not exceed p, and so web crippling may be conveniently provided for by ensuring that the average shearing stress comes within the requirement of Formula (1). This stress is to be regarded as uniformly distributed over the area dt, where d = the depth of the beam.

By applying the Carnegie column formula, $p = 19{,}000 - 100\dfrac{l}{r}$ to a diagonal strip of web in the same manner as above, the formula for the safe shearing stress based on diagonal buckling becomes

$$p = 19{,}000 - 490\frac{h}{t} \tag{2}$$

The American Railway Engineering Association formula, $p = 15{,}000 - 50\dfrac{l}{r}$ becomes

$$p = 15{,}000 - 245\frac{h}{t} \tag{3}$$

The A.R.E.A. formula for safe compressive resistance of webs on which the formula for stiffener spacing is based may also be adapted to apply to the webs of beams. This formula, to which reference is made in Art. **52,** is

$$p = 12{,}000 - 40\frac{d}{t}$$

where d = the distance between rivet lines of stiffeners in inches, and p and t are as previously defined. For the average case, d is approximately 1.07 h', where h' is the clear distance between stiffeners, and if the measurement be a vertical one, d, the distance between near lines of rivets, is also approximately $1.07h$, where h is the clear distance between flange angles. Consequently, the above formula when expressed in terms of $\frac{h}{t}$ becomes

$$p = 12{,}000 - 43\frac{h}{t} \tag{4}$$

A study of web crippling of I-beams was made by H. F. Moore in Bulletin No. 68 of the University of Illinois Engineering Experiment Station in connection with a general inquiry into the strength of I-beams in flexure. This study was continued by Mr. Moore and W. M. Wilson in Bulletin 86 of the same series. From the records of web failures there presented, the diagonal compressive stresses at mid-web accompanying failure and the ratio of the depth of the web between fillets to its thickness are seen to be as follows:

Size of beam	Thickness ratio of web $\frac{h}{t}$	Computed diagonal compressive stress at mid-web (lb. per sq. in.)	Size of beam	Thickness ratio of web $\frac{h}{t}$	Computed diagonal compressive stress at mid-web (lb. per sq. in.)
Bulletin 68:			Bulletin 86:		
12-in., 31.8-lb. I........	30.1	25,800	12-in. (web planed thin)	48.0	18,700
12-in., 31.8-lb. I (web planed thin)..........	37.4	27,200	12-in (web planed thin).	55.0	19,500
12-in., 31.8-in. I (web planed thin)..........	55.3	27,400	24-in. special built-up girder..............	101.0	20,000
12-in., 31.8-lb. I (web planed thin).........	65.9	21,400	12-in. (web planed thin).	56.0	17,300
30-in., 175-lb. Bethlehem girder beam.........	38.6	14,800	12-in (web planed thin).	53.0	17,100
20-in. special built-up girder...............	99.7	26,500	12-in. (web planed thin).	52.0	18,300
			12-in. (web planed thin).	54.0	18,800
			24-in. special built-up girder	101.0	22,200

In Fig. 11 the diagonal compressive stresses at failure divided by 2¾—a reasonable factor of safety for columns—have been plotted with respect to the thickness ratio of the web, h/t. Except for the extraordinary strength of the special built-up girders, the general trend is clearly downward with increasing thickness ratio.

On this diagram several formulas for permissible stresses in web crippling have been plotted. The Euler formula for fixed ends, assuming E = 28,000,000 lb. per sq. in., has been expressed in terms of the thickness ratio h/t and divided by 2¾, to give the plotted working formula

$$p = \frac{16{,}700{,}000}{\left(\frac{h}{t}\right)^2} \tag{5}$$

This formula fits the results for the lower values of h/t very well, but gives stresses that are too low for the higher values of h/t.

The Cambria formula (1), when plotted in relation to the tests appears to be too severe.

The adapted Carnegie formula (2) when plotted shows relatively very low values and the adapted A.R.E.A. formula (3) values that are nearly as low. The A.R.E.A. stiffener formula (4) gives values that appear too high for I-beams. None of these formulas conform at all closely with all the test results shown.

A straight line formula with two segments might be made to conform very well to the test results for I-beams without departing too radically from accepted practice. Such a formula has also, as compared with those of the Euler or the

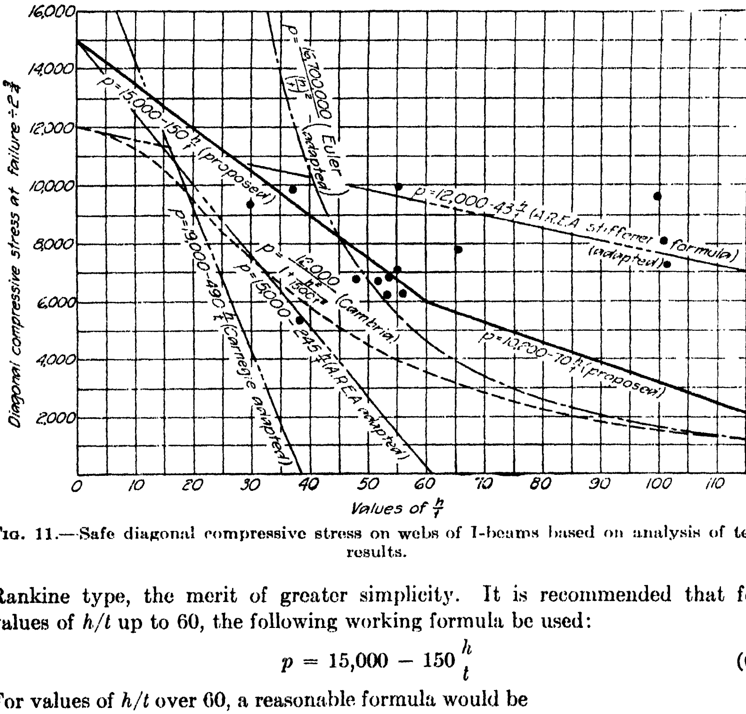


FIG. 11.—Safe diagonal compressive stress on webs of I-beams based on analysis of test results.

Rankine type, the merit of greater simplicity. It is recommended that for values of h/t up to 60, the following working formula be used:

$$p = 15,000 - 150\frac{h}{t} \tag{6}$$

For values of h/t over 60, a reasonable formula would be

$$p = 10,200 - 70\frac{h}{t} \tag{7}$$

The relation of the graph of these formulas to the tests and to other formulas is shown in Fig. 11.

If the shearing stress on a given beam is found to be in excess of that permitted by the particular web crippling formula used, the difficulty is best met by selecting a beam with a thicker web. It is not economical to use stiffeners to stay the webs of rolled beams, although it may be for plate girders, and the reinforcement of the web by riveting plates to it is only justifiable where the rolled section used is the heaviest available.

Illustrative Problem.—A 12-in., 31.8-lb. I carries a total uniformly distributed load of 70,000 lb. Investigate its safety against web crippling using the Cambria formula.

Thickness of web = 0.35 in.

Average shearing stress on web = $\frac{35,000}{(12)(0.35)}$ = 8,340 lb. per sq. in.

Clear distance between fillets = 9.75 in.

Safe shearing stress to provide against web crippling is

$$p = \frac{12,000}{1 + \frac{(9.75)^2}{(1,500)(0.35)^2}} = 7,900 \text{ lb. per sq. in.}$$

Hence the beam is not safe, according to the specification. A beam with a thicker web should be selected.

Illustrative Problem.—Select a 15-in. I-beam capable of taking safely an end shear of 50,000 lb. without giving rise to dangerous web crippling stresses, according to Formulas (6) and (7).

Assume a 15-in., 37.3-lb. Carnegie I. Area of web = (15)(0.332) = 4.98 sq. in. Average shearing stress = 50,000/4.98 = 10,050 lb. per sq. in.

Permissible shearing stress (based on web crippling) = p = 15,000 − (150) $\left(\frac{12.25}{0.332}\right)$ = 9,460 lb. per sq. in., using Formula (6) which applies, as $\frac{h}{t}$ is less than 60.

The section is, therefore, not quite adequate.

Assume a 15-in., 41-lb. Bethlehem I.

Average shearing stress = 50,000/(15)(0.34) = 9,800 lb. per sq. in. Permissible shearing stress = 15,000 − (150) $\left(\frac{12.875}{0.34}\right)$ = 9,320 lb. per sq. in. This section is also slightly below the requirement for web crippling.

Try a 15-in., 42.9-lb. Carnegie I. Average shearing stress = 50,000/(15)(0.41) = 8,130 lb. per sq. in. Permissible shearing stress = 15,000 − (150) $\left(\frac{12.50}{0.41}\right)$ = 10,430 lb. per sq. in. This section is adequate.

18. Vertical Buckling of Web.—While the diagonal buckling effect in the web considered above exists and must be provided for even at points remote from the supports or from concentrated loads, a beam to be safe, so far as the web is concerned, must be capable of safely withstanding concentrated loads or loads distributed over only a short length of the beam. Concentrated loads may be applied to the compression flange, to the web by means of brackets or connection angles, or, as occurs in every beam no matter how loaded, as a vertical (and usually upward) load at the support.

Based upon a series of unpublished tests on beams of various depths and web thicknesses, the safe end reaction R and the safe interior concentrated load W are given in the Carnegie Pocket Companion. The formulas, with slight modification, are as follows:

$$R = pt\left(a + \frac{d}{4}\right) \tag{1}$$

$$W = pt\left(a_1 + \frac{d}{2}\right) \tag{2}$$

In these formulas, t = web thickness, d = depth of beam, a = distance over which reaction is applied, a_1 = distance over which concentrated load is applied, p = safe compressive resistance of web = $19,000 - 173\frac{d}{t}$. This permissible stress is not limited to 13,000 lb. per sq. in. in the tables of allowable buckling resistance given in Carnegie.

An examination of these formulas indicates that they are based on the conception of the vertical loads R or W being resisted through column action by a section of the web of height d and width parallel to the beam equal to the distance (a or a_1) over which the load is applied plus one-quarter of the depth of the beam in the case of the reaction, or one-half the depth of the beam in the case of an interior load. Formula (2) is supposed to apply strictly only to the case of a single load concentrated at the center of the span. Some designers prefer the more conservative rule of considering the effective strip of the web as equal to not over the length of bearing of the load. The permissible compressive stress in the above formulas is based on the Carnegie column formula

$$p = 19{,}000 - 100\frac{l}{r},$$

l being taken as $d/2$, since the compression in the strip of web is not constant throughout its depth, but varies from a maximum at one end to zero at the other. It is, however, more convenient to base the formula on d/t than on d/r.

In a similar manner other web buckling formulas might be evolved using any accepted column formula as the basis—for example, the formula of the American Railway Engineering Association

$$p = 15{,}000 - 50\frac{l}{r}$$

Letting $l = d/2$ and $r = \sqrt{1/12}\,t$, the safe buckling resistance of the web would be

$$p = 15{,}000 - 87\frac{d}{t} \tag{3}$$

The width of the column might be assumed as the length of bearing of the load plus any approved fraction of the depth of the beam.

By replacing d in the Carnegie formula, $p = 19{,}000 - 173\frac{d}{t}$ and in the adapted A.R.E.A. formula $p = 15{,}000 - 87\frac{d}{t}$ by its approximate average value in terms of h—that is $1.25\ h$—these formulas in terms of $\frac{h}{t}$ become respectively

$$p = 19{,}000 - 216\frac{h}{t} \tag{4}$$

and

$$p = 15{,}000 - 109\frac{h}{t} \tag{5}$$

Although Formulas (6) and (7) of Art. **17** are intended to give only the safe diagonal buckling stress, they are sufficiently severe to be applied to vertical buckling. Up to a value of $\frac{h}{t} = 60$, Formula (6) of Art. **17**, that is

$$p = 15{,}000 - 150\frac{h}{t}$$

gives smaller values of the permissible compressive stress than do Formulas (4) and (5) of this article.

Due to exceptionally heavy concentrated loads applied to the compression flange, or perhaps even to the web itself, the thickness of web, although possibly adequate for ordinary diagonal compression arising from the combination of

shearing and flexural stresses, may need to be increased. An alternative procedure is to use stiffeners or to reinforce the web immediately under the load. For rolled beams it is generally desirable to use a beam with a thicker web, where such does not involve a very great increase in weight, rather than to use stiffeners. Reinforcement of the web is adopted only where certain restrictions respecting depth or availability of sections make the employment of a given section necessary.

Illustrative Problem.—A rolled beam resting on two columns 20 ft. apart supports two symmetrically placed loads of 70,000 lb., each 1.5 ft. from the supports, as shown in Fig. 12. Find the required size of the beam, if the permissible stresses are as follows: Flexure, 16,000 lb. per sq. in.; maximum shearing stress at neutral axis, 12,000 lb. per sq. in.; average shearing stress based on diagonal web buckling, $p = \dfrac{12,000}{1 + \dfrac{h^2}{1,500t^2}}$; maximum vertical compression at support to be according to Carnegie formula (1) of this article. Assume a 12-in. support.

Assume an 18-in., 48.2-lb. I, having web thickness of 0.38 in. and section modulus of 81.9.

Maximum moment due to concentrated loading = (70,000)(1.5) = 105,000 ft.-lb.

Maximum moment due to weight of beam = $(\frac{1}{8})(48.2)(20)^2$ = 2,410 ft.-lb.

Total maximum moment = (107,410)(12) = 1,289,000 in.-lb.

Section modulus required 1,289,000/16,000 = 80.5. The section selected is, therefore, adequate for moment.

Maximum end shear = 70,000 + (10)(48.2) = 70,480 lb.

Average shearing stress

$$v_a = \frac{70,480}{(18 \times 0.38)} = 10,310 \text{ lb. per sq. in.}$$

Increasing this by correction of 17 per cent (Table 6, Art. **16**), the maximum shearing stress

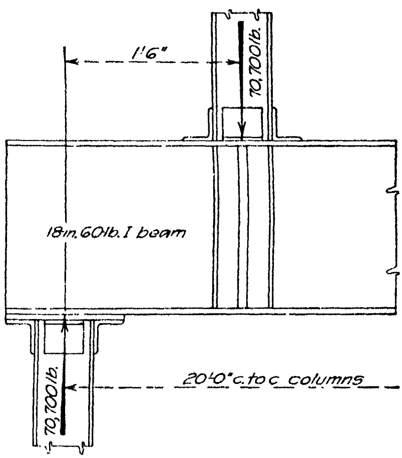


Fig. 12.—Shearing and web buckling capacity of I-beam.

$$v_m = (10,310)(1.17) = 12,070 \text{ lb. per sq. in.}$$

The section is, therefore, sufficiently strong for shear also.

Safe shearing stress based on diagonal buckling of web

$$p = \frac{12,000}{1 + \dfrac{(14.75)^2}{(1,500)(0.38)^2}} = 6,000 \text{ lb. per sq. in.}$$

This is very much below the existing average shearing stress of 10,310 lb. per sq. in. and the section must be increased.

Try an 18-in., 60-lb. I which has a 0.547-in. web.

Average shearing stress, taking the revised end shear as 70,700 lb., is

$$v_a = \frac{70,700}{(18)(0.547)} = 7,180 \text{ lb. per sq. in.}$$

Safe shearing stress based on buckling diagonally

$$p = \frac{12,000}{1 + \dfrac{(15.25)^2}{(1,500)(0.547)^2}} = 7,900 \text{ lb. per sq. in.}$$

This section is, therefore, adequate for diagonal buckling.

Vertical compressive stress at support, from Formula (1)

$$f = \frac{R}{\left(a + \frac{d}{4}\right)t} = \frac{70{,}700}{\left(12 + \frac{18}{4}\right)(0.547)} = 7{,}820 \text{ lb. per sq. in.}$$

Permissible vertical compression in web

$$p = 19{,}000 - 173\frac{d}{t} = 19{,}000 - (173)\left(\frac{18}{0.547}\right) = 13{,}310 \text{ lb. per sq. in.}$$

which is much greater than the existing stress.

The 18-in., 60-lb. I is, therefore, adequate in all respects.

It is assumed that the web is reinforced, if necessary, to take the concentrated superimposed loads.

Illustrative Problem.—A 6-in. H-column with a load of 85,000 lb. is supported on the top flange of a 20-in., 59-lb. Bethlehem I as shown in Fig. 13. Determine whether the web will carry the concentrated load without stiffening or reinforcement if the concentration be considered to be distributed over a length equal to the width of the column shaft plus one-half the depth of the beam. $p = 19{,}000 - 173\frac{d}{t}$ (Carnegie formula).

Length of web over which concentration of 85,000 lb. is concentrated $= 6 + \frac{20}{2} = 16$ in.

Compressive stress on web

$$f = \frac{85{,}000}{(16 \times 0.50)} = 10{,}630 \text{ lb. per sq. in.}$$

Permissible compressive stress

$$p = 19{,}000 - (173)\left(\frac{(20)}{0.375}\right) = 9{,}780 \text{ lb. per sq. in.}$$

Stiffeners are therefore required under the load. It is recommended that two $3 \times 2\frac{1}{2} \times \frac{3}{8}$-in. angles be placed vertically on each side of the web so that the outstanding 3-in. legs are directly under the flanges of the column. Four $\frac{3}{4}$-in. rivets in each angle are sufficient, as with the stiffeners ground to fit the beam flanges, most of the load in them is transferred by end bearing.

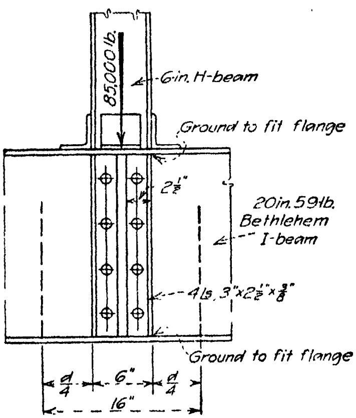


FIG. 13.—Transmission of concentrated load to beam web.

Illustrative Problem.—A 15-in., 33.9-lb. channel carries at one point the ends of two 12-in. I-beams in the manner shown in Fig. 14. The reaction of each beam is 13,000 lb. Investigate the compressive stresses in the web under the combined load, assuming that tl

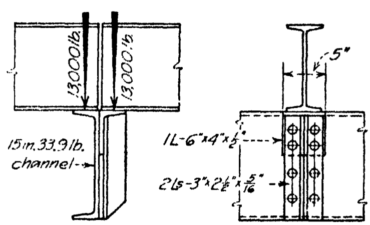


FIG. 14.—Transmission of two concentrated loads to web of channel.

angle. Permissible compressive stress $= p = 15{,}000 - 109\frac{h}{t}$, the adapted A.R.E.A. formula (5).

Total concentrated load on 5 in. of web $= 26{,}000$ lb.

Vertical compressive stress in web due to concentration, and neglecting compression due to combination of shearing and flexural stresses

$$= \frac{26{,}000}{(5)(0.40)} = 13{,}000 \text{ lb. per sq. in.}$$

Permissible compressive stress on unstiffened web $= p = 15{,}000 - (109)\left(\frac{12.25}{0.40}\right) =$ 11,660 lb. per sq. in. It will, therefore, be necessary to stay the web by two stiffener angles under the shelf angle, as shown in Fig. 14. These would not be necessary for stiffening the shelf angle under the existing load, with an angle ½ in. thick or over, but they are required to prevent the web from buckling. Two ¾-in. rivets in each angle under the shelf angle will be sufficient, under any ordinary specification. The stiffeners may be two 3 × 2½ × 5⁄16 in. angles.

Illustrative Problem.—A double-tier steel grillage carrying a total load (including the weight of the 20- × 20-in. column base) of 600,000 lb. has to be made up of 12-in., 31.8-lb. I-beams, reinforced if necessary. Three beams, 5 ft. long, constitute the upper tier, and 7 lines of 8-ft. beams of the same size the lower tier, as shown in Fig. 15. Investigate the web crippling and vertical compressive stresses in the beams of the two tiers, assuming that

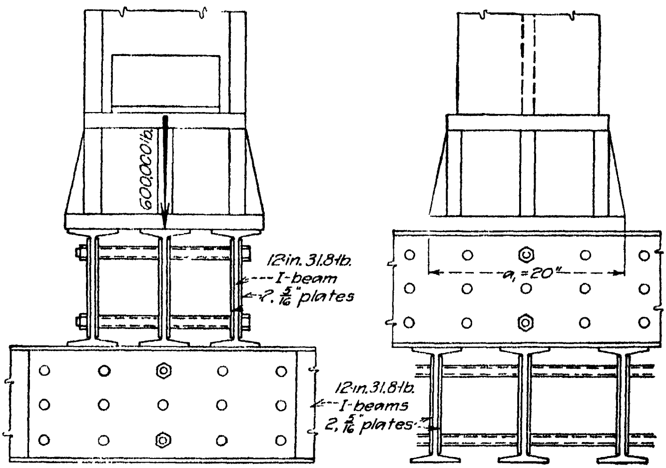


Fig. 15.—Shear and web crippling in grillage tier.

the beams without reinforcement are sufficient for bending moment. Permissible diagonal web crippling and web vertical compressive stress $= p = 15{,}000 - 150\,\frac{h}{t}$. Assume the direct compression to be distributed over a length of web equal to the length of bearing.

Total shear across beams of upper tier at edge of column base $= (\frac{1}{3})(600{,}000) =$ 200,000 lb.

Average shearing stress on unreinforced beam webs

$$\frac{200{,}000}{(3)(12)(0.35)} = 15{,}900 \text{ lb. per sq. in.}$$

Total area of webs of beams of upper tier resisting 600,000-lb. vertical compression, a_1 being 20-in. $= (3)(20)(0.35) = 21$ sq. in.

Compressive stress on beam webs $f = 600{,}000/21 = 28{,}600$ lb. per sq. in.

Permissible shearing stress in order that web crippling stresses may not be excessive, and also permissible vertical compressive stress under concentrated loading $= p =$ $15{,}000 - (150)\left(\frac{9.75}{0.35}\right) = 10{,}820$ lb. per sq. in. It is thus evident that the beams of the upper tier, without reinforcement, would be overstressed by ordinary web crippling due to the combination of shearing and flexural stresses, and very seriously overstressed by the direct compression of the column load.

Add one $\frac{5}{16}$-in. plate—the thinnest practicable plate—to each side of the web, of sufficient width, when ground at the edges, to fit tightly against both upper and lower flanges. Thickness of reinforced web = 0.35 + (2)(0.3125) = 0.97 in.

Average shearing stress on reinforced web now becomes

$$v_a = \frac{200{,}000}{(3)(12)(0.97)} = 5{,}720 \text{ lb. per sq. in.}$$

or much below that allowed.

Compressive stress on reinforced webs under column base

$$f = \frac{600{,}000}{(3)(20)(0.97)} = 10{,}320 \text{ lb. per sq. in.}$$

Permissible compressive stress for reinforced web, $p = 15{,}000 - (150)\left(\frac{9.75}{0.97}\right) = 13{,}490$ lb. per sq. in. The reinforcement provided is adequate.

The reinforced plates may be dispensed with, in theory, at a distance out from the edge of the column base where the total shear is such as to give a shearing stress on the reinforced web within the permissible stress given by the formula specified.

Shearing capacity of three webs = (3)(12)(0.35)(10,820) = 136,400 lb. This shear would exist at a distance from the edge of the column base

$$x = 20 - \left(\frac{136{,}400}{200{,}000}\right)(20) = 6.4 \text{ in.}$$

The plates should, however, be carried far enough beyond this point of theoretical ending to accommodate at least one vertical row of rivets, so that the projection would probably be at least 11 in. In such cases the plates are often carried to the ends of the beams.

As the vertical compressive stress at the junction of the web and flange of the unreinforced beam is 28,600 lb. per sq. in. and in excess of the usual permissible stress in bearing—20,000 to 24,000 lb. per sq. in.—the horizontal edges of the reinforcing plates should be ground to fit both the upper and lower flanges of the beams.

Since the 5-in. flanges of the 3 lines of beams of the upper tier cross all 7 lines of the lower tier, the area of webs of the beams of the lower tier resisting the 600,000-lb. load = (21)(5)(0.35) = 36.8 sq. in.

Compressive stress, f = 600,000/36.8 = 16,300 lb. per sq. in.

To provide for this vertical compressive stress due to the load from the upper tier, the beams of the lower tier would need to be reinforced similarly to those of the upper tier for a central length of about 27 in. If beams with heavier webs were available, it would be much more economical to employ them.

The diagonal web crippling stress in this case of the beams of the lower tier is well within the permissible limit.

The reinforcing plates on the webs of the beams should be attached by rivets spaced vertically not over 5 in. apart and preferably less.

19. Proportioning for Deflection.

19a. Beams with Constant Section.—Although a beam may be strong enough to ensure that under the greatest loads ever likely to be applied, it will not fail, the proportions may be such as to bring about objectionable and even alarming deflection. No harm may come to the beam itself because of the excessive deformation, but any very apparent sag, especially if it visibly increases during the imposition of a load, is likely to convey the impression of weakness to the observer. It is possible, too, that through large deflections, plastered ceilings may be cracked, tile, stone, or concrete floors may open out at the beam supports in a direction transverse to the beams, supported walls may crack, glass in nearby windows may be broken, doors may jam, or shafting or attached equipment or machinery may be thrown seriously out of line or level. If the deflection due to live load is large and frequently occurring, it promotes excessive vibration which may rock the structure, loosen rivets and necessitate constant and troublesome repairs.

Observation and experience show that the maximum deflection that it is safe to allow in a beam supporting plastered ceilings, after the ceilings have been plastered, is $\frac{1}{30}$ in. per ft. or $\frac{1}{360}$ of the span. Stone or tile floors are likely to crack when the deflection of the supporting beams is less than this. Where such floors are carried, it is desirable to limit the deflection to about $\frac{1}{600}$ of the span. Obviously, only the deflection produced by loads applied to the beam *after* the ceiling is plastered, or the floor laid, need be considered, so far as the possibility of cracking is concerned. Wherever the situation will permit, it is desirable to keep down the deflection by using the deepest practicable beams. If no dependent construction will be affected by deflection—as, for example, the interior panels of a building with timber flooring not supporting mechanical equipment—the amount of deflection permissible is entirely a matter of appearance. It might for example, be $\frac{1}{250}$ or $\frac{1}{300}$ of the span.

The maximum deflection of a beam is[1]

$$\Delta = K \cdot \frac{Wl^3}{EI}$$

where K is a constant depending upon the nature of the supports at the ends and the system of loading. Knowing K and the other quantities involved, it is easy to compute the deflection. Wherever a steel handbook is available, however, it is much more convenient to calculate the deflection by means of coefficients than to use the deflection formula for the type of beam and loading under consideration. For example, if in the case of the simply supported uniformly loaded beam, for which the maximum deflection

$$\Delta = \frac{5}{384} \cdot \frac{Wl^3}{EI}$$

the load W be replaced by its value $\frac{8If}{lc}$, and the span be expressed in feet, that is $l = 12L$, the deflection may be expressed as

$$\Delta = \frac{30fL^2}{E} \cdot \frac{1}{2c}$$

Values of the coefficient $\frac{30fL^2}{E}$ for various spans are tabulated for the simply supported, uniformly loaded beam in the handbooks for $E = 29{,}000{,}000$ and for various common values of f, so that to determine the maximum deflection of a uniformly loaded beam of either symmetrical or unsymmetrical section it is only necessary to divide the appropriate coefficient by twice the distance from the neutral axis to the extreme fiber. For beams of symmetrical section, the divisor is obviously the depth of the beam.

Illustrative Problem.—A lintel consisting of two $6 \times 3\frac{1}{2} \times \frac{3}{8}$ in. angles, placed with the 6-in. legs vertical and back to back, has a span of 10 ft., and carries a uniformly distributed load which produces an extreme fiber stress of 16,000 lb. per sq. in. What is the center deflection?

From the table in either Carnegie, Cambria or Bethlehem, the coefficient of deflection for a uniformly loaded beam simply supported at the ends and stressed in flexure to 16,000 lb. per sq. in. = 1.655.

Distance from neutral axis to extreme fiber = 3.96 in.

Hence center deflection

$$\Delta = \frac{1.655}{(2)(3.96)} = 0.209 \text{ in.}$$

[1] See Sec. 1, Art. 66

For conditions other than those assumed in the tables, either the coefficients, or the resulting deflections, may be readily adjusted. From the nature of the coefficient it is seen that the deflection varies directly as the fiber stress, directly as the square of the span and inversely as E. For other systems of loading than the uniform load and for other end conditions, the deflection found in the tables may be multiplied by the factors given in Table 7 to give the deflection under the same fiber stress for the same span.

TABLE 7.—RELATION OF MAXIMUM DEFLECTIONS OF TYPICAL BEAMS UNDER SAME FIBER STRESS AND FOR SAME SPAN

System of loading	Factor by which deflection for simply supported uniformly loaded beam is to be multiplied	System of loading	Factor by which deflection for simply supported uniformly loaded beam is to be multiplied
Simply Supported Beam		Cantilever Beam	
Uniform load........	1.00	Uniform load.....	2.40
Single central load...	0.80	Single end load...	3.20
Two loads at ⅓ points	1.20		

Useful tables are given in the Cambria handbook for simplifying the calculation of the deflection of a beam of any section having been given the load and the span. The coefficients N and N' there tabulated are the deflection for a simply supported beam 1 ft. long, loaded respectively by a uniformly distributed load of 1,000 lb. and a concentrated central load of the same amount. The deflection for a beam supporting any load of either of these types, and of any span, is found by multiplying the appropriate coefficient by the number of 1,000-lb. units in the load and by the cube of the span in feet.

In fixing the sizes of beams for a given situation, it is most desirable to select a depth that under the adopted working stress will be sure to give a deflection within the prescribed limit. This result will be attained if the depth of the beam is made not less than a certain fraction of the span, depending on the nature of the material, the end conditions, and the system of loading. If the limiting deflection be $\frac{1}{360}$ of the span, and this be equated to the deflection expressed in terms of the fiber stress, the limiting depth ratio may be readily obtained. Thus, for uniform loading

$$\Delta = \frac{5}{48} \cdot \frac{fl^2}{Ec} = \frac{l}{360}$$

from which the maximum permissible span for the stated deflection $\frac{l}{360}$ is found to be

$$l = \frac{Ec}{37.5f}$$

If $E = 29{,}000{,}000$, $f = 16{,}000$ lb. per sq. in., and the beam has a symmetrical cross-section, so that $c = \frac{d}{2}$,

$$l = 24.1d$$

If the section be unsymmetrical,

$$l = 48.2c$$

c being the distance from the neutral axis to the extreme fiber.

For other types of loading, and for the values of E and f given above, the maximum span for which the deflection will be $\frac{l}{360}$ is given in Table 8 for beams of symmetrical cross-section.

TABLE 8.—MAXIMUM PERMISSIBLE RATIO OF SPAN TO DEPTH FOR TYPICAL BEAMS

$f = 16{,}000$ lb. per sq. in.
$E = 29{,}000{,}000$ lb. per sq. in.

System of loading	Ratio of $\frac{l}{d}$ for $\Delta = \frac{l}{360}$	System of loading	Ratio of $\frac{l}{d}$ for $\Delta = \frac{l}{360}$
Simply Supported Beam		Cantilever Beam	
Uniform load..............	24.1	Uniform load............	10.1
Single central load..........	30.2	Single end load...........	7.05
Two loads at ⅓ points.....	23.6		

In order that the maximum permissible flexural stress and the maximum permissible deflection of $\frac{l}{360}$ may be attained at the same time, the span must, in general, not exceed

$$l = \frac{Ec}{Kf}$$

the constant K depending on the type of beam and loading. For a simply supported load carrying a uniform load K has been shown to be $= 37.5$, while for the same type of beam carrying a single central load, $K = 30$, and for a beam loaded at the third points it is 38.3.

Limitation of the depth ratio is frequently prescribed in structural specifications. For example, in Schneider's "General Specifications for Structural Work of Buildings," the following clauses occur:

The depth of rolled beams in floors shall be not less than one-twentieth of the span, and, if used as roof purlins, not less than one-thirtieth of the span.

In case of floors subject to shocks and vibrations, the depth of beams and girders shall be limited to one-fifteenth of the span. If shallower beams are used, the sectional area shall be increased until the maximum deflection is not greater than that of a beam having a depth of one-fifteenth of the span, but the depth of such spans and girders shall in no case be less than one-twentieth of the span.

The building code of Philadelphia contains the following restrictions:

The allowable deflection for beams or girders shall not exceed one-thirtieth of an inch per foot of span where the ceiling is to be plastered, or one-twenty-fifth of an inch per foot of span, where the ceiling is not to be plastered.

In what has been said above concerning the calculation of maximum deflection, the effect of the shear in producing deflection has been neglected. This is justifiable for all except precise calculations and for short beams and girders carrying heavy loads. As may be shown, the calculation of shearing deflection for rectangular, or nearly rectangular, sections must take into account the fact that the shearing stress is not uniformly distributed. For I-sections—the most commonly employed ones for flexural members—it may be assumed as pointed

out in Art. 16 that the shearing stress is uniformly distributed over the web only, and the shearing deflection computed accordingly.

A useful comparison of the deflections resulting from flexure and shear, made by R. Fleming in Engineering News-Record, May 27, 1920, is reproduced in Table 9 with some modifications and additions. The deflection due to shear was computed for uniformly loaded beams by the formula

$$\Delta_s = \frac{6Wl}{40E_s dt}$$

and for the centrally loaded beams by the formula

$$\Delta_s = \frac{6Pl}{20E_s dt}$$

In these expressions

W = total uniformly distributed load.
P = concentrated load at center of span.
l = span in inches.
E_s = shearing modulus of elasticity (= 12,000,000 lb. per sq. in.).
d = depth of beam.
t = thickness of web.

An examination of the last column of Table 9 shows that for very short spans—five or six times the depth of the beam—loaded to capacity in bending, the deflection due to shear may be between 30 and 50 per cent of that due to flexure. It is relatively more important for beams carrrying concentrated loads than for those carrying uniformly distributed loads. For beams with a span of from 20 to 24 times the depth (a ratio that is likely to be closely approached in most designs), the shearing deflection is in the neighborhood of 2 or 3 per cent of the deflection due to flexure. It is therefore evident that only for short spans loaded to capacity in bending is there necessity of taking the shearing deflection into account. Should it be desired to include it, for spans of ordinary proportion a close approximation to the total deflection may be made by increasing the deflection due to flexure by a percentage taken from Table 9, interpolating if necessary. Another method of taking account of the shearing deflection is to compute the deflection due to flexure by using a value of the modulus of elasticity somewhat lower than the usual value assumed, say from 10 to 25 per cent.

Since for beams of the usual depth ratios, the shearing deflection is relatively small as compared with that due to flexure, the shearing deflection may with sufficient accuracy be calculated on the assumption that the shearing stress is uniformly distributed over the web and is entirely borne by the web.

19b. Beams with Variable Section.—In computing the deflection of reinforced steel beams, account must be taken of the fact that the moment of inertia is not constant throughout the length of the beam. For such cases the total deflection may be computed by summing a number of partial deflections. The beam is first divided up into a number of short segments, so chosen that any abrupt changes in sectional area or moment will take place at the dividing lines between segments. If E be constant, the deflection due to flexure only is found by applying to the whole reinforced beam the summation

$$\Delta = \frac{1}{E} \Sigma \frac{M}{I}(m)(dx)$$

TABLE 9.—RELATION OF SHEARING AND FLEXURAL DEFLECTIONS FOR SIMPLY SUPPORTED ROLLED I-SECTIONS. $f = 16{,}000$ lb. per sq. in.

Section (in.) (lb.)	Span (feet)	Ratio of depth to span	Total load (pounds)	Distribution of load	Deflection due to flexure Δ_f (inches)	Deflection due to shear Δ_s (inches)	$\frac{100\Delta_s}{\Delta_f}$ (per cent)
Standard I-beams							
8 × 18.4	5	$\frac{1}{7.5}$	30,300	Uniform	0.050	0.011	22.0
8 × 18.4	10	$\frac{1}{15}$	15,150	Uniform	0.200	0.011	5.5
8 × 18.4	15	$\frac{1}{22.5}$	10,100	Uniform	0.449	0.011	2.5
8 × 18.4	5	$\frac{1}{7.5}$	15,150	Middle	0.040	0.011	27.5
8 × 18.4	10	$\frac{1}{15}$	7,600	Middle	0.160	0.011	6.9
8 × 18.4	15	$\frac{1}{22.5}$	5,050	Middle	0.359	0.011	3.1
12 × 31.8	5	$\frac{1}{5}$	76,700	Uniform	0.033	0.014	42.5
12 × 31.8	10	$\frac{1}{10}$	38,350	Uniform	0.133	0.014	10.5
12 × 31.8	20	$\frac{1}{20}$	19,175	Uniform	0.534	0.014	2.6
12 × 31.8	5	$\frac{1}{5}$	38,350	Middle	0.027	0.014	51.9
12 × 31.8	10	$\frac{1}{10}$	19,175	Middle	0.107	0.014	13.1
12 × 31.8	20	$\frac{1}{20}$	9,580	Middle	0.427	0.014	3.3
20 × 65.4	10	$\frac{1}{6}$	124,700	Uniform	0.080	0.019	23.8
20 × 65.4	20	$\frac{1}{12}$	62,350	Uniform	0.321	0.019	5.9
20 × 65.4	40	$\frac{1}{24}$	31,175	Uniform	1.281	0.019	1.5
20 × 65.4	10	$\frac{1}{6}$	62,350	Middle	0.064	0.019	29.7
20 × 65.4	20	$\frac{1}{12}$	31,175	Middle	0.256	0.19	7.4
20 × 65.4	40	$\frac{1}{24}$	15,580	Middle	1.024	0.19	1.9
Bethlehem I-beams							
30 × 120.0	15	$\frac{1}{6}$	248,400	Uniform	0.124	0.035	28.2
30 × 120.0	30	$\frac{1}{12}$	124,200	Uniform	0.496	0.035	7.1
30 × 120.0	50	$\frac{1}{20}$	74,600	Uniform	1.379	0.035	2.5
30 × 120.0	15	$\frac{1}{6}$	124,200	Middle	0.099	0.035	35.4
30 × 120.0	30	$\frac{1}{12}$	62,100	Middle	0.396	0.035	8.8
30 × 120.0	50	$\frac{1}{20}$	37,300	Middle	1.102	0.035	3.2
Bethlehem girder beams							
30 × 180.0	15	$\frac{1}{6}$	388,480	Uniform	0.124	0.042	33.9
30 × 180.0	30	$\frac{1}{12}$	194,240	Uniform	0.496	0.042	8.5
30 × 180.0	50	$\frac{1}{20}$	116,600	Uniform	1.379	0.042	3.1
30 × 180.0	15	$\frac{1}{6}$	194,240	Middle	0.099	0.042	42.4
30 × 180.0	30	$\frac{1}{12}$	97,120	Middle	0.396	0.042	10.6
30 × 180.0	50	$\frac{1}{20}$	58,300	Middle	1.102	0.042	3.8

In this expression,

Δ = required maximum deflection.

E = modulus of elasticity.

M = bending moment at the center of any segment distant x from a support for simply supported beams and from the free end in the case of cantilevers.

I = moment of inertia of the beam at the center of any segment.

m = bending moment at center of any segment due to load of 1 lb., acting at the point where the deflection is required.

dx = length of any short segment.

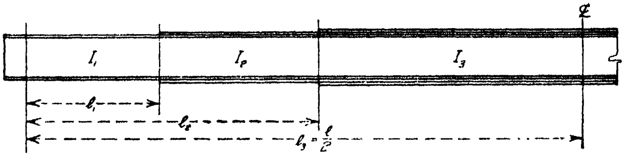


FIG. 16.—Calculation of deflection of reinforced or built-up beam.

Applying this to a uniformly loaded beam with several reinforcing plates on each flange, as shown in Fig. 16, the summation by means of the Calculus for the entire span gives a deflection of

$$= \frac{wl}{6E}\left[\frac{l_1^3}{I_1} + \frac{l_2^3 - l_1^3}{I_2} + \frac{l_3^3 - l_2^3}{I_3}\right] - \frac{w}{8E}\left[\frac{l_1^4}{I_1} + \frac{l_2^4 - l_1^4}{I_2} + \frac{l_3^4 - l_2^4}{I_3}\right] \qquad (1)$$

where

w = uniform load per unit of length.

E = modulus of elasticity of material.

l = span length.

l_1, l_2, l_3 = distances of successive points of change of moment of inertia from support.

I_1, I_2, I_3 = moment of inertia of successive sections from support.

This formula may be used to cover any number of abrupt changes by the inclusion of more terms.

For purposes of computing deflections, the moment of inertia of the gross section, which is the predominant section, should be used.

Computations made for beams of constant depth and section so varied as to give constant strength show deflections from 20 to 100 per cent greater than for beams with constant moment of inertia.

Illustrative Problem.—Compute the maximum center deflection of a 25-ft. 12-in., 31.8-lb. I with one 6 × 3/8-in. plate, 18 ft. long, riveted to each flange. (See Fig. 9 and the problem under Art. **12.**) The beam carries a total uniformly distributed load of 900 lb. per lin. ft. E = 29,000,000 lb. per sq. in.

The half beam will be divided into two segments, the first of which comprises the 3.5-ft. unreinforced portion of the end, and the second the remaining 9-ft. portion of the half span. The values of l_1 and l_2 are, therefore, 3.5 and 12.5 ft. respectively.

Moment of inertia of gross unreinforced section, I_1 = 215.8.

Moment of inertia of gross reinforced section, I_2 = 388.2.

For the case in hand, Formula (1) becomes

$$\Delta = \frac{wl}{6E}\left[\frac{l_1^3}{I_1} + \frac{\left(\frac{l}{2}\right)^3 - l_1^3}{I_2}\right] - \frac{w}{8E}\left[\frac{l_1^4}{I_1} + \frac{\left(\frac{l}{2}\right)^4 - l_1^4}{I_2}\right]$$

Using the inch as unit of length, and inserting the appropriate numerical quantities,

$$\Delta = \frac{(75)(300)}{(6)(29{,}000{,}000)}\left[\frac{(42)^3}{215.8} + \frac{(150)^3 - (42)^3}{388.2}\right]$$
$$- \frac{75}{(8)(29{,}000{,}000)}\left[\frac{(42)^4}{215.8} + \frac{(150)^4 - (42)^4}{388.2}\right]$$
$$= 0.721 \text{ in.}$$

If it be assumed that the gross moment of inertia of the reinforced section applies for the whole span, the deflection would be

$$\Delta = \frac{5}{384} \cdot \frac{Wl^3}{EI}$$

or, for the beam under consideration.

$$\Delta = \frac{5}{384} \cdot \frac{(900)(25)(300)^3}{(29{,}000{,}000)(388.2)} = 0.702 \text{ in.}$$

or but slightly less than the deflection found by the correct method. The close correspondence of these results is due to the fact that the flange plates run nearly the full length of the beam and the stresses in the central reinforced section influence the deflection much more than those in the unreinforced section near the ends.

20. Combined Stresses.—Cases frequently arise in practice of members subjected to flexure and at the same time to an axial tensile or compressive force.[1] These are in most cases, however, primarily tension or compression members and are discussed as such in this volume. One characteristic case of a flexural member being subjected to axial loading is the trussed beam. This type of member is discussed in Art. **21.**

21. Trussing of Beams.—If it happens that the heaviest rolled section available is not sufficiently strong to carry the stipulated load, and there is no restriction with respect to headroom, a rolled section may be trussed so as to enable it to carry a load. Two common methods of trussing are used, the king-post, Fig. 17(*a*) and the queen-post, Fig. 17(*b*) and (*c*). With the first, a single strut is connected to the primary beam at the center and a rod is carried from the bottom of it to each end. With the second, two struts are used, dividing the beam into three segments not necessarily equal. The struts may be of angles or castings and the ties may be single or multiple rods. Where cast struts are used, they may be at right angles to the top chord, as in Fig. 17(*b*), but if angles are used they should be battered so as to bisect the angle between the horizontal and sloping sections of the tie rod to give axial stress only in the struts.

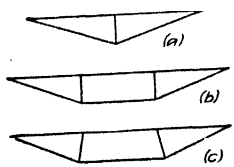


Fig. 17.—Types of trussed beams.

A common use of trussed steel beams is in roof construction, as rafters or purlins. If they are used as purlins, the bending will not be in a principal plane of the trussed section and hence in designing the principles pertaining to unsymmetrical bending (Art. **22**) must be observed.

While the accurate design of a trussed beam should be carried out in accordance with the method of least work, a sufficiently accurate procedure for most purposes is to regard the structure as a beam continuous over the struts. This involves the erroneous assumption that the beam does not settle at the struts with respect to the end supports—an assumption that is, however, justified for approximate design.

[1] See chapter on "Bending and Direct Stress" in Sec. 1.

In accordance with this assumption, the primary beam is not only subjected to the moments and shears existing in a continuous beam of the same number of spans as there are panels, but must also resist the axial thrust due to the pull of the tie rod. The end connections of this tie should be such that the thrust is applied centrically, thus avoiding secondary stresses.

The approximate method of design outlined above, may best be studied by means of an example.

Illustrative Problem.—An opening of 18 ft. center to center of bearings is to be spanned by a beam carrying a total uniformly distributed load of 600 lb. per lin. ft. For this situation there are available only minimum weight channels of depths up to 9 in., angles, and soft steel rods. There is no restriction as to headroom. Lateral support to the beam is afforded at the center and at points 3 ft. from each end. Design a trussed beam to carry the load if the permissible stresses are as follows:

Bending, 16,000 lb. per sq. in.

Compression on struts, $p = 19{,}000 = 100\frac{l}{r}$ where l = unsupported length and r = least radius of gyration.

Combined compression and bending, $p = 19{,}000 - 300\frac{l}{b}$, where l = unsupported length of flange and b = breadth of flange.

Shear, 10,000 lb. per sq. in., gross area of web.

Tension on soft steel rods, 15,000 lb. per sq. in.

Bearing, on soft steel, 15,000 lb. per sq. in.

As only very light channel sections are available, an arrangement will be adopted favorable to the primary beam, or what is really the top chord of the resulting truss.

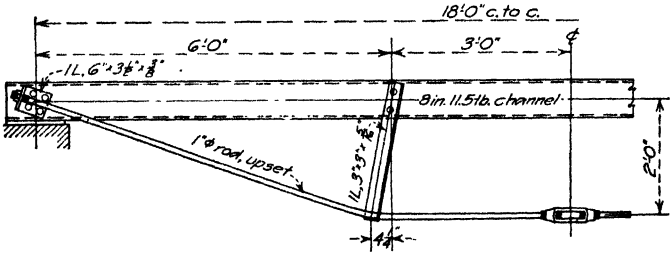


FIG. 18.—Details of trussed channel.

Two struts will therefore be used, symmetrically placed and 6 ft. apart at their intersection with the center line of the channels, as shown in Fig. 18, and the depth from the enter of the top chord to the center of the tie rod will be 2 ft., giving a slope of the end sections of tie rod of 1 vertical to 2.83 horizontal, with the struts bisecting the angle between the horizontal and inclined portions of the tie rod, which is desirable in order to give only axial stress in the struts.

Shear.—From the theory of continuous beams, the maximum shear in the top chord occurs at the two struts on the sides nearest the end supports and is

$$V = \frac{6}{10}wp$$

where w = total uniform load per unit of length; and p = panel length.

For this case

$$V = \left(\frac{6}{10}\right)(600)(6) = 2{,}160 \text{ lb.}$$

Assuming one 8-in., 11.5-lb. channel as the top chord, the average shearing stress on the web is

$$v_a = \frac{2{,}160}{(8)(0.220)} = 1{,}230 \text{ lb. per sq. in.}$$

which is very much below the allowed limit.

Combined Bending and Compression.—For a continuous beam of 3 equal spans, assuming no restraint at the end supports, the maximum moment occurs at an intermediate support and is

$$M = \frac{1}{10}wp^2$$

which for this problem becomes

$$M = \left(\frac{1}{10}\right)(600)(6)^2(12) = 25{,}920 \text{ in.-lb.}$$

Extreme fiber stress assuming the top chord to be one 8-in., 11.5-lb. channel with a section modulus of 8.1 is

$$f_1 = \frac{M}{S} = \frac{25{,}920}{8.1} = 3{,}200 \text{ lb. per sq. in.}$$

Horizontal or axial compression in top chord, neglecting the horizontal component of the stress in the strut,

$$H = \frac{11}{10}wp \cot a,$$

where a = angle of slope with the horizontal of the end sections of the tie rod.
Numerically,

$$H = \left(\frac{11}{10}\right)(600)(6)(2.83) = 11{,}200 \text{ lb.}$$

Maximum axial compressive stress,

$$f_2 = \frac{H}{A} = \frac{11{,}200}{3.36} = 3{,}340 \text{ lb. per sq. in.}$$

Total maximum compressive stress,

$$f_1 + f_2 = 3{,}200 + 3{,}340 = 6{,}540 \text{ lb. per sq. in.}$$

Permissible compressive stress on chord

$$p = 19{,}000 - (300)\left(\frac{72}{2.26}\right) = 9{,}450 \text{ lb. per sq. in.}$$

Since the effect of the necessarily eccentric application of the axial thrust has been neglected, the margin of safety is not too great.

Tie Rod.—Tension in tie rod

$$T = \frac{11}{10}wp \operatorname{cosec} a = \left(\frac{11}{10}\right)(600)(6)(2.99) = 11{,}850 \text{ lb.}$$

Required area = 11,850/15,000 = 0.79 sq. in.

Use one 1-in. rod upset, having an area of 0.79 sq. in. in the body and of 1.054 sq. in. at the root of the thread of the 1⅜-in. upset ends. A turnbuckle will be needed at the center of the span for adjustment.

Struts.—Compression in struts,

$$P = \left(\frac{11}{10}\right)wp = \left(\frac{11}{10}\right)(600)(6) = 3{,}960 \text{ lb.}$$

Assume one 3 × 3 × 5⁄16-in. angle, for which

$$A = 1.78 \text{ sq. in. and least } r = 0.59 \text{ in.}$$

Compressive stress = 3,960/1.78 = 2,220 lb. per sq. in.

Permissible stress,

$$p = 19{,}000 - 100\frac{l}{r} = 19{,}000 - (100)\left(\frac{24}{0.59}\right) = 14{,}930 \text{ lb. per sq. in.}$$

The outstanding leg of the lower end of the struts will be notched to semi-circular form so as to receive the rod.

Bearing area required for rod,

$$A = \frac{P}{p} = \frac{3{,}960}{15{,}000} = 0.26 \text{ sq. in.}$$

Area provided = (1.00)(0.3125) = 0.31 sq. in., which is adequate.

Details.—Details may be arranged as shown in Fig. 18. The connection of the tie rod and the struts to the top chord must be sufficient in strength to transmit the stresses in them to the channel.

22. Proportioning for Unsymmetrical Bending.—Beams subjected to bending not operating in the plane of one of the principal axes cannot properly be designed by the simple flexure formula

$$f = \frac{M}{S}$$

in which S is the ordinary section modulus about the principal axis most nearly at right angles to the plane of loading. If, however, the true section modulus applicable under the circumstances be employed, either the maximum stress at the extreme fiber or the safe capacity may be computed accurately by the common flexure formula. This quantity known as the *flexural modulus* is, as has been shown elsewhere in this volume,

$$S' = \frac{I_x I_y}{I_y y \sin\theta + I_x x \cos\theta}$$

where

I_x = moment of inertia of section about the x-axis.
I_y = moment of inertia of section about the y-axis.
x, y = coördinates of the most highly stressed fiber.
θ = angle between the plane of the moment and the x-axis.

For the purposes of practical design it is more convenient, however, to resolve the moment into two components, parallel respectively to the two principal axes of the section, and then add together the stresses produced by them at the critical fiber. The correctness of this method of procedure has been established elsewhere in this volume.

A frequent case of unsymmetrical bending is that of a beam subjected to both vertical and transverse moment, as a floor beam supporting a vertical load and at the same time resisting the thrust of an arch. Here a resultant oblique moment really exists in the form of two principal components. Investigation of such a beam may, therefore, be carried out as explained above.

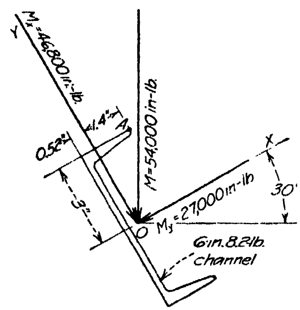


Fig. 19.—Flexural capacity of channel purlin.

Illustrative Problem.—A 6-in., 8.2-in. channel purlin of 15-ft. span with web inclined 30 deg. to the vertical, as shown in Fig. 19, carries a vertical roof load of 160 lb. per lin. ft. Express an opinion as to its safety if the permissible stress in bending is 16,000 lb. per sq. in.

Vertical moment on purlin

$$M = \frac{Wl}{8} = \frac{(160)(15)(180)}{8} = 54,000 \text{ in.-lb.}$$

Component of moment in plane of purlin web, or about the axis of x

$$M_x = (54,000)(\sin 60°) = 46,800 \text{ in.-lb.}$$

Component of moment at right angles to plane of web, or about the axis of y,

$$M_y = (54,000)(\cos 60°) = 27,000 \text{ in.-lb.}$$

The fiber at point A is evidently the most highly stressed one. Its coördinates are: $x = 1.4$ and $y = 3.0$. The moment of inertia about the x-axis is $I_x = 13.0$ and about the y-axis it is $I_y = 0.70$. Resultant fiber stress at point A is therefore

$$\begin{aligned} f_x + f_y &= \frac{M_x y}{I_x} + \frac{M_y x}{I_y} \\ &= \frac{(46{,}800)(3.0)}{13.0} + \frac{(27{,}000)(1.4)}{0.70} \\ &= 10{,}800 + 54{,}000 \\ &= 64{,}800 \text{ lb. per sq. in.} \end{aligned}$$

The purlin is therefore stressed to its ultimate strength. By supporting it laterally at short intervals the stress f_y could be greatly reduced and the resultant stress $f_x + f_y$ brought within the safe limit.

Had the loading been assumed as acting in the plane of the web, as is sometimes erroneously done, the fiber stress obtained would be 12,470 lb. per sq. in. The stress calculated in this manner may, therefore, give no real indication as to the actual existing stress.

Illustrative Problem.—A floor beam of 18-ft. span, consisting of one 12-in., 31.8-lb. I, carries a total uniformly distributed vertical load of 900 lb. per lin. ft. and a resultant horizontal arch thrust of 500 lb. per lin. ft. If the beam is divided into three 6-ft. segments by tie rods, as shown in Fig. 20, find the maximum fiber stress, assuming perfect lateral restraint at the points of attachment of the tie rods.

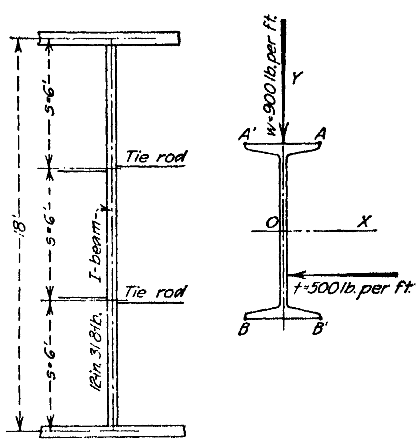


Fig. 20.—Floor joist subjected to vertical and horizontal moment.

Section modulus S_x, of 12-in., 31.8-lb. I about x-axis (normal to web) = 36.0, and section modulus about axis lying in center of web, $S_y = 3.8$. Flange width = 5 in.

Vertical moment at center of span,

$$M_x = \frac{wl^2}{8} = \frac{(900)(18)^2(12)}{8} = 437{,}400 \text{ in.-lb.}$$

Horizontal moment at center of span

$$M_y = \frac{ts^2}{24} = \frac{(500)(6)^2(12)}{24} = 9{,}000 \text{ in.-lb.}$$

where t = lateral thrust per lin. ft., and s = spacing of the rods.

Maximum fiber stress on fiber at A or B, at center of span,

$$f_m = \frac{M_x}{S_x} + \frac{M_y}{S_y} = \frac{437{,}400}{36.0} + \frac{9{,}000}{3.8} = 14{,}540 \text{ lb. per sq. in.}$$

Vertical moment at a tie-rod connection,

$$\begin{aligned} M_x' &= \tfrac{1}{2} wls - \tfrac{1}{2} ws^2 = \tfrac{1}{2}(900)(18)(6) - \tfrac{1}{2}(900)(6)^2 \\ &= 32{,}400 \text{ ft.-lb.} = 388{,}800 \text{ in.-lb.} \end{aligned}$$

Horizontal moment at the rod connection, assuming perfect restraint,

$$M_y' = \frac{ts^2}{12} = \frac{(500)(6)^2(12)}{12} = 18{,}000 \text{ in.-lb.}$$

Maximum fiber stress on fiber A′ or B′, at tie rod connection,

$$\begin{aligned} f_m' &= \frac{M_x'}{S_x} + \frac{M_y'}{S_y} = \frac{388{,}800}{36.0} + \frac{18{,}000}{3.8} \\ &= 15{,}530 \text{ lb. per sq. in.} \end{aligned}$$

The beam is, therefore, more seriously stressed at the tie rod connections than at the center.

23. Proportioning for Torsion.—Wherever beams are curved horizontally or are of such shape in plan that the applied loads do not lie on a straight line joining the two supports, a torsional moment is set up.

A typical case of this kind is the circular girder supporting an elevated tank. The arc of the girder between two adjacent posts must withstand a torsional moment of the magnitude that may be computed by the methods explained in discussions of elevated tanks.

If long, flexible beams connect to one side only of a girder, the girder is thereby subjected to torsional stresses which in severe cases should be investigated. A girder with a narrow flange, such as a single channel, is likely to be highly stressed in torsion. The torsional moment produced in a girder by a beam attached to it by a web connection may be considered as equal to the moment of restraint of the beam at the end. While such moment of restraint is disregarded in fixing the section of steel beams, the practice is common to assume that there is a moment of restraint that offsets the apparent moment of eccentricity in the end connection and renders it necessary to proportion the rivets through the beam web for direct shear only. Based on the character of the end connections of the beams framing into the girder subjected to torsion, an estimate may be made of the probable torsional moment applied at each loading point. Such torsional moments may be regarded as divided between the two segments of the beam on the two sides of the loading point in the inverse ratio of their length. With two symmetrically-applied torsional moments, there will be equal torsions in the two end segments and zero torsion in the center segment.

In determining the maximum existing torsional shearing stress on the cross section of an I-beam or channel section, it is incorrect to assume that the common torsion formula for circular shafts applies.

This formula is

$$\frac{q}{c} = \frac{T}{J} \qquad (1)$$

where q = torsional shearing stress at the extreme fiber.
c = radial distance from center of gravity of section to extreme fiber.
T = torsional moment.
J = polar moment of inertia (see treatise on mechanics).

Experimental determination of the torsional elastic limit of I-beams made by the author indicate that this is reached at a torque less than 20 per cent as great as Formula (1) would indicate. The relatively thin metal of the web has little torsional resistance itself and does not effectively prevent the flanges from twisting around under a combination of shear and bending. Beams designed for torsion should only be proportioned by Formula (1), provided the allowable stress selected is not over 20 per cent the usual permissible stress in shear.

MULTIPLE BEAM GIRDERS

By C. R. Young

24. Types and Uses.—Where a single rolled beam or girder with adequate bending capacity for the situation in hand is not available, it is frequently advantageous to use two or more rolled sections placed side by side a short distance

apart and suitably connected together. Such construction is particularly useful for the support of walls, on account of the broad bearing offered for the load. The number of sections varies from two or three, used for the support of walls, to as many as 10 or 12 in the case of a tier in a grillage foundation.

While the component sections are frequently of the same type, depth and weight, it is by no means necessary that they should be so. If three sections are used it may be advantageous to make the outer two somewhat lighter than the inner one; or if the latter section be an I-beam, to make the outer two channels of the same depth. Characteristic sections for multiple beam girders are shown in Fig. 21. Those shown in (*a*), (*b*) (*c*) and (*d*) are frequently employed for the support of walls, beams, and columns, while the use of a large number of sections as in (*e*), is confined to grillage tiers. Rolled beams in groups of from two to four are frequently employed as girders supporting timber decks in railway bridges. One group is placed under each rail. Modification of some of the types shown by the addition of shelf angles at the bottom is frequently made in

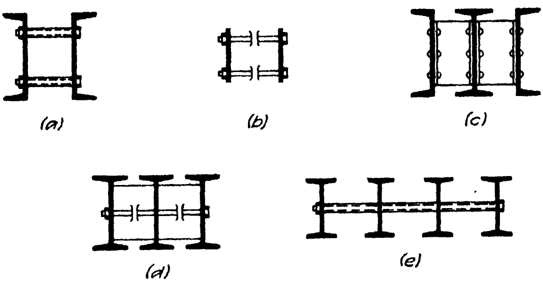


FIG. 21.—Cross-section of typical multiple beam girders.

order to adapt multiple beam girders to use as lintels or spandrel girders. Examples of these are shown in Fig. 24.

In making up the section of a multiple beam girder, regard must be had to the character of the determining stress. If the bending moment is relatively more important than the shear, I-beams should be employed, rather than channels, since the flexural efficiency of the I-beam is greater than that of the channel, as has been pointed out in the discussion of beams, Art. **6**. On the other hand, if the shear is large enough to influence the design, channels are preferable for economic reasons as the amount of shearing area per square inch of total section is greater for channels than for I-beams. This latter condition also gives channels an advantage in resisting local transverse compression or web crippling.

In selecting the sections to be utilized in a multiple beam girder, the bearing area that must be provided for the applied load should be considered. For well-bonded brick walls, there is no reason why the brick work in a wall of any thickness likely to be carried on multiple beam girders should not arch laterally over the clear space between the flanges of two supporting beams or channels.

25. Advantages and Disadvantages.—The use of multiple beam girders is only advantageous where large flexural strength with small depth is required. The broad bearing afforded by such a girder for the support of walls and for the transmission of loads to the end supports is also an advantage, as is the consider-

able lateral stiffness of the combined beam. It is, too, very convenient to be able to utilize a series of available light beams for building up a girder for the support of heavy loads.

On the other hand, the use of shallow beams or girders is highly uneconomical so far as flexure is concerned, as has been pointed out in the discussion of beams in Art. 5. Where the shear is relatively more important than the moment, as in short, heavily-loaded tiers of grillage beams, sections of small depth may be found more desirable because of their greater aggregate web area. They are deficient in vertical stiffness, however, and unless care is taken to limit the ratio of span to depth the deflection may be so large as to be objectionable in appearance or, in the case of foundation girders, to lessen the bearing at the outer ends of the beams. In no case should girders of the multiple beam type be used in damp situations without being properly protected from corrosion on the interior surfaces. Such protection is naturally afforded by the encasing concrete in grillages, but girders above ground are frequently left without such protection, and from the nature of the construction cannot be inspected or painted after erection.

26. Separators.—In order that the assemblage of sections may act as a unit in the support of loads and may possess adequate lateral rigidity, separators of various types are employed. These maintain the spacing of the component beams, and when loads are not applied equally to all of them should be able to distribute it equally amongst the various elements composing the girder, unless they are designed for unequal loads.

Three types, indicated in Fig. 21, are in common use—gas pipe, cast iron, and built-up or riveted separators.

The gas pipe type, Fig. 21 (*a*) and (*e*), consists of a series of short lengths of gas pipe fitting closely between the webs, with bolts passing through them from one side of the girder to the other. The number of tiers of bolts and pipe sections used, varies with the depth of the girder, but in average practice conforms approximately to the following table:

Depth of Girder (inches)	Number of Tiers of Bolts
3–10	1
12–18	2
20–30	3

Separators of the gas-pipe type are cheaper than any others, but are incapable of transferring any load from one beam of the compound girder to another. Wherever the applied load is known to be equally applied to the component sections or where the individual sections are designed for definite parts of the load, as in some lintels or spandrel girders, gas pipe separators may be used to advantage. They are particularly desirable in tiers of grillage beams since they do not interfere with the placing of the encasing concrete or break it up, as would cast iron or built-up separators. The American Bridge Company's standards call for gas pipe separators for all girders composed of beams under 6 in. in depth. The size of gas pipe and bolts should conform to the size of the prevailing rivets in the work. Generally, ¾-in. bolts and 1-in. gas pipe are used.

Cast iron separators, Fig. 21 (*b*) and (*d*), consist of cast plates, usually from ⅜ to ⅝ in. thick, with one or two lugs cast on the face of the plates to receive

the bolts which secure the separator in a transverse position between the connected beams. Two types of separator are commonly used, (1) a rectangular plate of width such as to maintain the component beams at the desired distance apart, and of height sufficient to clear the fillets (Fig. 21), and (2) a plate shaped to fit tightly against the webs and flanges of the beams (Fig. 22). If the second type be made to bear properly against the flanges, it is superior to the first, for the transfer of load from one beam to the other would then not depend solely upon the shearing and bending value of long flexible bolts, as with gas pipe separators. If a load be applied to one beam and not to the other, as in Fig. 22, the deflection of the loaded beam causes the top flange to transfer part of the load in bearing through the separator to the bottom flange of the other beam. With this type of separator, properly fitted, the tendency is for applied loads to be equally distributed amongst the component sections if they be of equal stiffness. If the sections are of unequal stiffness, the stiffest would receive the greatest loads. In situations where distribution of the load by separators is counted upon, therefore, cast iron separators of the second type may be advantageously employed. They should not be used in grillages for the reasons already given.

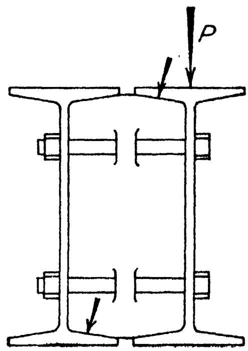


Fig. 22.—Distribution of load by fitted cast-iron separators.

According to the standards given in the handbooks of the steel companies, one bolt only may be used in each separator if the beams be not over 10 or 12 in. deep. Two bolts are used for beams from 12 to 24 in. deep, and 3 bolts for beams over 24 in. The prescribed spacing of the bolts and the dimensions and weights of separators and bolts is given in the standards mentioned. The width of the separators is so fixed that when they are used with the maximum weight of beams for the depth to which they conform, the flanges will clear.

Built-up separators or diaphragms are employed in situations where very rigid bracing is required between the component sections of a girder or where provision must be made for distributing unequally applied loads. They may be made up of a plate and two or four angles to form a built-up channel or I-section with flanges riveted to the webs of the beams, or if the desired spacing of the component sections will permit, of a piece of channel or I-beam placed with its flanges vertical and in contact with the webs of the connected sections. This riveted construction ensures the action of the assembled sections as one unified girder.

While separators serve to stay the top flanges of the component sections of the girder to some extent, their effectiveness in this regard, except in the case of the shaped cast iron separator, is considerably reduced through the attachment being to the web rather than to the flanges. The spacing of separators is therefore generally less than would be obtained by applying such rules as that the compression flange of beams must be stayed at intervals of 10 or 20 flange-widths if the customary flexural working stress is to be employed. It is good practice to place separators at the ends of the girder and at, or near, all points of concentrated loading. In addition they are placed at intermediate points, distances

apart varying with the depth of the beams. The spacing adopted where the position of points of concentrated loading does not determine it, is usually about as follows:

Depth of girder (inches)	Spacing of separators (feet)
3–10	3
12–15	4
18–30	5

27. Proportioning of Multiple Beam Girders.—The design of multiple beam girders differs in no way from that of single beams. Having found the maximum bending moment and maximum vertical shear, such component sections must be selected as will give the desired width for the effective support of the applied load and will supply the total section modulus and shearing area required. Deductions for any flange holes that may be near the critical section for moment should be made as described for beams in Arts. **8** to **11** inclusive, but web holes may be neglected so far as moment is concerned. If a built-up separator chances to

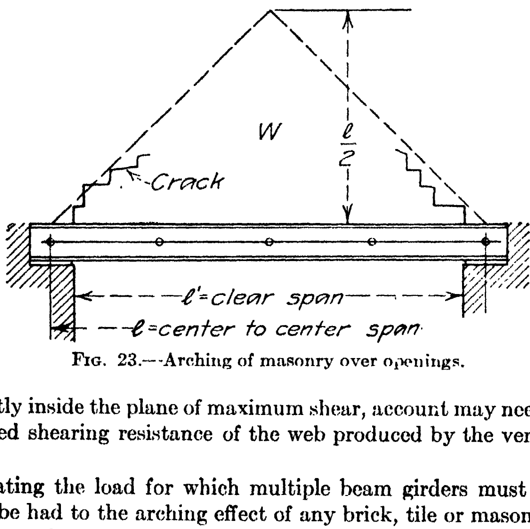


Fig. 23.—Arching of masonry over openings.

be at or slightly inside the plane of maximum shear, account may need to be taken of the lessened shearing resistance of the web produced by the vertical lines of holes.

In calculating the load for which multiple beam girders must be designed, regard must be had to the arching effect of any brick, tile or masonry walls that may be supported. Observations of the cracking of such walls above a supporting girder or lintel that has partially failed, or sagged excessively, show that under certain favorable conditions only a relatively small triangular portion of masonry is really carried by the girder. The height of this triangle is variously assumed as from ⅓ to ⅔ of the span. While the cracks mentioned trend upward and inward from the junction of the top of the girder with the faces of the supports, as shown in Fig. 23, it is more convenient and just as accurate, to assume the height of the supported triangle as based on the center to center span—the span on which the calculation of moments and shears must be based.

It is only safe to assume the arching effect as relieving the girder of all wall load, except the weight of the triangular portion mentioned, (1) when the supports

are capable of taking thrust in a direction parallel to the girder, or the supported wall continues past the supports for some distance, (2) when there is a height of brickwork not weakened by openings, for a distance above the girder about equal to the span length, (3) when the depth ratio of the girder is large enough to prevent excessive and disrupting deflection, and (4) when the masonry is well seasoned. Under such conditions the area of wall supported may safely be taken as contained within a triangle having base equal to the center to center span and height above the top of girder equal to ½ of the span. If the existing conditions depart measurably from those outlined, the full height of wall to the next support above should be taken. Piers or concentrated loads carried into the wall above the opening must be specially provided for. If the loads be relatively large, it is not safe to depend much on arch action in the masonry.

Care must be taken to add to the weight of the wall any floor loads that may be carried into it or into the girder direct.

If the wall above the opening spanned is cut up by windows or other openings, the weight of the existing sections of wall must be computed and the point of application of such weights carefully determined.

Illustrative Problem.—A solid, well-seasoned 13-in. brick wall weighing 120 lb. per cu. ft. is to be carried over a clear opening of 17 ft. The wall continues on for some distance past the supports on either side. Design a suitable multiple beam girder to carry the wall. Permissible stresses in bending and shear = 16,000 and 10,000 lb. per sq. in., respectively, the shearing stress to be the average on gross section of the web. Permissible shearing stress to safeguard against web crippling $= p = 15{,}000 - 150\dfrac{h}{t}$.

To ensure that the deflection will not be great enough to destroy the arching effect which the stated conditions would permit, the depth of beam, assuming the center to center span to be 18 ft. should not be less than about $(\frac{1}{24})(18)(12) = 9$ in.

Weight of brick work supported, taking the wall as weighing $\left(\frac{13}{12}\right)(120) = 130$ lb. per sq. ft., is

$$W_1 = (18)(\tfrac{9}{2})(130) = 10{,}530 \text{ lb.}$$

Moment due to brickwork, allowing for triangular loading, is

$$M_1 = \frac{W_1 l}{6} = \frac{(10{,}530)(18)}{6} = 31{,}590 \text{ ft.-lb.}$$

Moment due to weight of girder, assuming it to be made up of two 9-in., 21.8-lb. I-beams, the whole including gas-pipe separators, weighing 44 lb. per lineal ft., is

$$M_2 = (\tfrac{1}{8})(44)(18)^2 = 1{,}780 \text{ ft.-lb.}$$

Total moment, M, = 31,590 + 1,780 = 33,370 ft.-lb. = 400,400 in.-lb.

From tables of bending capacity, it is seen that a girder of two 9-in., 21.8-lb. I-beams would have a moment of resistance of 2 × 25,160 = 50,530 ft.-lb. at a fiber stress of 16,000 lb. per sq. in. A girder built up of two such beams would therefore be much stronger in bending than is necessary. Two 10-in., 15.3-lb. channels with a combined bending capacity of 35,680 ft.-lb. will be sufficiently strong and weigh less than the I-beams. Two such channels spaced 5 in. back to back, as shown in Fig. 21*a* will be adopted, subject to their being adequate in shear.

Total end shear $= \frac{1}{2}(W_1 + W_2) = \frac{1}{2}(10{,}530) + (18)(44) = 5{,}660$ lb.

Average shearing stress on webs $= v_a = \dfrac{5{,}660}{(2)(9)(0.24)} = 1{,}310$ lb. per sq. in. The girder is therefore evidently ample against both shear and diagonal buckling of the web.

A single tier of 1-in. gas pipe separator spaced 3 ft. apart, will be used.

A type of multiple beam girder requiring great care in design is a tier of beams in a grillage. In this case shearing and web crippling stresses are likely to be

very important, if not the determining factor in the design of the tier. The compressive stresses in the webs due to the application of heavy concentrated loads to the flanges must also be investigated. For a problem of this kind see **Art. 18.**

METALLIC LINTELS

By C. R. Young

28. Types and Uses.—Beams which carry walls over openings and deliver their loads to masonry walls or piers rather than to columns are called lintels. While structurally simple, their design is rendered somewhat uncertain by differences of opinion as to how much of the weight of the wall supported is really borne by the lintel, and how much that does go to the lintel is borne by the component sections thereof. The matter of loading from masonry walls has been discussed in detail in Art. **27.** The clear spans may vary from the width of an ordinary window or door to more than 20 ft. Metallic lintels may be of structural steel or of cast iron.

29. Steel Lintels.—Some types of structural steel lintels commonly employed are shown in Fig. 24. An essential feature of these members is that they must

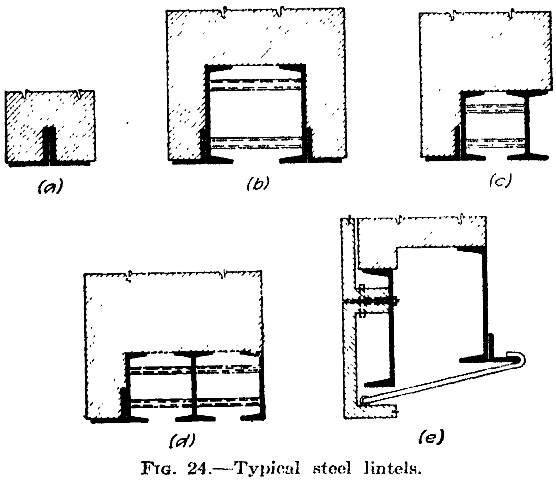


Fig. 24.—Typical steel lintels.

be so constructed as to give proper support to every part of a supported body of masonry, the bottom of which may be irregular in outline and at different levels, as shown in Fig. 24 (*e*). The support offered may be in part through hook bolts or anchors attached to convenient flanges of angles specially riveted to the primary elements of the lintel. Several shapes, specially arranged for each particular case, are often required for the support of walls with stone or terra cotta facing which must be tied in to the mass.

In the design of steel lintels the same principles are observed as in the design of multiple beam girders. The angles riveted to the sides of primary shapes, as in Fig. 24 (*b*), (*c*), and (*d*), are attached by sufficient rivets to support the

column of masonry bearing on them for only a few feet above, or up to such height as the projecting masonry may be considered as thoroughly bonded into the principal mass and deriving its support therefrom. These angles are not regarded as contributing to the flexural strength of the lintel as a whole. Provision must be made for any floor loads applied to the lintel or to the wall carried by it.

In fixing the composition and lateral dimensions of the lintel, regard must be had to the necessity for supporting the mass of masonry above so that no cracking will occur. A continuous surface for the bearing of the supported wall is not required, as the masonry will arch laterally over a space of several inches between the component sections. The shapes employed must not be so shallow as to deflect to such an extent as to prove unsightly or cause cracking in the masonry. For a discussion of deflection see Art. **19.**

30. Cast Iron Lintels.—Occasionally use is made of cast iron for lintels, although much less frequently than a few years ago, due to the greater reliability of structural steel. Common forms of such lintels are shown in Fig. 25. They consist essentially of a flat plate, or soffit, surmounted by a vertical rib or ribs. The number of ribs required will depend on the span and loading. These ribs are encased in the masonry or form exposed surfaces which may be ornamented (Fig. 25*d* and *e*). While theoretically they should be so proportioned that the factor of safety against compression on the upper fibers of the ribs would be the same as the factor of safety against tension on the lower fibers of the soffit, the liability to shrinkage cracks at the junction of highly unequal masses of cast iron, prompts designers to use the same thickness of rib as of soffit. The dimensioned requirements and limiting deflections also tend to modify any proportions that might be fixed by the stresses and the properties of the material. The New York and Boston building codes both specify that cast iron lintels shall not be less than ¾ in. thick and shall not be used for spans exceeding 6 ft.

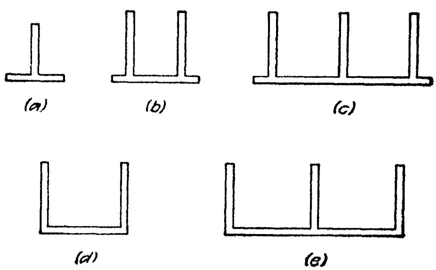


Fig. 25.—Typical cast-iron lintels.

In calculating the capacity of cast iron lintels, account should be taken of the fact that the flexural capacity may be fixed by the tensile or the compressive stresses on the corresponding extreme fibers. However, with ordinary proportions the capacity is much more likely to be limited by tensile than by compressive flexural stresses.

Typical permissible working stresses for gray cast iron are those prescribed in the New York building code, namely: bending on extreme compressive and tensile fibers = 16,000 and 3,000 lb. per sq. in., respectively; shear = 3,000 lb. per sq. in.

Due to the lack of symmetry of the section in a vertical direction, it is necessary to find the center of gravity of the section preliminary to finding the moment of inertia. The procedure to be followed in fixing a typical lintel section is one of trial and error—that is, a section is assumed and its capacity found. If it is

inadequate, the section is increased until it is sufficient to carry the specified load. The method is best elucidated by an example.

Illustrative Problem.—Find the total safe uniform load for a cast-iron lintel of 6-ft. span, center to center, having the section shown in Fig. 26, if the permissible stresses in flexure on the extreme compressive and tensile fibers are 16,000 and 3,000 lb. per sq. in. respectively, and in shear 3,000 lb. per sq. in.

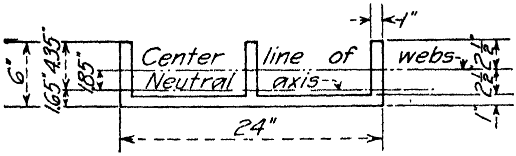


Fig. 26.—Design of a cast-iron lintel.

To find the center of gravity of the cross section—that is, the position of the neutral axis—it is convenient to take moments about the center line of the ribs. The calculations are then as follows:

Part	Area	Arm	Statical Moment
3 webs....................	15.0	0	0
Flange....................	24.0	−3.0	−72.0
	39.0		−72.0

Position of center of gravity *below* center line of webs $= 72/39 = 1.85$ in.

Moment of inertia of 3 webs about the neutral axis, or the gravity axis of whole section $= 3(I_0 + Ay_0^2) = 3[(1/12)(1)(5)^3 + (5)(1.85)^2] = 82.5$.

Moment of inertia of flange about neutral axis $= [(1/12)(24)(1)^3 + (24)(1.15)^2] = 33.8$.

Total moment of inertia $= 116.3$.

Section modulus, S_c, with respect to extreme compressive fiber $= 116.3/4.35 = 26.7$.

Section modulus, S_t, with respect to extreme tensile fiber $= 116.3/1.65 = 70.4$.

Safe resisting moment based upon permissible extreme fiber stresses in compression and tension is $(26.7)(16{,}000) = 427{,}000$ in.-lb. and $(70.4)(3{,}000) = 211{,}000$ in.-lb. respectively.

The safe capacity is, therefore, dependent on the tensile stress in flexure and the total safe uniform load is

$$W = \frac{8M}{l} = \frac{(8)(211{,}000)}{(6)(12)} = 23{,}400 \text{ lb.}$$

To facilitate the selection of cast iron lintel sections, tables may be prepared giving the properties of all sections likely to be employed.

BOX GIRDERS

By C. R. Young

31. Types and Uses.—In situations where a broad, comparatively shallow beam of great strength is required, such as for the support of walls or columns, and a multiple beam girder of sufficient capacity cannot be devised, resort is had to the box girder. This may consist of two or more rolled beams or channels arranged as in a multiple beam girder, with cover plates on their flanges, or it may be composed of an assemblage of built-up channels or beams with cover plates, as shown in Fig. 27.

The form of section adopted depends on the character and magnitude of the load carried. For moderate wall loads and for spans of such length that the

design depends on bending moment rather than on shear, the sections shown in Fig. 27 (a) and (b) utilizing channels and I-beams are satisfactory. By the use of the deeper and heavier I-beams, reinforced by one or more cover plates, a large moment of resistance may be developed with comparative cheapness. As has been pointed out in Arts. **6** and **24**, the use of I-beams is preferable where the predominating stress is flexural, but where shear and web crippling are determining factors, channels may be employed to advantage. In the latter case, the girder sections shown in Fig. 27 (a) and (c) are suggested.

If the situation calls for a special depth that cannot be made up by the use of reinforced rolled sections, or the required resistance cannot be readily developed in that manner, built-up channels or I-beams, as shown in Fig. 27 (e) and (f), are employed with the necessary number of cover plates. By this method it is possible to place the material where it will be most effectively used for moment and shear. Only one angle can be employed to connect the cover plates to the outer webs of the girder shown in Fig. 27 (f), since, unless the girder is large enough to allow a man to crawl through it, the riveting to the inside angles could not be done. The flanges are riveted to the center web before the center webs are assembled in place.

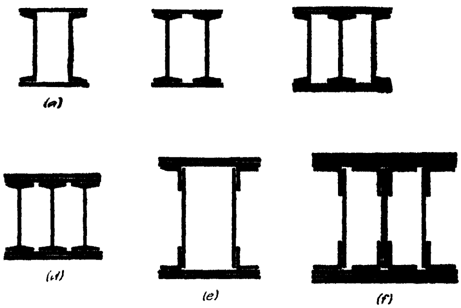


FIG. 27.- Cross-sections of typical box girders.

It is with the object of resisting very heavy shears, rather than moments, that the sections with three webs, Fig. 27 (c), (d), and (f), are employed. Within certain limits, an increased bending moment might be met by increasing the number or thickness of the flange plates, but, for increased shear, additional webs or thicker ones must be used. If the load applied to the top flange, or cover plates, be uniformly distributed laterally, it is reasonable to assume that, for a section such as (d), the three I-beams would bear the load equally, if they are of equal strength and stiffness. If the outer ones are lighter, or if the section be as in (c), with channels on the outside, the lesser stiffness of the outer component parts would, under the same load as could be borne by the center section, bring about a greater yielding in them and a transfer of a larger proportion of the load to the center section. Consequently, it is common practice to make the center web of girders of the type of (f) twice the thickness of each of the outer webs. Two angles at both the upper and the lower edges of this web are needed to receive and transfer the flange stress that is passed on to them by the web. Under the above assumption, it is reasonable to assume that one-half the cover plate area is tributary to the center web and one-quarter to each of the outer webs. Since the center web is also twice as thick as the side webs, the values of the flange rivets through it will be twice as great as for those through the outer webs. It is thus possible by this arrangement to keep the rivet spacing equal in the inner and outer flange angles.

It is desirable to keep the composition of the flanges practically alike so that the neutral axis may not be in any case far from the center of the webs. By so doing the rivet spacing may in general be made the same in the two flanges.

32. Advantages and Disadvantages.—The advantages and disadvantages of box girders are the same as those pertaining to multiple beam girders. They give high strength with shallow depth; they afford broad bearing for applied loads and at the supports; and they are stiff laterally. On the other hand, they are uneconomical of material; their pound cost is greater than for single or multiple rolled sections because of the extra work of fabrication involved; they lack vertical stiffness because of their small depth; and they are subject to corrosion on the interior faces in damp situations.

Box girders are superior to multiple beam girders because of the better tie between the compression flanges and the better bearing for the applied load. The rigid connection of parts reduces the tendency to flange buckling and increases the factor of safety in compression.

33. Proportioning for Moment.—Whether a box girder consists of rolled beams or channels with flange plates riveted thereto, Fig. 27, (*a*), (*b*), (*c*) and (*d*), or of an assemblage of plates and angles, Fig. 27 (*e*) and (*f*), it should, because of the rigid attachment of the parts to each other be regarded as essentially a built-up beam. Its moment of resistance, or capacity to resist bending moment should, therefore, be computed from the common flexure formula, $f/c = M/I$, or $M = fI/c$. This necessitates the computation of the moment of inertia, I, of the section, unless this fortunately chances to be listed in available tables, as in those given in Cambria Steel. Because of the innumerable combinations of shapes and plates worked into box girder sections, the properties of the particular section most suitable for the work in hand are frequently not listed and so must be specially determined.

It is of great advantage to use an approximate method of design at times, particularly in making rough estimates or in making the first trials for an exact design. For such purpose the approximate method of designing plate girders (Art. **44**) may be used. If the box girder be much over 3 ft. deep, such a method may be sufficiently accurate for final design.

In proportioning box girders, the permissible working stress is commonly assumed at either 15,000 or 16,000 lb. per sq. in. The tables of capacity of box girders given in Cambria Steel are based on the former stress, but the rivet holes are assumed as only $\frac{1}{16}$ in. larger than the diameter of the rivet. The capacity tables for riveted beam girders given in the Carnegie Pocket Companion are based on a working stress of 16,000 lb. per sq. in., but the section modulus used is that for the gross section.

In making exact designs by what is called the "moment of inertia" method, no reduction in section modulus need be made for any type of beam on account of holes in the compression part of the section. The neutral axis of a box girder, therefore, lies somewhat above the center of gravity of the gross section, if the flanges have the same gross area, but for the reasons set forth in Art. **7**, the shift cannot be so great as consideration only of the net area through the weakest section would indicate. In view of the uncertainty as to the exact position of the neutral axis, and in view of the simplification of work introduced by computing the net section modulus with respect to the neutral axis of the gross section, this

method will be adopted, the results being corrected as recommended in Art. 10 to compensate for the error in the assumption of the position of the neutral axis. For further substantiation of the percentage corrections specified in the latter article, see the second problem under Art. 37.

For girders with heavy flanges so supported laterally that no allowance need be made for flange buckling, it is desirable to make the net area of the tension flange as nearly as possible equal to the gross area of the compression flange, which means that the gross area of the tension flange must be greater than the gross area of the compression flange. By so doing, the neutral axis is kept practically at the center line of the webs, thus improving the flexural efficiency of the girder and making it possible to keep the rivet spacing in the tension and compression flanges equal (except for local transverse loading). An approximate compensation for the loss of section due to rivet holes on the tension side may be made by adding to the cover plates on the tension side sufficient area to offset the rivet holes in the flange material. The high working stress of the material added to the plates will roughly offset the neglect of any web holes.

In the computation of net section modulus of box girders built up with rolled sections as their primary component parts, such as shown in Fig. 27 (*a*), (*b*), (*c*) and (*d*), the work may be carried out by making use of Table 3. To the net section modulus in the compound section of the beams or channels employed, based on the stationary axis theory, may be added the net section modulus of the added plates, deducting holes for the plates on the tension side only. The sum may then be reduced by the appropriate percentage to compensate for the erroneous assumption of fixed neutral axis.

Illustrative Problem.—Calculate the net section modulus of the compensated, 3-web, built-up box girder section shown in Fig. 28, assuming that rivet holes are to be deducted on the tension side only and that the neutral axis is at the center line of the webs. Rivets ⅞ in. and rivet holes 1 in. diameter.

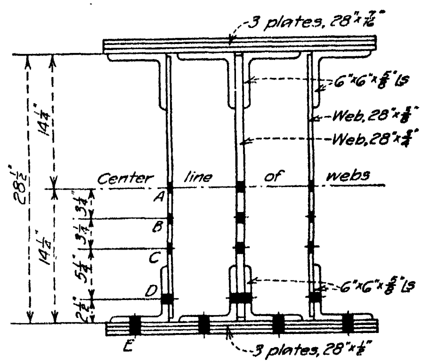


Fig. 28.—Section modulus of compensated built-up box girder.

In finding the net moment of inertia of the compensated section, it is convenient to tabulate the quantities as done in Table 10, the gross areas being taken first and the moment of inertia of the holes being listed below. The moment of inertia of the holes about their own gravity axes is neglected, since it is relatively too small to affect the result appreciably. Areas of metal are marked "plus" and areas of holes "minus." Distances above and below the assumed neutral axis, which is at the center line of the webs, are "plus" and "minus" respectively. The summations at the foot of the second, fourth, fifth and sixth columns are algebraic. The summation of the fourth and fifth columns when added should equal the summation of the sixth.

Net section modulus of girder = net moment of inertia divided by the distance from the assumed neutral axis to the extreme fiber, which in this case (because of the thicker flange plates on the tension flange) is to the extreme tensile fiber.

Hence

$$S = \frac{I}{c} = \frac{27{,}028.5}{15.75} = 1{,}717$$

TABLE 10.—NET SECTION MODULUS OF COMPENSATED BUILT UP BOX GIRDER

Part	Area of part (sq. in.)	Distance (y_0) from assumed neutral axis to gravity axis of part (in.)	Moment of inertia (I_0) of part about its gravity axis (in.^{4})	Ay_0^2 (in.^{4})	$I = I_0 + Ay_0^2$ (in.^{4})
3 webs	+ 42.00	0.0	+2,744.0	0.0	+ 2,744.0
4 top angles	+ 28.44	+12.52	+ 96.8	+ 4,460.0	+ 4,556.8
4 bottom angles	+ 28.44	−12.52	+ 96.8	+ 4,460.0	+ 4,556.8
3 top covers	+ 36.75	+14.91	+ 5.3	+ 8,175.0	+ 8,180.3
3 bottom covers	+ 42.00	−15.00	+ 7.9	+ 9,450.0	+ 9,457.9
3 Holes A	− 1.50				
3 Holes B	− 1.50	− 3.25	....	− 15.9	− 15.9
3 Holes C	− 1.50	− 6.50	....	− 63.4	− 63.4
3 Holes D	− 4.00	−11.75	....	− 553.0	− 553.0
4 Holes E	− 8.50	−14.69	....	− 1,835.0	− 1,835.0
	+160.63	....	+2,950.8	+24,077.7	+27,028.5

As compensation has been made by the extra thickness of the flange plates on the tension flange for the loss of section occasioned by rivet holes on the tension side of the neutral axis, no correction need be made because of the assumption that the neutral axis is at the center line of the webs.

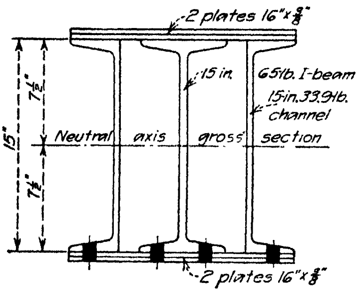


FIG. 29.—Capacity of rolled shape box girder.

Illustrative Problem.—Find the total safe uniformly distributed load that may be carried by a box girder of 21-ft. span of the form shown in Fig. 29, consisting of one 15-in., 65-lb. I, two 15-in., 33.9-lb. channels, and two 16 × ⅜-in. flange plates on each flange. If ¾-in. rivets are used, and two lines are employed in each flange of the I-beam in addition to one line in each flange of the channels, compute the effective section modulus of the combined section, assuming the neutral axis as at the center of gravity of the gross section, and correcting the result as explained in Art. **10**. $f = 16{,}000$ lb. per sq. in.

Gross moment of inertia of one 15-in., 65-lb. I and two 15-in., 33.9-lb. channels about their own gravity axes = 632.1 + (2)(312.6) = 1,257.

Gross moment of inertia of two pairs of 16 × ⅜-in. flange plates

$$= 2\left[\left(\frac{1}{12}\right)(16)(0.75)^3 + (12.0)(7.875)^2\right] = 1{,}490$$

Total gross moment of inertia = 1,257 + 1,490 = 2,747.

I of two ⅞-in. diam. holes through tension flange of beam and flange plates, if grip of beam is ⅞ in. = $(2)(1.63)(0.88)(7.44)^2 = 158$.

I of two holes through channel flanges and plates = $(2)(1.38)(0.88)(7.56)^2 = 138$.

Total I of 4 holes = 296.

Net I of entire section = 2,747 − 296 = 2,451.

Net section modulus = 2,451/8.25 = 297.

This corrected by the coefficient 0.95, as recommended in Art. **10** is (0.95)(297) = 282.

Hence total safe uniformly distributed load

$$W = \frac{8Sf}{l} = \frac{(8)(282)(16{,}000)}{(21)(12)} = 143{,}000 \text{ lb.}$$

34. Length of Flange Plates.—As in the case of rolled beams reinforced for bending, discussed in Art. **12**, it is possible, if there is more than one plate on each

flange, to vary the section of a box girder by terminating some of the flange plates where they are no longer needed. The theoretical length of any flange plate, of a uniformly loaded girder is, as was established in Art. **13**,

$$x_n = l\sqrt{\frac{s_1' + s_2' + \ . \ . \ . \ s_n'}{S}} \tag{1}$$

where x_n theoretical length of the nth flange plate from the outside.

l = span of girder.

$s_2' = \ . \ . \ . \ s_n'$ = section moduli contributed to the total required section modulus of the girder by successive pairs of cover plates from the outside beginning with the second plate.

s_1' = section modulus required to be contributed by outside plate.

S = total required net section modulus of girder.

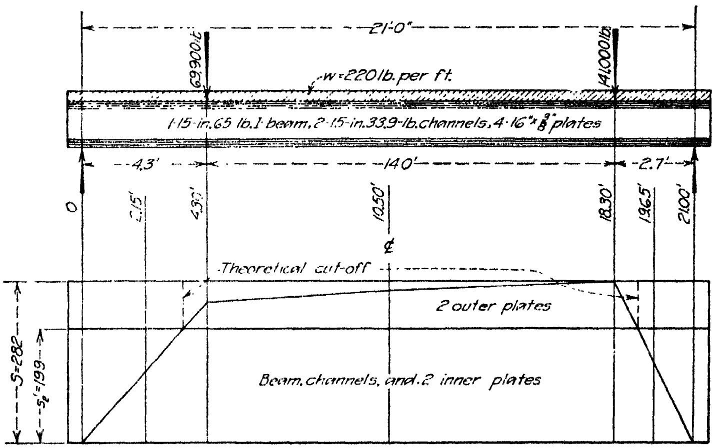


FIG. 30. Graphical determination of length of flange plates for box girder.

In a box girder one cover plate on each flange must run full length to form the necessary tie between the main component parts of the girder, and often in the case of the top flange, to receive and distribute the applied load.

While it is commonly specified that the thinnest of the flange plates shall be put on the outside, there appear to be better reasons for placing the thinnest on the inside, as pointed out in Art. **46**.

The practical rule respecting the addition of 9 or 12 in. to the theoretical length of the cover plates at each end stated for reinforced beams in Art. **13** applies also to the box girder.

If the loading on a box girder be not uniform, it is necessary to make use of a graphical method for the determination of the length of cover plates. The bending moment is computed at critical or determining points and from it the required section modulus is derived. A diagram is then prepared for the half span if the loading be symmetrical, and for the full span if it be unsymmetrical, showing the requirement for section modulus at all points of the span. On this diagram is

laid off vertically from the base the section modulus provided (1) by the primary component parts of the box girder, (2) by each of the successive plates from the inside plates outward. The point where the upper horizontal boundary line of the rectangle laid off to represent a component part of the girder cuts the curve will locate the point of theoretical cut-off for the *next* (outside) part represented.

Illustrative Problem.—Assuming that the loading to which the 21-ft. girder of Fig. 29 is subjected, is not uniform, but is as shown in Fig. 30, and that the span is 21 ft., find the theoretical and practical lengths of the flange plates.

The moments and required section moduli at the points of concentrated loading and certain intermediate points are as shown in the accompanying table.

Distance of point from left support (ft.)	Uniform load moment (ft.-lb.)	Moment from concentrated (ft.-lb.)	Combined moment (ft.-lb.)	Required section modulus (in.^{3})
2.15	4,460	158,500	162,960	122
4.3	7,895	317,000	324,895	244
10.5	12,100	342,000	354,100	266
18.3	5,400	369,500	374,900	282
19.65	2,900	185,100	188,000	141

Plotting the required section moduli vertically on a diagram for the full span, Fig. 30, and laying off also vertically the section moduli provided by the primary beam and channels, and the successive cover plates, the theoretical required lengths of the plates may be readily scaled off.

In the previous problem on the girder of Fig. 29, the effective section modulus at the maximum section was found to be 282.

The effective section modulus of the beam and channels plus *one* cover plate on each flange needs to be found.

Gross I of beam and channels $= 1{,}257$.

Gross I of two plates (one on each flange) is approximately $(2)(6)(7.1875)^2 = 620$.

Total gross $I = 1{,}877$.

I of two holes through beam flange and plate and two through the channel flanges and plate

$$= [(2)(1.25)(0.88)(7.25)^2] + [(2)(1.00)(0.88)(7.75)^2] = 115 + 105 = 220.$$

Net I of section with two flange plates only $= 1{,}877 - 220 = 1{,}657$.

Corrected net section modulus $= (0.95)(1{,}657)/7.875 = 199$.

Laying off this distance vertically on the required section modulus diagram of Fig. 30 and drawing a horizontal line across the diagram, the points of theoretical cut-off are found where this line cuts the curve. Since the stress which these outer plates must carry must be transferred to them in distances of 4.3 and 2.7 ft. at the left and right hand ends respectively, the plates will need to be carried full length to accommodate the necessary rivets.

35. Stiffeners.—In order that concentrated loads may be transferred to the webs of a box girder without exceeding the permissible buckling stress in the webs, it may be necessary to use stiffeners, as in the case of beams, Art. **18.** If it is not practicable to make the thickness of the webs great enough to obviate danger from crippling due to ordinary diagonal compression, stiffeners spaced at suitable intervals throughout the length of the girder will need to be used. For the principles governing their proportioning and spacing, see Art. **52.**

36. Diaphragms.—To ensure that the principal component elements of a box girder act together as a unit and that any excess of loading received by one ele-

ment is distributed to the others, diaphragms should be inserted at certain points. It is not prudent to count on the stiffness of the cover plates in a vertical direction to transfer load from one vertical rib to another. Diaphragms are essentially the same as the built-up separators, described in Art. **26**, being attached to the webs of the box girder preferably by rivets, or if such cannot be driven, then by, bolts if possible. If web stiffeners are used for the girder, the attachment of the diaphragm plates may be to them. If stiffeners are not used, the attachment may be by vertical connection angles.

The difficulty of fastening diaphragms to the webs arises from the fact that unless the girder be very large, rivets cannot be driven through the outside webs of a three-web girder or through either web of a two-web girder after the flange plates are riveted on. For a two-web girder the diaphragms may in theory at least, be riveted to the webs *before* the flange plates are riveted on, though there are shop difficulties entailed by this procedure. For a three-web girder, the cover plates must be connected to the inner web before the outer webs are assembled in place, thus making it impossible to rivet the diaphragms to the outer webs. One way of overcoming the difficulty is to rivet the diaphragms to the inner web, as shown in Fig. 31, and then provide on the inside of each of the outer webs (at top and bottom where diaphragms occur), short bracket angles "*A*" between which the diaphragm would fit. Before these bracket angles are riveted in place, the diaphragms should be fitted in between them to ensure close bearing when the whole girder is riveted up. By this device, the excessive deflection of one web with respect to the others could be obviated—that is, the load might be distributed transversely.

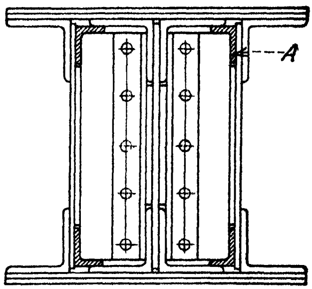


Fig. 31.—Distributing diaphragm for box girder.

Where the spacing of the girder webs permit, single pieces of channel run vertically, as shown in Fig. 21*c*, may be used. This is usually only practicable for girders composed of rolled sections as the primary elements, and is not desirable for the heavier girder because of the lack of stiffening of the horizontal edges of the diaphragm. For such girders, diaphragms with both horizontal and vertical edge angles are desirable. The thickness of the diaphragm plate is commonly fixed by experience, though an indication of a suitable thickness may be gained by considering a load equal to one-quarter of the total load carried by the girder, divided by the number of diaphragms connecting to one outer web as applied at the upper outside corner of each diaphragm. The web should then be proportioned by a web crippling formula such as

$$p = 15{,}000 - 150\frac{h}{t}$$

or any other of those mentioned in Art. **17**. The connection and stiffening angles should be $\frac{5}{16}$ in. for the lighter girders and $\frac{3}{8}$ in. for the heavier ones.

Diaphragms should be placed at all points of concentrated loading to ensuer the proper lateral distribution of the load. They should be placed also at the

ends to give lateral support where the web buckling tendency is pronounced, and at such other points as they might appear desirable in view of probable inequalities of the loading.

37. Flange Riveting.—A simple but indirect method of determining the spacing of rivets in the flanges of a box girder is that followed in the problem on the reinforced beam, Art. **14.** This consists of finding the difference in total stress in the added flange material at two sections, and placing between the sections sufficient rivets to develop the difference in stress. The two sections may be conveniently taken at the end of the attached flange element and at the point of maximum stress therein. For most box girders, such as those with only one or two plates of moderate thickness, where the theoretical spacing is much greater than would be permissible by the practical restrictions relating to the maximum rivet spacing, this method is sufficiently accurate. In computing the total stress in the plate, the stress per sq. in. may, without material error, be taken as the maximum permissible fiber stress in bending.

For box girders with relatively heavy flanges, in which the adopted rivet spacing will depend on stress conditions rather than on practical rules for maximum spacing, it is desirable to employ a more exact method than that described in the previous paragraph. Although in the case of deep box girders, the flange rivet spacing may be determined by the approximate methods usually adopted for plate girders (discussed in Art. **51**), the generally applicable method is that based on the true horizontal shear between faces of connected parts.

It has been established in Sec. 1, Art. **51***b*, that the intensity of horizontal shearing stress at any point in a beam or girder is given by the formula

$$v = \frac{QV}{It}$$

where v = intensity of horizontal (or vertical) shearing stress in lb. per sq. in.

Q = statical moment of area on either side of the point considered, taken about the neutral axis.

V = total vertical shearing force at the section considered.

I = moment of inertia of the area of the entire section about the neutral axis.

t = thickness of the section at the point considered.

For a lin. in. of girder the horizontal shearing area = $(1)(t) = t$, and hence the total horizontal shear per lin. in. is

$$H = vt = \frac{QV}{I}$$

While the applicability of this formula to joints that lie in a horizontal plane is clear, it may not be so evident that it applies to joints in a vertical plane. For example, let it be required to determine the total horizontal shear per lin. in. between the web plates of the girder shown in Fig. 32 and the flange angles riveted thereto. The total horizontal shear between that portion of the section which lies above the horizontal plane BB, and the portion lying below it, is obviously

$$H_B = \frac{Q_B V}{I}$$

Part of H_B is borne by the portion of the two web plates above the level BB and part by the flange angles and cover plate, the division being on the basis of relative statical moments. Consequently if in Q_B only the statical moment of the angles and cover plate is included, the result, H_B, will be the amount of total horizontal shear between the flange angles and the web plates.

Having found the horizontal shear per lin. in., H_1 between an element of the flange which is riveted to the remainder of the girder, the rivet pitch in the flange element, provided it does not bear any local transverse load, may be expressed by the formula

$$p = \frac{r}{H_1}$$

where r = safe resistance of one rivet in the situation under consideration.

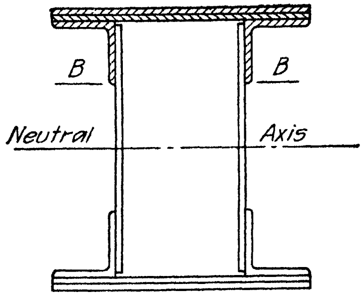


Fig. 32.—Horizontal shear between web and flanges.

Illustrative Problem.—If, at a certain section, the box girder shown in Fig. 28 is subjected to a total vertical shear of 348,000 lb., find the required pitch of the rivets in the tension flange. Rivets, $\frac{7}{8}$ in. diameter. Safe shearing and bearing stresses = 12,000 and 24,000 lb. per sq. in., respectively.

Total horizontal shear per lin. in. on planes of contact between web plates and angles of tension flange riveted thereto is

$$H_1 = \frac{QV}{I}$$

The statical moment of the net area of the tension flange about the assumed neutral axis (the center line of the webs) may be readily determined from the figures given for this girder in Table 10, p. 236, respecting the moment of inertia of this section. Arranged in tabular form, the computation is as below. For this purpose the areas of holes are considered negative, but all distances, though measured downward from the neutral axis, are taken as positive. From the summation of the last column, the statical moment is seen to be 831.8.

Part	Area of part (sq. in.)	Distance (y_0) from assumed neutral axis to gravity axis of part (in.)	Statical moment of part (Ay_0) (in.^{3})
4 bottom angles	+28.44	+12.52	+356.0
3 bottom covers	+42.00	+15.00	+630.0
Part 3 holes (D)	− 2.50	+11.75	− 29.4
4 holes (E)	− 8.50	+14.69	−124.8
	+59.44	....	+831.8

Net moment of inertia of section, from Table 10 = 27,028.5.

Horizontal shear, per lin. in. of girder, transferred to flanges by web

$$H_1 = \frac{(831.8)(348,000)}{27,028.5} = 10,700 \text{ lb.}$$

Since the outer webs are each one-half the thickness of the center web, each will be assumed as taking one-quarter of the total horizontal shear per lin. in.—that is,

$$H_1 = (0.25)(10{,}700) = 2{,}675 \text{ lb.}$$

The minimum value of one ⅞-in. rivet connecting the flange angles to the center web is its single shearing value $r = (0.60)(12{,}000) = 7{,}200$ lb.

Hence theoretical pitch should be

$$p = \frac{r}{H_1} = \frac{7{,}200}{2{,}675} = 2.69 \text{ in.}$$

To give a well distributed connection between the web and the flange angles it is best to use two gage lines in the vertical legs of the angles, so that the pitch of 2.69, or say 2⅝ in., would be a staggered pitch.

Illustrative Problem.—Determine the necessary section for a symmetrically made up double-beam box girder, of the form shown in Fig. 33 (illustrating a girder designed in accordance with assumption *D*), to the data given below under the following alternative assumptions respecting allowance made for rivet holes and the position of the neutral axis:

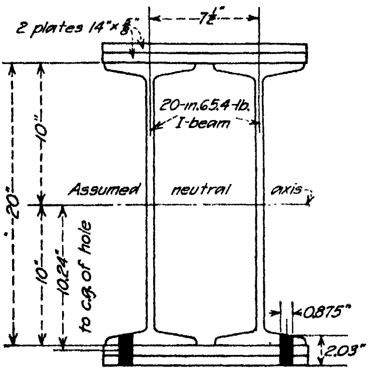


FIG. 33.—Proportioning of box girder cross-section—Design D.

(*A*) No deduction for rivet holes; neutral axis at center of gravity of gross section.

(*B*) Rivet holes in both flanges deducted; neutral axis at center of gravity of gross (or net) section.

(*C*) Rivet holes in tension flange only deducted; neutral axis at center of gravity of net area.

(*D*) Rivet holes in tension flange only deducted; neutral axis at center of gravity of gross section.

Data.—Span, 22 ft. clear, or 23 ft. 4 in. center to center of bearings.

Section must be not over 23 in. deep nor over 14 in. wide.

Load to be 10,270 lb. per lin. ft., uniformly distributed, consisting of 10,000-lb superimposed load and 270 lb. due to weight of girder.

Permissible stresses to be as follows: Bending on extreme fiber = 16,000 lb. per sq. in.; shearing on beam webs = 10,000 lb. per sq. in. on gross area.

Vertical buckling stress on beam webs

$$f = \frac{R}{\left(a + \frac{d}{4}\right)t}$$

must not exceed $p = 19{,}000 - 173\frac{d}{t}$, where

R = end reaction of one beam in lb.
a = length of bearing in in.
d = depth of beam.
t = thickness of beam web.

Rivets = ¾ in. dia.; holes for stress calculations = ⅞ in. diam.

Design A

Shear.—End reaction, or end shear,

$$V_1 = (\tfrac{1}{2})(23.33)(10{,}270) = 119{,}800 \text{ lb.}$$

Required section for shear = 119,800/10,000 = 11.98 sq. in.

As the limitation for depth would permit 20-in. beams, it would be in the interests of flexural economy to use beams of this depth, although shallower ones with thicker webs would be desirable if the shear happened to determine the design. Assume two 20-in., 65.4-lb. I's. Shear area provided = $2dt = (2)(20)(0.5) = 20$ sq. in.

These are, therefore, sufficient.

Web Buckling.—Existing vertical compressive stress on the web of *one* beam at the support, assuming a 14-in. bearing,

$$f = \frac{(\frac{1}{2})(119{,}800)}{\left(14 + \frac{20}{4}\right)(0.5)} = 6{,}300 \text{ lb. per sq. in.}$$

Permissible buckling stress on web,

$$p = 19{,}000 - 173\left(\frac{20}{0.5}\right) = 12{,}100 \text{ lb. per sq. in.}$$

Hence, assumed section is safe against web buckling.

Bending.—Maximum bending moment,

$$M = \frac{Wl}{8} = \frac{(10{,}270)(23.33)(280)}{8} = 8{,}387{,}000 \text{ in.-lb.}$$

Required maximum section modulus,

$$S = \frac{M}{f} = \frac{8{,}387{,}000}{16{,}000} = 52$$

Assume as section the following:

Two 20-in., 65.4-lb. I's
Four 14 × $\frac{9}{16}$-in. plates,

arranged as shown in Fig. 33 (which in detail applies only to case *D*), the outer plate on each flange to be cut off at the point where it is no longer necessary.

Gross moment of inertia of two 20-in., 65.4-lb. I's about neutral axis of girder = (2)(1169.5) = 2,339.

Gross moment of inertia of two pairs of 14 × $\frac{9}{16}$-in. plates,

$$\begin{aligned} I_p &= 2(I_0 + Ay_0^2) \\ &= 2[(\tfrac{1}{12})(14)(1.125^3) + (14)(1.125)(10.563^2)] \\ &= 2(2 + 1{,}755) = 3{,}514 \end{aligned}$$

Total gross moment of inertia = 5,853.

Gross section modulus provided = 5,853/11.125 = 526. The section assumed is therefore adequate for bending.

Design B

Shear and Web Buckling.—The stresses and necessary sections are the same as for Design *A*.

Bending.—To compensate for the loss of area due to rivet holes in *both* flanges, assume the following section:

Two 20-in., 65.4-lb. I's
Two 14 × $\frac{5}{8}$-in. plates (inside)
Two 14 × $\frac{3}{4}$-in. plates (outside).

Gross moment of inertia of two I's = 2,339 in.^{4}

Gross moment of inertia of two pairs of plates, each pair comprising one 14 × $\frac{5}{8}$-in. plate and one 14 × $\frac{3}{4}$-in. plate, the thinner plate being on the inside for the reason given in Art. **46**,

$$I_p = 2[(\tfrac{1}{12})(14)(1.375^3) + (14)(1.375)(10.688^2)] = 4{,}404 \text{ in.}^4$$

Total gross moment of inertia = 2,339 + 4,404 = 6,743 in.^{4}

Moment of inertia of four $\frac{7}{8}$-in. holes through 0.78 + 1.38 = 2.16 in. of metal, neglecting the moment of inertia of the holes about their own gravity axis,

$$I_h = (4)(0.875)(2.16)(10.30)^2 = 803 \text{ in.}^4$$

Net moment of inertia = 6,743 − 803 = 5,940 in.^{4}

Net section modulus = 5,940/11.375 = 522.

This is sufficiently near the requirement, 524.

Design C

Shear and Web Buckling.—Same as for Design *A*.

Bending.—In this case, only the holes on the tension side are to be deducted and the neutral axis is assumed to take up a position at the center of gravity of the resulting net area.

The eccentricity, e, or distance of the center of gravity of the net area from the center of gravity of the gross area may be readily found by the formula,

$$e = \frac{Q_h}{A_n}$$

where Q_h = statical moment of the holes deducted, taken about center of gravity of the gross section;

A_n = net area of whole section.

For this case assume that the section adopted and the gross area is as follows:

Two 20-in., 65.4-lb. I's	= 38.16 sq. in. gross
Four 14 × ⅝-in. plates	= 35.00 sq. in. gross
Total gross area	= 73.16 sq. in.

The statical moment of two holes about the neutral axis of the gross section

$$Q_h = (2)(0.875)(2.03)(10.24) = 36.4 \text{ in.}^4$$

Net area of whole section = 73.16 − (2)(0.875)(2.03) = 69.60 sq. in.

Hence, eccentricity

$$e = \frac{36.4}{69.6} = 0.52 \text{ in.}$$

Moment of inertia, I_n, of net section about the neutral axis established above may be found from

$$I_n = I_g + A_g e^2 - a y_o^2$$

where

I_g = moment of inertia of gross section about neutral axis of gross section.

A_g = gross area of section.

a = area of holes deducted.

y_o = distance of center of gravity of holes deducted from neutral axis of net section.

I_g, for the present case $= 2{,}339 + 2[(\frac{1}{12})(14)(1.25^3) + (14)(1.25)(10.625^2)] = 6{,}297$ in.^{4}

$$A_g = 73.16 \text{ sq. in.}$$
$$e = 0.52 \text{ in.}$$
$$a = (2)(0.875 \times 2.03) = 3.55 \text{ sq. in.}$$
$$y_o = 10.24 + 0.52 = 10.76 \text{ in.}$$

Hence,

$$I_n = 6{,}297 + (73.16)(0.52)^2 - (3.55)(10.76)^2 = 5{,}906 \text{ in.}^4$$

Net section modulus = 5,906/11.77 = 502 in.^{3}

As this is somewhat below the requirement, 524, the section will need to be increased. Assume that the outer plates are each increased $\frac{1}{16}$ in. in thickness. The increased moment of inertia for the addition to the compression flange is approximately

$$I = (\tfrac{14}{16})(10.73 + 0.03)^2 = 101.1 \text{ in.}^4$$

The increase in moment of inertia for the added thickness to the tension flange is

$$\Delta I' = \frac{14 - (2)(0.875)}{16}(11.77 + 0.03)^2 = 106.4$$

Total increase in moment of inertia = 207.5.

Total net moment of inertia of increased section = 5,906 + 207.5 = 6,113.5. Hence, net section modulus of increased section = 6,113.5/11.83 = 518, which is the nearest approach that can be made to the requirement, 524. The outer cover plates will therefore each be 14 × $\frac{11}{16}$ in.

Design D

Shear and Web Buckling.—Same as for Design *A*.

Bending.—Assume as section,

Two 20-in., 65.4-lb. I's
Four 14 × ⅝-in. plates

From Design *C*, the gross moment of inertia of this section about the gravity axis of the gross section = 6,297 in.^{4}

Moment of inertia of two holes about the neutral axis of the gross section (assumed in this case as the neutral axis of the girder as built) is

$$I_h = (2)(0.875)(2.03)(10.24)^2 = 373 \text{ in.}^4$$

Net moment of inertia of girder = 6,297 − 373 = 5,924 in.^{4}
Net section modulus = 5,924/11.25 = 526. This is adequate.

Comparison of Designs

In order to facilitate the comparison of the results reached by designing according to the four alternative assumptions respecting the effect of rivet holes, the following table has been prepared:

Design	Holes deducted	Assumed position of neutral axis	Net I (in.^{4})	Net S (in.^{4})	Max. gross area, A (in.^{2})	$\frac{S}{A}$	Relative efficiencies
A	None	Center of gravity of gross section..................	5,853	526	69.66	7.55	1.00
B	Two from each flange	Center of gravity of gross section..................	5,940	522	76.66	6.81	0.90
C	Two from tension flange	Center of gravity of net area.	6,114	518	74.91	6.92	0.92
D	Two from tension flange	Center of gravity of gross area..................	5,924	526	73.16	7.18	0.95

In the next to the last column of the table is given the amount of section modulus developed for each square inch of gross area in accordance with the four basic assumptions of design. In the last column the relative efficiencies are given—that is, the relative amount of section modulus developed per sq. in. of gross area. From this column it is seen that there is a loss of about 10 per cent where all holes are deducted, but only from 5 to 8 per cent, depending on the assumption respecting the position of the neutral axis, when only the holes in the tension flange are deducted.

The figures given in the table show the possibility of saving time in the design of box girders by making the calculation of section modulus by one of the simpler assumptions, such as A or D, and then correcting the results in accordance with the actual assumption made respecting deductions and position of the neutral axis. For example, if the section modulus for a given section were found according to assumption A, reducing it by 8 per cent would give the section modulus for the same gross section according to assumption C. If the section modulus were found by assumption D, reducing it by 4 per cent would give the section modulus according to assumption C. It is best to base the calculation on assumption D, as the amount of correction is less than required if it is based on method A.

Illustrative Problem.—Find the theoretical and practical lengths of the cover plates for the box girder designed in the last problem, according to assumption B.

Although the outer cover plates are required for only a fraction of the girder length, the inner cover plate on the compression flange must be carried full length to stay the flanges of the beams against buckling and to provide a satisfactory bearing for the applied load. While in theory the inner cover plate on the bottom flange may be cut off short of the end, it is customary to carry it full length also. This practice has the incidental advantage of keeping the neutral axis near the center of the beam webs.

From Formula (1) of Art. **34**, the length of the outer cover is given by the formula

$$x_1 = l\sqrt{\frac{s_1'}{S}}$$

where

x = the theoretical length of the cover plate.
l = length of span.
s_1' = section modulus required to be contributed to the girder by the *two* outer covers.
S = total required net section modulus.

Net section modulus of beams with two *inner* 14 × ⅝-in. cover plates only is found by the methods already elucidated to be 328.

Difference between total required section modulus S and the section modulus provided by two beams and the two inside flange plates is

$$s_1' = 524 - 328 = 196$$

Hence, theoretical length of *outer* flange plates is

$$x_1 = 23.33\sqrt{\frac{196}{524}} = 14.3 \text{ ft.}$$

To this length about 2 ft. would be added for the reasons given in Art. **13**, so that the plates would be about 16 ft. long.

Illustrative Problem.—Determine the rivet spacing in the cover plates of the girder of the last problem given the following data: Rivets, ¾ in. diam.; permissible shearing and bearing stresses on shop rivets = 10,000 and 20,000 lb. per sq. in. respectively.

The number of rivets required in each flange from the center of the bearing to the center line of the girder is the number necessary to transmit the total stress in the two plates from the beams into those plates.

Since the fiber stress increases uniformly from the neutral axis out to the extreme fiber and the thickness of plates on each flange is ⅝ + ¾ = 1.38 in., the average fiber stress in the flange plates will be

$$f_a = (16{,}000)\left(\frac{10.69}{11.38}\right) = 15.000 \text{ lb. per sq. in.}$$

Total stress in two plates at center of girder equals net area of plates multiplied by average stress per sq. in.—that is,

$$P = [(14) - (2)(0.875)](1.38)(15{,}000) = 254{,}000 \text{ lb.}$$

Least safe resistance of one rivet is single shearing value,

$$r = (0.442)(10{,}000) = 4{,}420 \text{ lb.}$$

Number of rivets required from center of bearing to center line of girder

$$N = \frac{254{,}000}{4{,}420} = 57$$

or, say, 29 in each gage line.

Considering the outer one of the two cover plates alone—that is, the ¾-in. plate—the average working stress in it is

$$f_a' = (16{,}000)\left(\frac{11.00}{11.38}\right) = 15{,}450 \text{ lb. per sq. in.}$$

Total stress borne by one 14 × ¾-in. plate with two ⅞-in. holes out is

$$P' = [(14) - (2)(0.875)](0.75)(15{,}450) = 142{,}000 \text{ lb.}$$

Number of rivets required through outer cover plate from its end to center line of girder

$$N = \frac{142{,}000}{4{,}420} = 32$$

or 16 in each gage line.

The actual spacing should be arranged so as not to exceed 6 in. in either line and so that for a distance of about 2 ft. at each end of each plate the spacing is less than this, say 3 or 4 in.

PLATE GIRDERS

By C. R. Young

38. General Characteristics.—Whenever the situation calls for a beam or girder of greater flexural capacity than that of any single rolled beam or beam girder available, and the height conditions permit a girder of economic depth for bending, a single built-up beam, or plate girder, can be used to advantage. Although the pound price of such a girder is greater than for a multiple beam girder, or for a box girder utilizing rolled beams or channels for its primary elements, the saving in metal due to the use of a more favorable depth generally makes the plate girder cheaper for heavy loads.

Essentially, a plate girder is an I-beam built up of plates and angles, as shown in Fig. 34. Unlike the I-beam, which is of uniform section throughout its length, the plate girder may be varied in section should such prove desirable. It is, for example, easy to reduce the flange section at points where the smallness of the bending moment warrants it, and for very large girders to use a thinner web in the region of light shear than in those of heavy shear.

Like the I-beam, the plate girder developed naturally from the I-shaped cast-iron beams and girders that preceded it. An indication of the possibilities of long-span built-up girders was given in the successful completion of the great Britannia and Conway tubular bridges in Wales, the former containing two spans of 460 ft. each, built in 1850. These tubular spans were, in reality, nothing but very large box girders carrying the traffic through, rather than on top of them. The longest span ever built in single web plate girder construction was a through span of 170 ft. in the clear, built in 1864 over the Pilalee River on the Eastern

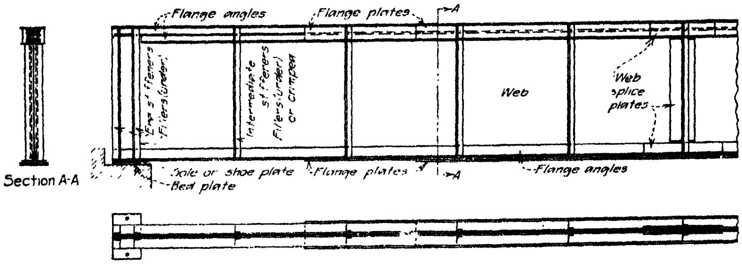


FIG. 34.—Typical plate girder.

Bengal Railway. The two main girders were 13½ ft. deep and 22 ft. apart. The longest simple plate girder spans in America occur in the double-track bridge of the Lehigh Valley Railroad over the Susquehanna River at Towanda, Pa., which contains 13 spans of 129½ ft., and one of 120 ft.

Although plate girder spans of 130 ft. or over may thus be successfully built, it is usually more economical to employ a truss span for lengths over about 120 ft. The pound price and the maintenance cost of plate girder bridges is relatively low, but beyond the 120-ft. limit the saving of material in truss spans is likely to offset the advantages of plate girder construction.

39. Composition of a Typical Girder.

39*a*. Web.—The primary element in the make-up of a plate girder is the web. This may be in one piece if the girder is not over about 30 ft. in span, or it may be in several pieces for longer girders. For very long and very deep girders, web splices, such as shown in Fig. 34, may be as close together as 10 ft. due to the difficulties in getting web plates of sufficient width for the depth of the girder.

39*b*. Flanges.—Flange angles are riveted to the upper and lower edges of the web so as to add to the flexural capacity of the web, and to these angles flange plates are riveted in turn. The most common type of flange is the T-flange, consisting of two angles arranged as a T, with or without attached plates, as in Fig. 35 (*a*) and (*b*). The bottom flange of the girder of Fig. 34 is of this type. Angles alone are used for comparatively light girders.

If more than about 50 per cent of the flange material occurs in the form of cover plates, it is customary, if a T-flange is being used, to connect flange plates directly to the web by placing them between the flange angles and the web and letting them extend past the inner edges of the flange angles. A flange of this sort is shown in Fig. 35 (*c*).

If it is desirable to maintain the top surface of the top flange as a plane—as, for example, in deck plate girder spans for railway work—a four-angle flange, arranged as shown in Fig. 35 (*d*) may be used to advantage. Although flange, or cover plates, are added to a T-flange on the backs of the outstanding legs, they are added to a four-angle flange in two vertical planes on the outer faces of the vertical legs. In either case, variation of the flange section is easily possible by cutting off the flange plates wherever desired. If it be desired to build up a particularly heavy flange, four angles may be used and flange plates may be added both in horizontal and vertical planes, as shown in Fig. 35 (*f*). Some of the latter may be placed between the angles and the web, and thus be classed as directly-connected material. Where a considerable lateral moment is exerted on the flange—as for a crane runway girder—or for any reason especial lateral rigidity is required in the flange, a channel with flanges turned down, Fig. 35 (*g*), is advantageously employed.

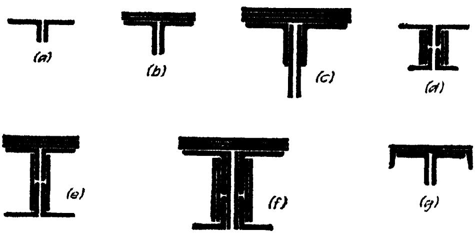


Fig. 35.—Typical flanges for plate girders.

If the total length of the girder is over about two car lengths, or about 65 ft., it is usually necessary to count on splices in the flange, as flange angles in single pieces of greater length than mentioned are not likely to be available in the average shop. These splices consist commonly of a short piece of angle of appropriate section riveted to the spliced flange angle, as described in Sec. 3, Art. **17**.

39c. Stiffeners.—If the unsupported depth of the web exceeds the limit mentioned in Art. **42,** stiffeners are riveted vertically to the web at intervals equal roughly to the depth of the web. These consist usually of a pair of angles, one on each side of the web with one line of rivets serving the two. Where the top flange is of the four-angle type, as in Fig. 35 (*d*), (*e*), and (*f*) pairs of short angles must be used between the upper and lower angles of the top flange to give support to the outstanding legs of the upper angles and to help transfer the concentrated applied loads to the web. These short angles should be ground to fit at both top and bottom, but the main angles, except in the case of the end ones, need to be fitted tightly only at the top. At the ends of the girder two pairs of stiffener angles are employed to prevent the web from buckling and to serve in a measure as a column for the transfer of the end reaction to the support. Fillers are always inserted under these stiffener angles between the flange angles on each side of the web to keep the stiffener angles straight. Fillers may be used under intermediate stiffeners, but it is generally more economical, particularly for the deeper girders, to crimp the ends of the stiffener angles to fit over the flange angles.

39d. Bearings.—At each end of the girder is a sole or shoe plate resting on a bed plate or bed casting. If the span is over about 80 ft. a bolster, such as described in Arts. **56** and **59**, is generally used between the shoe and the bed plates to overcome the tendency of the deflection to produce intensified pressures on the supports near the inner edge. Rollers may be used at the sliding end of the span, under the conditions described in Arts. **56** and **60**.

40. Stress Conditions to be Met.—Like an ordinary beam, a plate girder must be secure against failure by flexure, flange buckling, shear or web crippling, and at the same time must not deflect to such an extent as to cause damage or unsightliness to any construction or interfere with the easy operation of moving structures. As a plate girder may be built to conform closely to all the stress conditions, unlike a rolled beam which must be of some standard cross-section, greater economies of material are possible with the use of plate girders than by the utilization of rolled beams.

41. Proportioning for Shear.—Although with a rolled I-beam, investigating for shearing stresses is necessary only for short, heavily-loaded spans, with a plate girder the design concerns the web quite as much as the flanges. In the latter case, the relation of web to flange area is subject to almost indefinite variation, while for beams there are a few fixed standard proportions, one of which must be adopted.

The general formula giving the intensity of shearing stress at any point in a beam,

$$v = \frac{QV}{It}$$

discussed in Art. **16**, in connection with beams, applies to plate girders, as does the common approximate assumption that the shearing stress is borne entirely by the web and may be considered as uniformly distributed over the web. It is, of course, true, as has been shown in the discussion of Art. **16**, and in Table 6 giving the relation of maximum to average shearing stress in beams, that the above assumptions are in error. For plate girders the error is about the same as for beams—that is, the maximum shearing stress at the neutral axis is ordinarily from 10 or 20 per cent greater than that given by the method of average stress. As, in the case of beams, compensation for the error involved is made by selecting a conservative working stress, one which at the same time makes provision for the ordinary loss of section due to vertical lines of rivet holes.

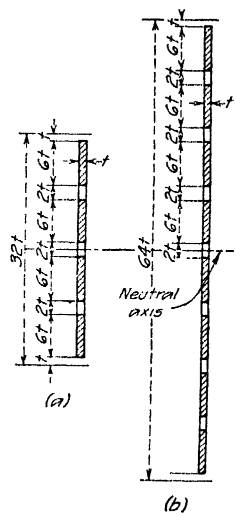


Fig. 36.—Effect of rivet holes on shearing strength of webs.

Although the effect on the shearing strength of a web plate produced by a vertical line of holes filled with rivets is not definitely known, some idea of the extreme limit of this effect may be gathered from considering the effect of a vertical line of open holes. Assume two plates each of thickness t, the first (a) being $32t$ deep and the second (b) $64t$ deep, as shown in Fig. 36. Let there be holes of diameter $2t$ in each, spaced $8t$ apart, center to center. For convenience, let half

of one hole in each case be deducted at each of the extreme edges. In each case the loss of area and the loss of statical moment of half the area about the neutral axis is 25 per cent. For case (*a*) the loss of moment of inertia is 28.2 per cent, and for case (*b*) it is 25.9 per cent. This shows that the reduction in moment of inertia of a plate due to punching holes at uniform spacing is very nearly proportional to the reduction of area, being almost exactly so for the deeper plates. Since the true shearing stress

$$v = \frac{QV}{It}$$

and both the statical moment, Q, and the moment of inertia, I, may be taken as proportional to the net area of the plate, the shearing stress will be increased in the same ratio as the area is reduced.

If the actual maximum shearing stress is therefore in excess of the average shearing stress on the gross area of the web by, say, 15 per cent on account of the error involved in assuming uniform distribution, and this in turn must be increased by as much as 25 per cent by reason of holes, the actual stress would be 44 per cent in excess of the apparent stress. If a girder web is designed, therefore, for an average stress on gross area of 10,000 lb. per sq. in., and allowance for the effect of lines of fairly closely-spaced holes must be made fully, the actual maximum stress would be over 14,000 lb. per sq. in. With a factor of safety of 4, the safe shearing stress in structural steel is about 12,000 lb. per sq. in., so that the 10,000 lb. on gross area would then be excessive. However, it is probably too severe to assume a line of vertical holes tightly filled with rivets as no better than a line of open holes for the resistance of shear. It is probably not necessary, therefore, to require webs to be designed for an average shearing stress of 10,000 lb. per sq. in. on net area, although there are cases when 10,000 lb. per sq. in. on gross area gives excessive results.

42. Proportioning for Web Buckling.—As has been pointed out in the discussion of web stresses in beams, Arts. **16, 17,** and **18,** the adequacy of a web is more frequently determined by its capacity to resist crippling or buckling than by its shearing value. The maximum crippling stress at a point arises from a combination of the flexural and the shearing stress. The location of the point where the diagonal compression reaches its absolute maximum is of importance, as well as the actual value of this maximum.

In order to show the nature and magnitude of the diagonal compressive stresses in typical plate girders, the stresses at various sections of two characteristic girders, Fig. 37, have been calculated in the two problems which follow. The intensity of the maximum stresses and their direction are computed by the appropriate formulas of Sec. 1, Art. **53.**

$$f_m = \tfrac{1}{2}f \pm \sqrt{\tfrac{1}{4}f^2 + v^2}$$

$$\tan 2\theta = -\frac{2v}{f}$$

The flexural stress at any point on a cross-section has been computed by applying to the section the ordinary flexure formula

$$f = \frac{My}{I}$$

where

M = moment in inch-pounds
y = distance of fiber from neutral axis of girder
I = *gross* moment of inertia of section.

The shearing stress was calculated by the exact formula of Art. **16**.

$$v = \frac{QV}{It}$$

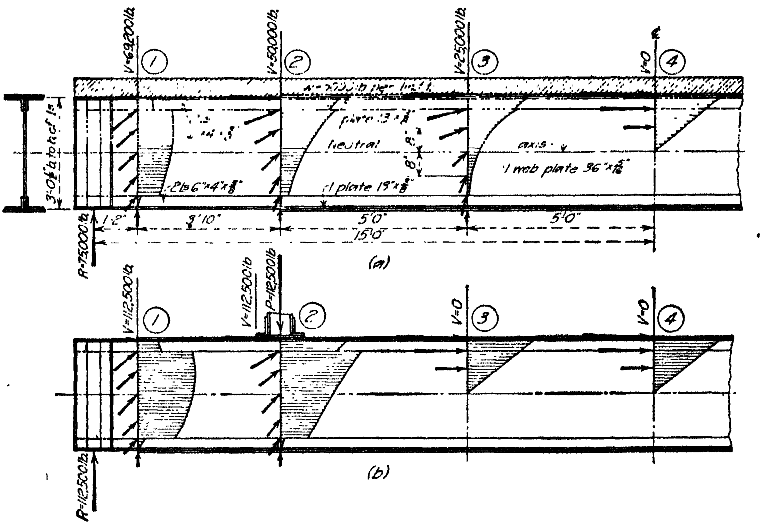


FIG. 37.—Diagonal compressive stresses at various sections on typical plate girders

Illustrative Problem.—A 30-ft. plate girder consisting of one $36 \times \frac{5}{16}$-in. web plate, four $6 \times 4 \times \frac{3}{8}$-in. flange angles, and two $13 \times \frac{3}{8}$-in. cover plates, 20 ft. long, as shown in Fig. 37*a*, carries a total uniformly distributed load of 5,000 lb. per lin. ft. Find the intensity and direction of the diagonal compressive stress at the points indicated in Fig. 37*a* at cross sections 1.17, 5, 10 and 15 ft. from one support, neglecting the local effect of the application of the superimposed load.

The bending moments and shearing forces at the section considered are listed in Table 11.

Gross moment of inertia of girder without cover plates = moment of inertia of web + moment of inertia of four flange angles or

$$I_1 = (\tfrac{1}{12})(0.3125)(36)^3 + 4[(4.9) + (3.61)(17.31)^2] = 5{,}565 \text{ in.}^4$$

Gross moment of inertia of girder with two cover plates,

$$I_2 = (5{,}565) + (2)(4.875)(18.438)^2 = 8{,}880 \text{ in.}^4$$

Applying the flexure and shearing stress formulas to each of the four cross-sections, the bending and shearing stresses at the seven stipulated points are calculated. Combining them, the intensities and direction of the maximum resultant stresses at the selected points are then found. In Fig. 37*a* the intensity of the resultant compressive stress at the seven points on each of the four cross sections is plotted horizontally to the right of the section, and curves drawn through the extremities of the lines so plotted. By scaling off the horizontal distance from the section plane to the appropriate curve at the level of the point

considered, the intensity of the maximum diagonal compression at the point may be found. The direction of the maximum compressive stress is indicated for the chosen points on each cross-section by arrows to the left of the section.

Figures for the actual intensity of the diagonal compressive stress are given in Table 11 in the case of each of the cross-sections for the extreme compressive fiber and for the fiber of the web immediately below the inner edge of the flange angles.

TABLE 11.—COMPARISON OF MAXIMUM COMPRESSIVE STRESSES AT SELECTED POINTS ON VARIOUS CROSS-SECTIONS OF TYPICAL GIRDERS

Section no.	Distance from support (ft.)	Moment M (ft.-lb.)	Shear V (lb.)	Moment of inertia I (in.^{4})	Point on cross-section	Flexural stress f (lb. per sq. in.)	Shearing stress v (lb. per sq. in.)	Maximum compressive stress (lb. per sq. in.)
				Girder with uniform load, Fig. 37 (a)				
1	1.167	84,000	69,200	5,565	Extreme compressive fiber...............	3,320	.....	3,320
					Web at inside edge of flange...	2,590	5,720	7,150
2	5.0	312,500	50,000	5,565	Extreme fiber	12,320	.....	12,320
					Inside edge of flange .	9,625	4,140	11,150
3	10.0	500,000	25,000	8,880	Extreme fiber.......	12,580	.....	12,580
					Inside edge of flange ..	9,620	2,100	10,060
4	15.0	562,500	......	8,880	Extreme fiber.. . . .	14,150	.....	14,150
					Inside edge of flange	10,830	.....	10,830
				Girder with two symmetrical concentrated loads, Fig. 37 (b)				
1	1.167	131,000	112,500	5,565	Extreme fiber.... .	4,300	.....	4,300
					Inside edge of flange...	3,360	9,300	11,130
2	5.0	562,500	112,500	8,880	Extreme fiber......	14,150	....	14,150
					Inside edge of flange..	10,830	9,460	16,310
3	10.0	562,500	. ....	8,880	Extreme fiber	14,150	.. ..	14,150
4	15.0	562,500	.. . .	8,880	Extreme fiber .	14,150	.....	14,150

Illustrative Problem.—A 30-ft. girder made up as for the last problem, but with cover plates 27 ft. 8 in. long, carries a 112,500-lb. load at each of two points 5 ft. from the supports. The dead weight of the girder is assumed to be included in these loads. Find the intensity and direction of the maximum compressive stresses at the same sections and points as those selected for the last problem.

The moments and shears for the sections considered are listed in Table 11. On the assumption that the weight of the girder is included in the two concentrated loads, the moment increases uniformly from zero at the supports to a maximum at the points of loading and in loads. The shear being assumed as zero for this section, the maximum stresses are all purely flexural. Diagrams similar to those plotted in Fig. 37 (*a*) are shown in Fig. 37 (*b*). The intensities of maximum compressive stress for the two points of particular interest on each cross-section are listed in Table 11. For simplicity, the maximum shear and maximum moment are both assumed to occur at the center of the concentrated load.

42a. Variation in Web Compression.—Study of the results obtained in the last two illustrative problems shows:

1. That the diagonal compressive stress tends to become measurably constant throughout the clear depth of the web as the shear becomes relatively large and the moment relatively small.

2. That the direction of the maximum diagonal stresses tends to become approximately at 45 deg. with the neutral axis throughout the clear depth of the web, wherever the shear is relatively large and the moment relatively small.

3. That near the ends of a girder and at points where the shear and moment are both large, the diagonal compressive stress in the web immediately inside the inner edge of the flange angles may be considerably in excess of the flexural stress at the extreme fiber. Such occurs at sections 1 and 2 for both girders considered.

42b. Thickness Ratio of Web.—Although some provision against failure of a girder web by buckling is involved in proportioning for shear according to the usual range of permissible shearing stresses, further limitations must often be made.

To keep the slenderness ratio of a vertical strip of web plate within reasonable bounds, it is sometimes specified that the thickness of the web shall not be less than $\frac{1}{60}$ of the unsupported depth between flange angles or side plates. This, for example, is the requirement in Schneider's "General Specifications for the Structural Work of Buildings." The rule adopted in the 1920 specification of the American Railway Engineering Association is that the thickness of the web plate shall not be less than $\frac{1}{20}\sqrt{h}$, where h is the unsupported distance between flange angles. Other specifications commonly fix the minimum thickness of webs as $\frac{3}{8}$ in. for girders of railway bridges and $\frac{5}{16}$ in. for highway bridge and building girders. Occasionally a $\frac{1}{4}$-in. web is used for building work, but the percentage loss through corrosion is so large and the possible damage in fabrication, transportation and erection so great that such thin webs are not to be recommended. In most cases a uniform thickness is used throughout the length of the girder, although for very heavy girders an increase of thickness in regions of large shear and diagonal compression is adopted.

A further restriction of the clear length of web plate that may buckle without hindrance is imposed by the requirement of intermediate stiffeners under certain conditions. If the thickness of web plate is less than $\frac{1}{50}$ or $\frac{1}{60}$ the unsupported distance between flanges, stiffeners are commonly specified. These, if spaced sufficiently close together, as explained in Art. **52**, break up, or limit the length of diagonal belts of web, along which compressive or buckling stresses may reach a high intensity in regions of large shear, or large shear and large moment combined.

42c. Limiting Buckling Stresses.—Although the limiting thickness ratio of a web may be observed and stiffeners used, dangerous buckling stresses may nevertheless arise in the webs at a section of heavy shear, unless the stiffeners are closely spaced. To provide for these, the shearing stress at the section should be computed and compared with the safe shearing stress based on web buckling.

Illustrative Problem.—Express an opinion concerning the safety against crippling of the web of the girder of the last problem under an end shear of 112,500 lb. Safe web crippling stress by adapted A.R.E.A. stiffener formula (Art. **17**).

$$p = 12{,}000 - 42\frac{h}{t}$$

Average existing shearing stress

$$v_a = \frac{112{,}500}{(36)(0.3125)} = 10{,}040 \text{ lb. per sq. in.}$$

This being considerably less than the existing average shearing stress on the cross-section, the web would need to be thickened or stiffeners used, spaced a distance apart in the clear about one-half the clear depth of the web, so that the longest unsupported strip of web making an angle of 45 deg. with the neutral axis would be one-half that assumed in the above formula for working stress.

42*d*. Proportioning for Diagonal Tension.—As has been pointed out in Art. **17**, there exist at points below the neutral axis of a simply supported beam, diagonal tensile stresses of the same magnitude as those existing at corresponding points above the neutral axis, acting in a direction at right angles to the maximum compressive stress existing at the point on the tension side being considered. If the upper half of each diagram of Fig. 37 were turned down about the neutral axis as a hinge it would correctly represent the intensity of the maximum tensile stresses below the neutral axis.

In spite of the existence of these high diagonal tensile stresses, however, it is usually unnecessary to investigate them, since if the web can safely resist the shearing stresses in it, it can safely resist any tensile stresses arising from the combination of flexure and shear. This is because the safe strength of steel is a third more in tension than in columnar compression. The only exception to this condition is for the web fibers immediately inside the inner edge of the flange angles in girders with very thin webs. There, as has been already pointed out in this article, the maximum diagonal stresses may exceed the maximum flexural stress at the extreme fiber. In such cases, the web may need to be reinforced along this line of weakness by longitudinal plates placed under the flange angles and extending inside them, or by using angles with wider vertical legs.

Indirectly, some advantage accrues to the compression half of the beam through large diagonal tensile stresses, whether brought about by the combination of ordinary flexural and shear stresses, or by the application of a concentrated load to the tension side. The greater the diagonal tension in the web, the better is the web restrained against buckling. Inadequate webs are thus often kept from failure in buckling by the existence of excessive tensile stresses.

The effect of concentrated loads on the web immediately below or above them is to increase the maximum diagonal compressive stress already existing. If the load is applied above the neutral axis, the diagonal compressive stress of the web below is increased; if it is applied below the neutral axis, the diagonal tensile stress is increased. On the assumption that the shearing stress is uniformly distributed over the web, it may be shown that the intensity of the vertical stress on horizontal sections arising from the concentrated load decreases uniformly with the vertical distance of the horizontal section from the point of loading. If this vertical stress at any selected point be q, the maximum diagonal stress, f_m, at this point may be shown to be

$$f_m = \frac{1}{2}(f + q) + \sqrt{\frac{1}{4}(f - q)^2 + v^2}$$

where f and v are the flexural and shearing stresses at the point.[1] The angle θ which this total maximum stress makes with the vertical is such that

$$\tan 2\theta = \frac{2v}{f - q}.$$

[1] See Sec. 1, Art. 53*c*.

Adoption of a given web thickness for a girder is influenced by other factors than the capacity to resist shear and crippling without exceeding certain prescribed stresses. If in order to obtain the required area, a thin web be made very deep, the extra width may entail a higher pound cost for the web material. In addition, there is considerable risk of damage to thin webs in fabrication, transportation, and erection and the percentage loss through corrosion is higher than for thicker webs. Wherever an effort is made to utilize a very thin web, it may necessitate stiffeners so closely spaced as to increase the cost of the whole girder materially.

43. Moment of Resistance, Exact Method.—By reason of the rigid attachment of the flanges to the web, a plate girder is essentially a built-up beam and its moment of resistance, or capacity to resist bending moment, should properly be computed from the common flexure formula, $f = M/S$, or $M = Sf$. To apply this formula accurately, it is necessary to compute the moment of inertia of the actual section, if such is not available in tables. However, because of the innumerable combinations of plates and shapes worked into plate girder sections, the properties of the particular section in hand are often not listed, and hence the value of I must be specially determined for the case under consideration.

In allowing for rivet holes on the tension side of the girder, the same differences of practice exist as have been mentioned in connection with the design of beams containing rivet holes. Some designers proportion by the moment of inertia of the gross section, as is recommended by the American Bridge Co. in specifications for steel structures. Some proportion for the gross section on the compression side and net section on the tension side and others for the net section on both sides. Some assume the neutral axis to be at the center of gravity of the gross section and others assume it as at the center of gravity of the net section.

As has been pointed out in the discussion of the moment of resistance of the net section of beams, it is reasonable to assume the gross section as operative on the compression side but only the net section on the tension side. While the neutral axis is shifted upward a small distance by the holes on the tension side, it is more convenient to assume it as at the center of gravity of the gross section and then apply a correction to the net moment of inertia or section modulus to compensate for the erroneous assumption respecting the position of the neutral axis. Since the position of the neutral axis must be somewhere between the gravity axes of the gross section and that of the net section, this correction cannot be large.

Computation of the moment of resistance of a plate girder section is simple, once the correct section modulus has been found. The determination of this quantity is tedious if the net section is considered, since the holes in the web as well as those in the flange must be considered. The necessary operations can best be explained by an example.

Illustrative Problem.—A plate girder section consists of a $36 \times \frac{3}{8}$-in. web plate, four $6 \times 6 \times \frac{3}{8}$-in. angles spaced $36\frac{1}{2}$ in. back to back, and four $14 \times \frac{3}{8}$-in. cover plates, as shown in Fig. 38 (*b*). Holes for $\frac{7}{8}$-in. rivets (counted as 1 in. diameter) occur in the flanges and web, as shown, those on the compression side not being counted. Compute the section modulus: (1) Of the gross section; (2) of the section with the holes on the tension side only deducted, assuming that the neutral axis is at the center of gravity of the net section; (3) of the net section as above, assuming the neutral axis as at the center of gravity of the gross section.

1. *Section Modulus of the Gross Section.*—This is found by dividing the moment of inertia, I, of the gross section by the distance, c, from the center of gravity of the gross section to the extreme fiber of the outer cover plate. The method has been already illustrated in the problems on box girders.

Gross I of section:

Web, $(\frac{1}{12})(0.375)(36)^3$	=	1,458
4 angles, $4(I_0 + Ay_0^2) = 4[(15.4) + (4.36)(16.61)^2]$	=	4,882
2 prs. plates, I_0 being negligible $= (2)(14)(0.75)(18.625)^2$	=	7,290
Total I	=	13,630

Section modulus of gross area,

$$S = 13{,}630/19.0 = 717.5$$

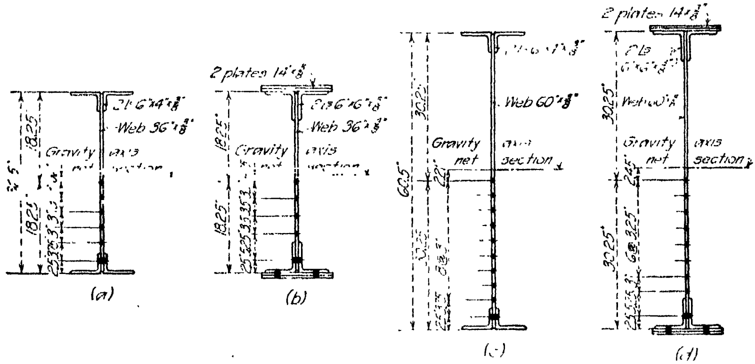


FIG. 38.—Comparison of the section moduli of typical plate girder sections.

2. *Section Modulus of Net Section, Neutral Axis at its Gravity Axis.*—The distance, e, of the gravity axis of the unsymmetrical net section above the center of gravity of the gross section may be found by the formula

$$e = \frac{Q_h}{A_n}$$

where

Q_h = statical moment of holes about neutral axis of gross section.
A_n = net area of girder section.

From Fig. 38 (b), Q_h is seen to be

$$0.375(0 + 3.5 + 7 + 10.5) + (1.125)(15.75) + (2)(1.125)(18.438) = 67.1$$

Net area of section = gross area − area of holes on or below gravity axis of gross section = 51.94 − 4.88 = 47.06 sq. in. Hence

$$e = 67.1/47.06 = 1.425 \text{ in.}$$

The moment of inertia of the unsymmetrical net section about its gravity axis may be found readily from the formula

$$I_n = I + Ae^2 - I_h'$$

where

I = moment of inertia of gross section about its own gravity axis.
A = area of *gross* section.
e = distance of gravity axis of net section above gravity axis of gross section.
I_h' = moment of inertia of holes about gravity axis of *net* section.

The latter will, following Fig. 38b, be

$$0.375(1.425^2 + 4.925^2 + 8.425^2 + 11.925^2) + (1.125)(17.175)^2 + (2)(1.125)(19.863)^2 = 1{,}313.$$

Hence

$$I_n = (13{,}630) + (51.94)(1.425)^2 - 1{,}313 = 12{,}423$$

Section modulus of net section,

$$S_n = 12.423/20{,}425 = 608 \qquad S_n/S = 0.848$$

3. *Section Modulus of Net Section, Neutral Axis at Gravity Axis of Gross Section.*—Net moment of inertia of unsymmetrical net section,

$$I_n' = I - I_h$$

where I_h = moment of inertia of holes about gravity, axis of *gross* section.

The latter is

$$0.375(0^2 + 3.5^2 + 7^2 + 10.5^2) + 1.125(15.75)^2 + (2)(1.125)(18.438)^2 = 1{,}108$$

Hence

$$I_n' = 13{,}630 - 1{,}108 = 12{,}522,$$

and

$$S_n' = 12{,}522/19.0 = 659. \qquad S_n'/S = 0.92$$

Comparing S_n and S_n' it is seen that $S_n/S_n' = 0.923$. That is, the section modulus obtained by assuming the neutral axis to be at the center of gravity of the net section is 92.3 per cent of that obtained assuming the neutral axis as fixed at the neutral axis of the gross section. However, as pointed out in the discussion of beams, Art. **7**, the neutral axis probably does not shift to the extreme position of the center of gravity of the net section and consequently if the convenient assumption of fixed neutral axis be made the result in the present instance need be reduced by not over 5 per cent.

Computations similar to those in the last problem have been made for the three other girders shown in Figs. 38*a*, *c*, and *d* and the results obtained are set forth in Table 12. An examination of this table shows that, for the four typical girders analyzed, the loss of section modulus under the most serious assumption—namely, that the neutral axis is at the center of gravity of the unsymmetrical net section—ranges from 12.7 to 17.4 per cent, being greater for sections with heavy flanges than for those with light ones. The loss under the simpler assumption of fixed axis ranges from 6.5 to 8.3 per cent, being, as before, greater for sections with heavy flanges than for sections with light ones. The ratio of section modulus determined under the first assumption to that determined under the second one varies from 0.90 to 0.934, being lowest for girders with heavy flanges. Since the true position of the neutral axis is somewhere between the two positions assumed, a reduction factor of $K = 0.95$ applied to S_n' will give sufficiently close results for even girders with heavy flanges.

TABLE 12.—MOMENTS OF INERTIA AND SECTION MODULI OF TYPICAL PLATE GIRDERS

Girder		Gross section		Unsymmetrical net section						$\frac{S_n}{S_n'}$	Average reduction factor "K" recommended to be applied to S_n' to give S_n''
				Neutral axis at gravity axis of net area			Neutral axis at gravity axis of gross area				
No.	Fig. No.	I	S	I_n	S_n	$\frac{S_n}{S}$	I_n'	S_n'	$\frac{S_n'}{S}$		
1	38*a*	5,810	318	5,395	278	0.873	5,430	298	0.935	0.934	0.95
2	38*b*	13,630	718	12,423	608	0.848	12,522	659	0.920	0.923	0.95
3	38*c*	19,170	634	17,155	538	0.848	17,615	583	0.919	0.924	0.95
4	38*d*	40,910	1,319	36,347	1,086	0.824	37,401	1,207	0.917	0.900	0.95

44. Moment of Resistance, Approximate Method.—To obviate the somewhat laborious computation of moment of inertia involved in employing the accurate

method for determining moment of resistance, it is customary to employ an approximate method giving sufficiently precise results for all but the shallower girders. This method is based on the concept of the girder as a truss. Virtual chord areas are determined from the area of the girder section and the disposition of the material therein. Knowing these chord areas and the distances between their centers of gravity, it is easy to compute the moment of resistance, which is merely the product of the total permissible stress in one chord and the distance between chord centers.

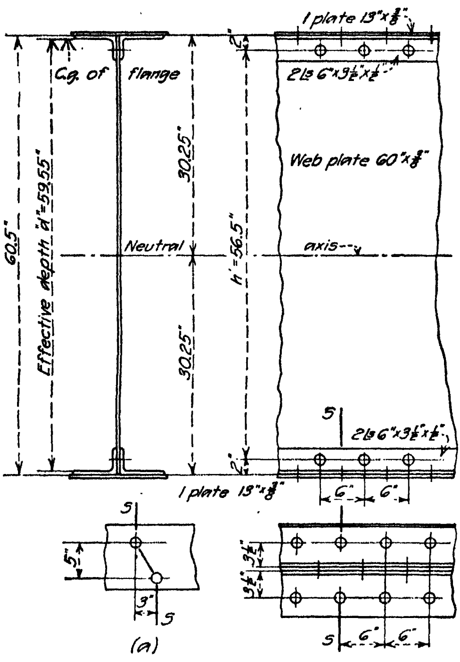


FIG. 39.—Moment of resistance of girder by approximate method.

Consider the flexure formula $M = fI/c$, as applied to any girder section, such as that shown in Fig. 39. The moment of inertia of the whole section is made up of the moment of inertia of the flanges plus the moment of inertia of the web—that is, $I = I_f + I_w$. Let A_f' be the net area of *one* flange proper—that is, the angles and flange plates riveted to the web at one edge as flange material; d the distance between centers of gravity of flanges, or the effective depth; t the thickness of the web plate; and h the depth of this plate. Then, approximately,

$$= 2A_f'\left(\frac{d}{2}\right)^2 \quad \frac{1}{12}th^3$$

In this expression the moment of inertia of the two flange areas A_f' about their own gravity axes parallel to the neutral axis of the whole girder have been neglected, since it is relatively unimportant when compared with the term $2A_f'\left(\frac{d}{2}\right)^2$

Where cover plates are employed, as in Fig. 39, the center of gravity of the flange is not far from the edge of the web plate—that is, $h = d$, approximately. If at the same time the distance c, be replaced by $\frac{d}{2}$, or in effect the prescribed stress f be assumed to act at the center of gravity of the flange, the expression for moment of resistance of the entire section becomes,

$$M = \frac{f\left\{2A_f'\left(\frac{d}{2}\right)^2 + \frac{1}{12}td^3\right\}}{\frac{d}{2}}$$

or simplifying,

$$M = fd(A_f' + \tfrac{1}{6}A_w'), \tag{1}$$

where $A_w' = td$ = approximate area of the web.

Formula (1) is an expression for the moment of resistance of a virtual truss, of which the area of one chord or flange is $A_f' + \frac{1}{6} A_w'$. It is evident, therefore, that the web contributes to the moment of resistance an amount equal to that which would be produced by concentrating approximately one-sixth of its area at the center of gravity of each of the flanges. This amount is commonly known as the web equivalent.

In applying Formula (1), care must be taken to give proper recognition to the presence of rivet holes. A_f', the area of *one* flange, must be the net, not the gross area, for while it is customary to make no deduction for holes in the compression flange if properly filled with well-driven rivets, the full deduction for a tension member must be made for the tension flange. The actual number of rivets to be deducted in a given case will depend on the number of rows of rivets in the vertical and horizontal legs of the flange angles and the pitch of the rivets. The method of computing the deduction will be in accordance with the provisions of the specification. It may be that outlined in Art. **65**.

If there be vertical lines of holes in the web, as at a stiffener or a web splice, the area of web, A_w', should be taken as the net area through the rivet holes. For such girders, it is convenient to make an average approximation of the relation of net to gross web area, and base the formula on gross web area. If it be assumed that $\frac{7}{8}$-in. rivets or 1-in. holes are spaced on an average 4 in. vertically apart in the stiffeners, the net area of the web is $\frac{3}{4}$ the gross area, and hence $\frac{1}{6}$ of the net area equals $\frac{1}{8}$ of the gross area. Formula (1) applied to girders with vertical lines of holes in the web then becomes

$$M = fd\,(A_f' + \tfrac{1}{8}A_w) \tag{2}$$

where A_w = gross area of the web. Some designers permit only $\frac{1}{10}$ or $\frac{1}{12}$ of the gross area of the web to be counted as web equivalent, but as no deduction really needs to be made for the compression half, $\frac{1}{8}$ is not excessive.

Formerly, it was common practice to disregard altogether the value of the web in contributing to the moment of resistance and require all of the flange area to be area in excess of the web. Under such a specification, Formula (2) would become

$$M = fdA_f' \tag{3}$$

The obvious severity of this requirement has led to its abandonment in nearly all specifications.

In computing the moment of resistance of a girder, it is common practice to assume the effective depth of plate girders with two T-flanges carrying cover plates as the depth of the web. No appreciable error is involved in such an assumption as will be shown in examples, but the rule does not apply with sufficient accuracy to girders with four-angle flanges or with T-flanges without cover plates. In these cases the center of gravity of the flange may be several inches inside the edge of the web plate, and its position must therefore be specially calculated.

Since neither the areas of the two flanges nor the permissible stresses in them are necessarily equal, the moment of resistance with respect to the two flanges

will, in general, be unequal. The strength of the girder will, of course, be governed by the lesser of the two.

Illustrative Problem.—Compute by the approximate method the moment of resistance of the plate girder section shown in Fig. 39, and made up of one web plate 60 × 3/8 in., four 6 × 3½ × ½-in. angles, and two 13 × 3/8-in. cover plates. Assume the web equivalent as 1/8, the permissible flexural stress as 16,000 lb. per sq. in., and rivets 7/8 in. diameter. Net section of tension flange will be computed by the exact method of Art. **65**, the rivet spacing being as shown.

From Fig. 39, it is evident that the dangerous section is *S-S*, cutting the holes through the cover plates and the horizontal legs of the flange angles, and located 3 in. from the centers of the holes through the vertical legs of the flange angles. Since the distance between gage lines of the developed flange angle is 5 in. the deduction for one angle is found from Fig. 55 to be 1 + 0.6 = 1.6 holes, and, for two angles, 3.2 holes. For the cover plate it is 2 holes.

Net area of two angles is (2)(4.50) − (3.2)(1.0)(0.5) = 7.4 sq. in.; net area of one cover plate is (13)(0.375) − (2)(1.0)(0.375) = 4.13 sq. in.

Total net area of one flange proper = 7.4 + 4.13 = 11.53 sq. in.

Web equivalent = (1/8)(60)(0.375) = 2.81 sq. in.

Assuming d = depth of web plate = 60 in.,

$$M = (16{,}000)(60)(11.53 + 2.81) = 13{,}760{,}000 \text{ in.-lb.}$$

Let it be required to find the moment of resistance using the exact value for the effective depth—that is, the computed distance between the centers of gravity of the flanges. Making the computation on the basis of gross flange area, the section that predominates, it is found that the center of gravity of the flange is 0.47 in. inside the back of the flange angles and that the true effective depth is 59.55 in. The moment of resistance using this value = M = (16,000)(59.55)(11.53 + 2.81) = 13,680,000 in.-lb., or 0.75 per cent less than by using the approximate value of d. In girders with a pair of flange plates on each flange, there is very little error involved in assuming the effective depth as equal to the depth of the web plate.

Illustrative Problem.—Find the moment of resistance of the girder in the last problem, assuming that the cover plates are omitted.

As there are holes in only one leg of the flange angles, the deduction from each angle will be one hole.

Net area of one flange = (2)(4.50) − (2)(1)(0.5) = 8.00 sq. in. The web equivalent is, as before, 2.81 sq. in.

Since there are no cover plates, it is best not to assume the effective depth as the depth of the web plate. The distance of the center of gravity of each flange being 0.83 in. from the backs of the flange angles, the effective depth is 60.5 − (2)(0.83) = 58.84 in.

The moment of resistance,

$$M = (16{,}000)(58.84)(8.00 + 2.81) = 10{,}170{,}000 \text{ in.-lb.}$$

If the effective depth had been assumed as the depth of the web, or 60 in., the error would have been 1.97 per cent on the unsafe side. It is thus evident that if the outstanding legs of the flange angles are considerably greater in length than the vertical legs, there is no serious error involved in assuming the effective depth as the depth of the web plate, even though there are no cover plates.

Illustrative Problem.—If in the last problem the flanges consist of two 6 × 6 × ½-in. angles without cover plates, the flange rivets having a staggered pitch of 6 in., as shown in Fig. 40, find the moment of resistance.

From Fig. 55 it is seen that the rivet hole deduction from one angle is 1 + 0 = 1 hole, and 2 holes for the two angles. The net flange area is therefore (2)(5.75) − (2)(1)(0.5) = 10.50 sq. in.

The distance of the center of gravity of a flange being 1.68 in. from the backs of the angles, the effective depth is therefore 60.5 − (2)(1.68) = 57.14 in.

The moment of resistance is, therefore,

$$M = (16{,}000)(57.14)(10.50 + 2.81) = 12{,}160{,}000 \text{ in.-lb.}$$

Had the effective depth been assumed as 60 in., the result would have been 5 per cent in error. With equal-legged flange angles and no cover plates, it is, therefore, necessary to compute the effective depth.

Illustrative Problem.—If the section of a plate girder be as shown in Fig. 41—that is, with a 60 × 7/16-in. web plate, bottom flange of two 6 × 6 × 11/16-in. angles and two 14 × 1/2-in. cover plates, and the top flange of four 6 × 4 × 3/8-in. angles, and four 7 × 1/2-in. plates, with flange riveting as shown—find the amount of resistance, assuming the web equivalent as 1/8. Permissible flange stress on tension flange = 16,000 lb. per sq. in. net area, and on compression flange 15,000 lb. per sq. in. gross area. Rivets 7/8 in.

Applying the method of calculating exact deductions from tension flanges, explained in Art. 65 to the riveting arrangement shown in Fig. 41, it is evident that the deduction should be two holes from each angle and two from each cover plate.

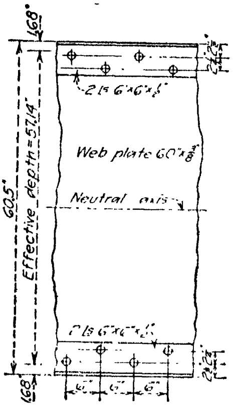


Fig. 40.—Effect of omission of cover plates on effective depth.

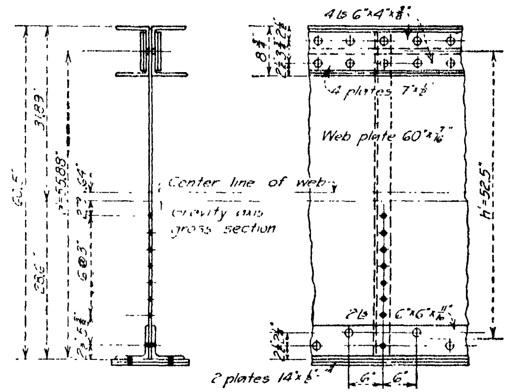


Fig. 41.—Moment of resistance of unsymmetrical girder section.

Net area of two 6 × 6 × 11/16-in. angles = (2)(7.78) − (4)(1)(0.688) = 12.81 sq. in.
Net area of two 14 × 1/2-in. cover plates = (2)(14)(0.5) − (4)(1)(0.5) = 12.00 sq. in.
Web equivalent = (1/8)(60)(0.4375) = 3.28 sq. in.
Total net flange area = 12.81 + 12.00 + 3.28 = 28.09 sq. in.

The center of gravity of the compression flange is seen by inspection to be 4 3/8 in. down from the backs of the outer angles, and if the center of gravity of the tension flange be assumed as at the edge of the web, the effective depth is then 60.5 − (4.375 + 0.25) = 55.88 in.

Moment of resistance of girder with respect to tension flange is

$$M = fd\left(A_f' + \frac{A_w}{8}\right) = (16{,}000)(55.88)(28.09) = 25{,}100{,}000 \text{ in.-lb.}$$

Gross area of compression flange is made up as follows:

4 angles, 6 × 4 × 3/8	= (4)(3.61)	= 14.44 sq. in.
4 plates, 7 × 1/2	= (4)(7)(0.5)	= 14.00 sq. in.
1/8 web = (1/8)(60)(0.4375)		= 3.28 sq. in.
	Total gross area	= 31.72 sq. in.

Moment of resistance with respect to compression flange is

$$M = (15{,}000)(55.88)(31.72) = 26{,}600{,}000 \text{ in.-lb.}$$

The girder is therefore, stronger in the compression flange than in the tension flange, but the make-up of the flanges is in accordance with the principle that the gross area of the compression flange should be as nearly as possible equal to the gross area of the tension flange (see Art. 46).

45. Comparison of Exact and Approximate Methods.—Before using the approximate or truss-chord method of calculating the moment of resistance, designers should be aware of the degree of accuracy attainable by it. Although for the ordinary cases of plate girders with T-flanges it is sufficiently exact, for very shallow girders it cannot safely be used, particularly if these girders have heavy or 4-angle flanges. The assumptions of uniform stress over the flanges and that the effective depth equals the depth of the web plate are too much in error. In such cases the method of the moment of inertia should be employed instead.

Two characteristic examples will make this point clear.

Illustrative Problem.—Find the moment of resistance of the plate girder shown in Fig. 38*b* by the approximate method and compare it with the moment of resistance for this girder already found by the moment of inertia method. Assume permissible bending stress on both flanges = 16,000 lb. per sq. in. Gross area of one flange is:

Two angles, $6 \times 6 \times \frac{3}{8}$ in.	$= (2)(4.36)$	$= 8.72$ sq. in.
Two plates, $(2)(14)(0.375)$		$= 10.50$ sq. in.
$\frac{1}{8}$ of web $= (\frac{1}{8})(36)(0.375)$		$= 1.69$ sq. in.
	Total gross area	$= 20.91$ sq. in.

Net area of one flange with the rivet pitch assumed = gross area less 2 holes out of each angle and 2 holes out of each cover plate $= 20.91 - (4)(1)(0.375) - (4)(1)(0.375) = 17.91$ sq. in.

Assuming the effective depth as equal to the depth of the web, as is commonly done the moment of resistance is

$$M = (16{,}000)(36)(17.91) = 10{,}320{,}000 \text{ in.-lb.}$$

Referring to Table 12, it is seen that the probable net section modulus of this girder, considering the effect of the holes on the position of the neutral axis is $(0.95)(659) = 626$ The moment of resistance would, therefore, be

$$M = S_n''f = (626)(16{,}000) = 10{,}025{,}000 \text{ in.-lb.}$$

The approximate method in this case gives a result in error on the safe side by 3 per cent.

Illustrative Problem.—Calculate the moment of resistance of the unsymmetrical girder shown in Fig. 41 by the exact method and compare it with the moment of resistance already found by the approximate method.

Taking statical moments of gross areas about the center line of the web, the center of gravity of the gross section is found to be 1.64 in. below the center line of the web.

Moment of inertia of gross section with respect to its gravity axis is found to be 52,827.

Moment of inertia of holes about gravity axis of gross section = 4,572.

Net moment of inertia = 48,255, and section modulus = 48,255/31.89 = 1,515. Multiplying this by the recommended correction factor (Table 12), adjusted net section modulus = 1,440.

Moment of resistance of net section by the exact method = (1,440)(16,000) = 23,000,000 in.-lb.

Least moment of resistance found by the approximate method (last problem under Art. 44), = 25,100,000 in.-lb.

Hence the approximate method gives in this case a result 9 per cent too great.

From the first problem it is evident that the approximate method gives reasonably accurate results for girders of moderate depth with T-flanges. Had the girder been much shallower with the same flange material, or if it had had much heavier flanges with the same depth, the error would have been more serious, justifying the use of the exact method of the moment of inertia.

From the second problem, it is seen that with a top flange of 4 angles, even where the girder is of moderate depth, the approximate method cannot be relied

upon. Had the flanges been heavier, or the girder shallower, the error involved in the use of the approximate method would have been greater.

46. Composition of Flanges.—In making up a flange section, such as any of those shown in Fig. 35, it is necessary to decide upon the proportion of the added flange material that must be directly connected to the web. Many specifications require that not over one-half of the total flange section must be in the form of plates not directly connected to the web, and this rule is very commonly observed. However, the 1920 A.R.E.A. specifications merely require that flange angles shall form as large a part of the area of the flange as practicable. If a very large amount of material in the form of cover plates were attached to relatively light flange angles, the latter might not be able to transmit safely to the plates the stress they should bear. For flanges of such large section that approximately half the area cannot be provided by the number and size of angles adopted, flange plates are added between the angles and the web, as shown in Fig. 35*c* and *f*. They are then directly connected material.

For a T-flange in which unequal-legged angles are used and there is no danger of concentrated loads such as ties, it is best to have the long legs outstanding, since this throws the center of gravity of the flange farther out than if the short legs were outstanding, and hence increases both the vertical and lateral flexural efficiency. Unequal-legged angles so placed are more efficient than equal-legged ones, but the latter are often required to accommodate the necessary flange rivets.

Four-angle flanges, such as shown in Fig. 35*d*, *e* and *f*, should be as shallow as possible, in order to throw the center of gravity well out, and if the angles are of unequal areas the inner pair of the two should be the smaller, since they are less effective than the other pair. Unequal-legged angles with the long legs outstanding are more efficient than equal-legged angles.

It has long been the custom to specify that if the flange plates on a flange are of unequal thickness, the thinnest plate should be on the outside. There does not appear to be any good reason for this rule. On the contrary, with a given rivet spacing there is less likelihood of a thick plate separating from the remainder of the flange than there is for a thin plate to do so. Besides, if one plate of the top flange must be carried the full length of the girder, it is more economical to employ the thinnest one for this purpose. The width of flange plates should be so fixed that they do not extend more than 6 in. outside the outer line of rivets nor more than 8 times the thickness of the thinnest plate on the flange.

Since no deduction need be made for rivet holes in the compression flange, the gross area of this flange might theoretically be made less than the gross area of the tension flange, even though the working stress on the former may be less than on the latter, due to allowance for buckling. However, as it is desirable to keep the neutral axis as nearly as possible in the center of the web, and its position depends largely on gross areas, it is frequently specified that the gross area of the compression flange shall be the same as the gross area of the tension flange.

For the purpose of determining the flange area required to resist a given moment by the approximate method, Formula (2) of Art. 44 may be written

$$A_f' + \frac{A_w}{8} = \frac{M}{fd} \qquad (1)$$

or

$$A_f' = \frac{M}{fd} - \frac{A_w}{8} \qquad (2)$$

From Formula (1) the total flange area required (including web equivalent) is found, while from Formula (2) the area of the angles and flange plates only is determined.

If no part of the web is counted as flange material, Formula (1) or (2) becomes

$$A_f' = \frac{M}{fd} \tag{3}$$

While this latter method of proportioning is too severe; it is useful in making rough preliminary approximations of the size of the flanges required.

In proportioning by the exact or moment of inertia method it is useful to make a preliminary calculation of section by the approximate method and then test it by the exact method. If the stresses are too large or too small, the section must then be revised. Proportioning by the exact method is much more laborious than by the approximate one.

Illustrative Problem.—Determine by the approximate method the required composition of the two flanges of a girder for which the maximum bending moment is 3,000,000 ft.-lb., if the web is 72 × ½ in. and the upper flange is to be of the 4-angle type, with vertical flange plates if necessary. Permissible flexural stress = 16,000 lb. per sq. in. on tension flange and 15,000 lb. per sq. in. on compression flange. Web equivalent, ⅛. Rivets ⅞ in. dia.

Assume the top flange angles as 6 × 6 in. the upper pair being set ¼ in. out from the edge of the web and the pairs 12½ in. back to back. Assume also that the two angles of the bottom flange are set out ¼ in. from the edge of the web plate. The effective depth may therefore be taken with sufficient accuracy as 72.5 − (6.25 + 0.25) = 66 in.

Required net area in angles and flange plates of bottom flange, from Formula (2)

$$A_f' = \frac{M}{fd} - \frac{A_w}{8} = \frac{(3,000,000)(12)}{(16,000)(66)} - \frac{(72)(0.5)}{8} = 29.6 \text{ sq. in.}$$

Required gross area in angles and flange plates of top flange,

$$A_f' = \frac{(3,000,000)(12)}{(15,000)(66)} - \frac{(72)(0.5)}{8} = 31.9 \text{ sq. in.}$$

For the bottom flange, the riveting arrangement will be assumed as such that the two holes should be taken out of each angle and two out of each flange plate. The flange may then be made up as follows:

Two angles, 6 × 6 × 13/16 in., less four 1-in. holes	= 14.93 sq. in. net
Two plates, 14 × ⅝ in., less four 1-in. holes	= 15.00 sq. in. net
	29.93 sq. in. net

The total gross area of the bottom flange is 35.68 sq. in.

The top flange will be made up as below, the small excess of area over stress requirements being provided to make the gross areas of the two flanges nearly equal.

Four angles, 6 × 6 × ⅜ in.	= 17.44 sq. in.
Four plates, 10 × ⅜ in.	= 15.00 sq. in.
	32.44 sq. in.

47. Moment of Resistance of Girders with Sloping Flanges.—Variation of the depth of a girder to suit the shear and moment requirements from point to point is sometimes carried out in girders for travelling cranes, turntables, bridge floor beams and viaduct girders. This is done by sloping one or both flanges, as shown in Fig. 42, or sometimes in the case of the lower flange by curving it

upward from the center towards the ends where it becomes horizontal. Such a girder is termed a "fish-bellied" girder.

To obtain the total stress in either flange of a girder with sloping flanges, it is only necessary to divide the moment at the section by the perpendicular distance to the center of gravity of the flange concerned from the intersection of the section plane with the neutral line of the other flange, reducing the result to allow for such portion of the moment as may be taken by the web.

Thus, in the case shown in Fig. 42*a*, the compressive stress in the top flange

$$C = K \cdot \frac{M}{d},$$

where

M = moment at the section.

d = effective depth at the section, measured vertically.

$$K = \frac{\text{Net area of angles and plates for one flange}}{\text{Total net area of one flange, including web equivalent}}.$$

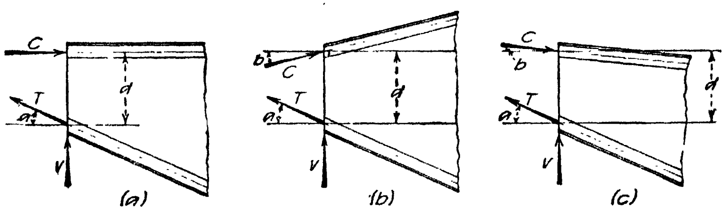


FIG. 42.—Shear and flange stresses in girders with sloping flanges.

The tension in the bottom flange is

$$T = K \cdot \frac{M}{d \cos a}$$

where

a = angle of slope of bottom flange with horizontal.

In the case of the girder shown in either Fig. 42 *b* or *c*, with the top as well as the bottom flange inclined, the total stresses in the top and bottom flanges respectively are:

$$C = K \cdot \frac{M}{d \cos b}$$

$$T = K \cdot \frac{M}{d \cos a}$$

where b = angle of slope of top flange with horizontal.

Where the flanges of a girder are inclined, they absorb some of the shear at the cross section, in the same manner as do the chords in curved chord trusses. The web shear is then less than it would be if the flanges were horizontal.

For the girder of Fig. 42*a*, if V be the total shear at the section, the shear that must be absorbed by the web

$$V_w = V - T \sin a$$

$$= V - K \cdot \frac{M \tan a}{d}$$

In the case of Fig. 42*b*

$$V_w = V - C \sin b - T \sin a$$
$$= V - K \cdot \frac{M}{d} (\tan a + \tan b)$$

For the girder of Fig. 42*c*, it is

$$V_w = V + C \sin b - T \sin a$$
$$= V - K \frac{M}{d} (\tan a - \tan b)$$

In dealing with fish-bellied girders, the flange at the section being investigated may be assumed to have the slope of the tangent to the curve at the section.

48. Flange Buckling.—To compensate for the columnar or buckling action of compression flanges of plate girders not supported continuously in a lateral direction, a reduction in the normal working stress should be made for them. In the discussion of flange buckling for beams, Art. **15**, reference was made, for purpose of comparison, to the method of providing for flange buckling in plate girders. The last six formulas of Table 5, p. 196, are applicable to such a situation. Reduction is required for all values of $\frac{l}{b}$ in the case of the A.R.E.A., C.E.S.A. and the writer's formulas, although such is not required for values of $\frac{l}{b}$ under 10 for any of the others.

Reduction of the working stress on the compression flange tends to bring the gross area of this flange somewhat nearer the gross area of the tension flange. It serves to offset in part the neglect to rivet holes in compression material.

Application of reduction formulas is carried out as in the problem under Art. **15**.

49. Length of Flange Plates.—As in the case of the box girder, the flange area of a plate girder may be readily varied to suit the moment requirement by terminating the flange plates where they are not needed to supplement the angles. The length of flange plates, whether they be placed in vertical planes or horizontally on the backs of the flange angles, may be determined by either analytical or graphical means. If the moment be computed at sections not over 5 ft. apart, the sections being taken at all points of concentrated loading, the flange area requirements at these points may be compared with the area of the flange, counting various numbers of plates, and the point of cut-off of the plates may then be found.

49*a*. Graphical Method.—Such comparison, except in the simple case where the loading is, or may be considered as uniform, is best made graphically, particularly if the loading is unsymmetrical. The simplest procedure is to plot a diagram of required flange areas at the different points where moments are determined, rather than to plot a moment diagram as is sometimes done. It is unnecessary for typical plate girders to plot a diagram of required section modulus as was done for reinforced beams (Art. **13**). When the approximate method of calculating moment of resistance is used, it is sufficiently accurate to work with a diagram of required areas. If the diagram be constructed by plotting required flange areas vertically and lengths horizontally, as in Fig. 43, and the assigned constituent areas be plotted on the same diagram, the length for which each is required may be readily scaled off. The points where each of the cover plates

should end theoretically are found by noting the points where the inner horizontal bounding line representing the area cuts the curve. This method may be applied whether the flange plates are in vertical planes over the vertical legs of the flange angles or are in horizontal planes on the outstanding legs of the flange angles.

It is customary to extend each plate far enough past the point where it might theoretically end to accommodate two transverse rows of rivets so that the plates may be capable of bearing stress at the points where they are first needed.

For deck plate girders in bridge work, the inner cover plate of the top flange, if it be a T-flange, is extended to the full length of the girder so as to protect the flange from corrosion and separation of the angles.

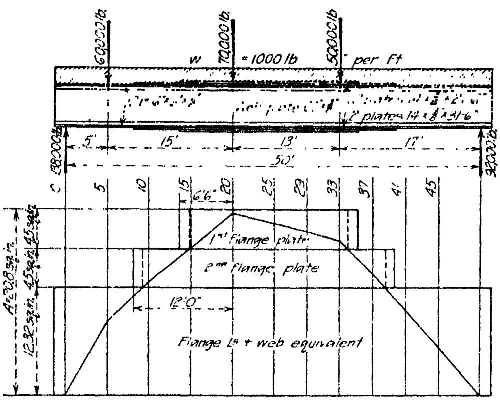


FIG. 43.—Length of plate girder flange plates by graphical method.

To keep the neutral axis at the same level throughout the girder and preserve the assumed distribution of stress, corresponding plates on the top and bottom flanges should, as far as possible, be cut off at the same points.

Illustrative Problem.—Consider a 50-ft. girder loaded with a uniform load of 1,000 lb. per lin. ft., which includes the weight of the girder, and a system of concentrated loads, as shown in Fig. 43. Find the points of theoretical and practical cut-off, if the section of each flange at the point of maximum moment is made up as follows:

1/8 of web	= (1/8)(60)(0.375) =	2.82 sq. in.
2 angles, 6 × 6 × 1/2 in., net	=	9.50 sq. in.
2 plates, 14 × 3/8 in., net	=	9.00 sq. in.
		21.32 sq. in.

Assume the permissible flange stress = 16,000 lb. per sq. in., and effective depth = 60 in.

The moments and required flange areas at points of concentrated loading and certain intermediate points are as shown in the accompanying table.

On Fig. 43, the required flange areas and the part areas assigned are shown plotted vertically. The points of theoretical cut-off are shown dotted, but the plates are extended somewhat more than a foot at each end past these points. The total lengths required and the position of the left hand end of each plate with respect to the 70,000-lb. load are shown on the diagram.

Distance of point from left support (ft.)	Uniform load moment (ft.-lb.)	Moment from concentrated loading (ft.-lb.)	Combined moment (ft.-lb.)	Required flange area (in.)
5	112,500	565,000	677,500	8.5
10	200,000	830,000	1,030,000	12.9
15	262,500	1,095,000	1,357,500	16.9
20	300,000	1,360,000	1,660,000	20.8
25	312,500	1,275,000	1,587,500	19.8
29	304,500	1,207,000	1,511,500	18.9
33	280,500	1,139,000	1,194,500	17.7
37	240,500	871,000	1,111,500	13.9
41	184,500	603,000	787,500	9.8
45	112,500	335,000	447,550	5.6

49b. Analytical Methods.—Where the loading is uniformly distributed, or is such that it produces practically the same moment curve and flange area curve (a parabola), the theoretical lengths of the flange plates may be readily determined by analytical means. In Fig. 44, let $\frac{l}{2}$ be the half span of the girder,

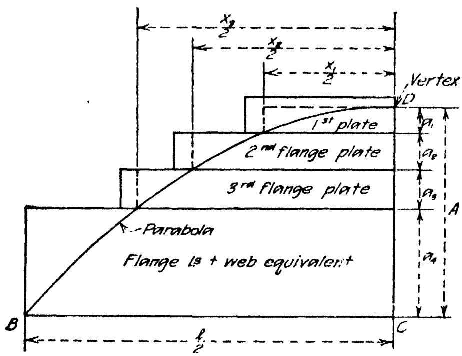


FIG. 44.—Analytical determination of length of flange plates.

and A, the maximum ordinate, be the total flange area required at the center A parabolic curve BD, with vertical axis and vertex at D, will then correctly represent the flange area requirement at all points of the half span. Let $\frac{x_1}{2}$, $\frac{x_2}{2}$, etc., be the half lengths of the flange plates, the areas of which are a_1, a_2, etc., numbered from the outside. The area a_1 is to be taken as the area required to be contributed by the outside plate.

For similar reasons to those set forth in Art. **13**, the lengths of the first or outer flange may be written

$$x_1 = l\sqrt{\frac{a_1}{A}}$$

For the second and the nth plate the lengths are

$$x_2 = l\sqrt{\frac{a_1 + a_2}{A}}$$

$$x_n = l\sqrt{\frac{a_1 + a_2 + \cdots}{A}}$$

To the theoretical lengths thus found there should be added about a foot at each end to ensure action of the plates where first needed.

The above simple method is based on the assumption that the next areas of component parts of the flanges are the same at points of cut-off as at the point of maximum moment. Aa more precise solution may be devised in the same form as that for the length of reinforcing plates for beams (Art. **13**).

Illustrative Problem.—If the total net area of one flange of a 50-ft. plate girder subjected to uniform loading be 18.67 sq. in., and of this 4.22 sq. in. is provided by each of two flange plates, find the theoretical and practical length of the two plates.

Theoretical length of first cover $= x_1 = 50\sqrt{\frac{4.22}{18.67}} = 23.75$ ft.

Adding a total of 2.25 ft., the practical length becomes 26 ft.

Theoretical length of second cover $= x_2 = 50\sqrt{\frac{8.44}{18.67}} = 33.6$ ft.

Practical length = 36 ft.

50. Economic Depth.—It may be shown by means of the calculus that for a girder to withstand a given bending moment, there is a depth which will give the minimum weight of material. This depth may be conveniently called the least-weight depth. In determining it, however, certain assumptions with respect particularly to the size and spacing of stiffeners have to be made which have an important influence on the result. If they are not realized in a given case—and such is possible, as there are very great differences in practice respecting stiffeners—the resulting depth may be materially in error.

As a result of such studies it may be shown that the least-weight depth varies from as much as $\frac{1}{4}$ to $\frac{1}{12}$ of the span, the former figure being for short, very heavily-loaded spans and the latter for long, slightly-loaded spans. For girders of from 40 to 80-ft. span with modern railway loading, the least-weight depth is not far from $\frac{1}{8}$ of the span.

Practical considerations often make it desirable to adopt a shallower depth than that giving the least weight. For example, for girders used in grade separation work where the upper grade must be kept as low as possible and the lower one as high as possible, it is economically wise to use the shallowest practicable girders. In long span girders the depth of web indicated by least weight calculations may entail extra cost per pound for wide plates and especial risk of damage in fabrication, shipment, and erection. In such cases, it is best to reduce the least weight to give what may be called the true economic depth.

Fortunately, little addition to the weight of a girder is caused by considerable changes in the depth. Thus, it may be shown that for a change of as much as 25 per cent, the increase in weight is only about 3 per cent. The advantages to be gained in other respects, and larger economies which may be effected, result in the true economic depth of plate girders being usually 15 or 20 per cent less than the least-weight depth.

Such depth ratios as have been given may be regarded without material error as referring to the depth of the web plate.

51. Flange Riveting.—In order that the flanges of a plate girder may act as an integral part of the whole girder, they must be attached to the web by sufficient rivets to transfer to them from the web the stress that they should properly bear. In this connection it is helpful to conceive of the web as the primary part of the girder and the flanges as edge reinforcement of the web. Up to the limit of the capacity of the flange rivets, therefore, the shortening and lengthening of the edges of the web will bring about a stress transference to the angles and plates of the flanges.

The rate of transference of stress to the flange, or the rate of increase of stress therein, on which flange rivet spacing depends, is directly proportional to the rate of increase of moment, as will be seen from the formula $F = M/d$, in which $F = f(A_f' + \frac{1}{8}A_w)$ = the total flange stress, M = the moment, and d = the effective depth (which may be assumed constant). But the rate of increase of moment, horizontal distances being denoted by x, is dM/dx, which may be shown as equal to V, the total vertical shear. Rivet spacing formulas may therefore be based upon the vertical shear at the section, a quantity that is more readily determined than the moment.

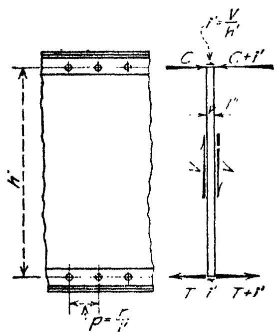


FIG. 45.—Pitch of flange rivets when web takes no moment.

51a. Unloaded Flange, Web Takes No Moment.—Consider the section of the plate girder shown in Fig. 45. Let the distance between the rivet lines in the two flanges be h' and the total vertical shear at the section be V. The increment in moment per lin. in. being $dM/dx = V$, the growth in total flange stress per lin. in. will be $i' = V/h'$, assuming the transference of stress to be at the rivet lines and assuming that no part of the flange stress is taken by the web. If the safe resistance of one flange rivet be r lb., then the number of

$$\text{inches that may be served by one rivet} = \frac{r}{\frac{V}{h'}}, \text{ or the pitch } p = \frac{rh'}{V} \qquad (1)$$

Illustrative Problem.—Find the rivet spacing for the bottom (unloaded) flange of the girder shown in Fig. 39, assuming that the web takes no share of the moment. Vertical shear at the section = 100,000 lb.; rivets, ⅞ in. dia.; safe shearing and bearing stresses on rivets = 10,000 and 20,000 lb. per sq. in. respectively. Distance between gage lines of flanges = 56.5 in.

Least safe resistance of ⅞-in. rivet is in bearing on the ⅜-in. web = (0.88)(0.38)(20,000) = 6,560 lb.

Theoretical pitch, p = (6,560)(56.5)/100,000 = 3.72 in.

51b. Unloaded Flange, Web Takes Moment.—If the web is assumed to take its proper share of the bending moment, the increment in flange stress per lin. in. will be less than if the web takes no moment, in the proportion of the respective effective flange areas. If, therefore, K = (net area of flange angles and covers of one flange) ÷ (net area of flange angles and covers of one flange + web equivalent), the actual increment of flange stress per lin. in.

going to the angles and covers will be $i = K \cdot \frac{V}{h'}$. The rivet pitch then becomes

$$r \div \frac{KV}{h'}$$

or

$$p = \frac{rh'}{KV} \tag{2}$$

The value of K defined above applies to rivets in the tension flange. For the compression flange, gross areas may be used in finding K, but no material error results from using the same value for both flanges. The value obtained by using net areas is somewhat smaller than that found by using gross areas.

Illustrative Problem.—Find the rivet spacing for the unloaded flange of the girder of the last problem if one-eighth of the gross area of the web is considered as effective flange area. Assume the rivet spacing for purposes of computing net section as 4 in., the rivets through the outstanding legs of the flange angles staggering exactly with those in the vertical legs, similar to those in Fig. 39

With a stagger of 2 in. and a distance between gage lines of 5 in. for a developed flange angle, the deduction for one angle is $1 + 0.85 = 1.85$ holes, and for two angles it is 3.70 holes. The net area of two angles is $(2)(4.5) - (3.7)(1)(0.5) = 7.15$ sq. in. For one cover plate it is, as previously found, 4.13 sq. in. The web equivalent is 2.81 sq. in.

$$K = \frac{7.15 + 4.13}{7.15 + 4.13 + 2.81} = 0.8$$

$$p = \frac{(6{,}560)(56.5)}{(0.8)(100{,}000)} = 4.64 \text{ in.}$$

51c. Loaded Flange, Web Takes No Moment.—Where the imposed load on a girder is applied directly to one or other of the flanges, as in deck plate girder bridges and also in occasional through plate girder spans with ties bearing on the outstanding legs of the flange angles, the flange rivets have a dual function to perform. Not only must they transmit to the flange angles and covers the same increments of flange stress as in the unloaded flange, but they must be able to carry the directly applied load into the web for proper distribution. The stress per lin. in. on rivets, therefore, is a resultant stress, compounded of the horizontal increment of flange stress i' and the vertical load per lin. in. w, or $R = \sqrt{i'^2 + w^2}$. The rivet pitch then becomes $r/R = r/\sqrt{i'^2 + w^2}$, or since $i' = V/h'$

$$p = \frac{r}{\sqrt{\left(\frac{V}{h'}\right)^2 + w^2}}, \tag{3}$$

Illustrative Example.—Consider the top flange of the girder of Fig. 39 as supporting a directly applied load of 7,200 lb. per lin. ft. and the web as taking no share of the moment. Other data will be as for the last two problems. Find the rivet spacing.

Increment of flange stress per lin. in. $= V/h' = 100{,}000/56.5 = 1{,}770$ lb.

Directly applied vertical load on flange rivets per lin. in. $= 7{,}200/12 = 600$ lb.

Resultant stress on rivets per lin. in. of girder $= R = \sqrt{(1{,}770)^2 + (600)^2} = 1{,}870$ lb.

Pitch, $p = 6{,}560/1{,}870 = 3.51$ in.

51d. Loaded Flange, Web Takes Moment.—If it be assumed that the web resists its appropriate share of the bending moment, the increment of flange stress per lin. in. will be $i = Ki' = K \cdot \frac{V}{h'}$, and Formula (3) becomes

$$p = \frac{r}{\sqrt{\left(\frac{KV}{h'}\right)^2 + w^2}} \tag{4}$$

This is the general form of the rivet spacing formula, reducing to Formula (1) when $K = 1$ and $w = 0$, to Formula (2) when $w = 0$, and to Formula (3) when $K = 1$.

Illustrative Problem.—Adding to the data of the last problem the stipulation that one-eighth of the gross area of the web is to be assumed as effective flange area, find the spacing. For deduction purposes assume the pitch as 4 in., the stagger being similar to that shown in Fig. 39. Assume $K = 0.80$, the same values as used for the tension flange.

$$R = \sqrt{\left[\frac{(0.8)(100{,}000)}{56.5}\right]^2 + (600)^2} = 1{,}540 \text{ lb.}$$

The pitch is, therefore,

$$p = 6{,}560/1{,}540 = 4.26 \text{ in.}$$

51e. Multiple Rows of Flange Rivets.—Formulas (1), (2), (3) and (4) apply directly to flanges containing one row of rivets, or to those containing two rows if they be staggered, as in Fig. 46a. In the latter case the distance h' is to be measured to the center of gravity of the rivet lines, as also when the flange contains two rows of rivets opposite each other, as in Fig. 46b. When the pitch is to be computed for rivets arranged as in Fig. 46b, it should be remembered that the rivets are in pairs and hence the general Formula (4) becomes

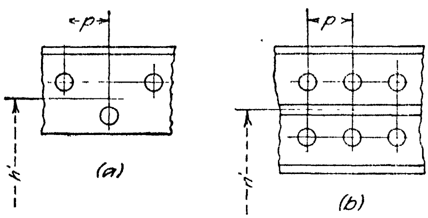


Fig. 46.—Pitch for multiple rows of flange rivets.

$$p = \frac{2r}{\sqrt{\left(\frac{KV}{h'}\right)^2 + w^2}} \tag{5}$$

In this case the directly applied load w is sometimes considered as taken up entirely by the upper pair of angles, since the load bears directly on them and the short stiffeners between the angles transfer load only at distances of several feet apart.

Illustrative Problem.—Find the pitch of the rivets in the top (loaded) flange of the girder shown in Fig. 41 if the total shear at the section is 80,000 lb. and a uniformly distributed load of 6,000 lb. per lin. ft. is applied to the top flange. Rivets, ⅞-in. dia.; safe shearing and bearing stresses on rivets = 10,000 and 20,000 lb. per sq. in. respectively. Web equivalent = ⅛. Consider vertical load as transmitted to both pairs of angles.

Least resistance of one rivet in bearing on 7/16-in. web = (0.875)(0.4375)(20,000) = 7,660 lb.

Ratio K for the compression flange from the problem under Art. 44 relating to this girder = 28.44/31.72 = 0.89.

Distance h' from Fig. 41 = 52.5 in.

$$w = 6{,}000/12 = 500 \text{ lb. per lin. in.}$$

Hence required pitch

$$p = \frac{(2)(7{,}660)}{\sqrt{\left[\frac{(0.89)(80{,}000)}{52.5}\right]^2 + (500)^2}} = 10.6 \text{ in.}$$

As the minimum permissible pitch in the line of stress is 6 in. for this case, the theoretical pacing would need to be reduced to the latter figure.

51f. Rivet Pitch in Flange Plates.—Since the function of the rivets connecting the flange plates to the other material of the flange is to transmit to them the part of the increment of flange stress per lin. in. which they should bear, a formula similar to (2) will apply. The part of the total increment of flange stress going to the flange plates or to any one plate will be some fraction K' of the whole increment, or

$$i'' = K'\frac{V}{h'}$$

The rivet value employed will in general be the single shearing value r' and not the bearing value r. If the rivets in any plate or group of plates are in pairs, as is usual, the pitch will be

$$p = \frac{2r'h'}{K'V} \qquad (6)$$

$$\text{where } K' = \frac{\text{Net area of plate (or plates) considered on one flange}}{\text{Total net area of one flange including web equivalent}}$$

Illustrative Problem.—Find the theoretical required pitch of rivets in the cover plates of the girder shown in Fig. 39 at a section where the shear is 100,000 lb., if $\frac{7}{8}$-in. rivets are used and the safe shearing and bearing stresses on rivets are 10,000 and 20,000 lb. per sq. in., respectively. Web equivalent = $\frac{1}{8}$.

Least value of rivet is single shearing value = (0.601)(10,000) = 6,010 lb.

h' from Fig. 39 = 56.5 in.

$$K = \frac{\text{Net area of one cover plate}}{\text{Total net flange area including web equivalent}} = \frac{4.13}{13.94} = 0.30,$$

counting four holes out of the flange angles and two out of the cover plate.

Hence,

$$p = \frac{(2)(6{,}010)(56.5)}{(0.30)(100{,}000)} = 22.7 \text{ in.}$$

This would need to be reduced to not over 6 in. for practical reasons.

51g. Rivet Spacing in Sloping Flanges.—Where a flange is inclined to the horizontal, as shown in Fig. 42, Formulas (1) and (2), Arts. **51a** and **51b** must be modified to take into account the fact that the incrememt of flange stress per lin. in. is not the total shear at the section divided by the vertical distance between rivet lines in the flanges, for part of the shear is resisted by the inclined flanges. The amount of this increment for typical cases will be found.

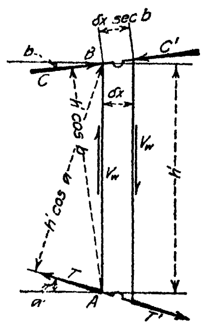


FIG. 47.—Rivet pitch in sloping flanges.

Consider a vertical strip of web of width dx and height h' between flange rivet lines at the section considered, as shown in Fig. 47. Let the shear actually absorbed by the web be V_w and let the total compressive and tensile flange stresses

to the left of the strip be C and T respectively, and to the right of the strip C' and T'. Then if i be the actual increment in the top flange stress per lin. in.,

$$C' - C = i\ dx \sec b.$$

Taking moments about A,

$$idx \sec b.\ h' \cos b = V_w\ dx$$

or

$$i = \frac{V_w}{h'}$$

Hence the pitch measured along the top flange is

$$p = \frac{rh'}{V_w}$$

or substituting the value of V_w from Art. **47,**

$$p = \frac{rh'}{V - K\frac{M}{d}(\tan a + \tan b)} \tag{7}$$

Proceeding in the same manner for the bottom flange the same formula may be shown to apply, indicating that for the case considered the pitch measured along the flanges is the same for top and bottom flanges.

51*h*. Practical Considerations in Rivet Spacing.—To simplify both office and shop work, it is desirable to use one spacing of rivets for a section of flange several feet in length. Accordingly, the theoretical pitch is computed for several points in the half span, usually at the center of the panels marked off by stiffeners, and a spacing to the nearest lower practicable fraction of an inch is adopted for the entire panel, putting any odd spaces near the stiffeners. If one flange is loaded, it may be desirable to make the spacing in the unloaded flange correspond with it so that the flange angles may be kept alike.

In no case must the adopted spacing depart from the usual minimum of 3 diameters of the rivet or the maximum of 16 times the thickness of the outside material, or 6 in. The restriction on the minimum side often necessitates increasing the thickness of the web to keep the rivets at the end of the girder from being too close.

Where the girder is shallow and has heavy flanges, the approximate method of determining spacing outlined above cannot safely be applied. The horizontal shear method explained and illustrated for box girders in Art. **37** must then be employed.

52. Intermediate Web Stiffeners.—One way of strengthening the web of a plate girder or of making, what otherwise might be too thin, a web available for a given girder is as has been pointed out in Art. **17,** to restrain it from buckling, in some measure at least by means of stiffeners. These consist usually of pairs of vertical angles, one on each side of the web, riveted firmly thereto at intervals of approximately the depth of the web. Stiffeners in place in a typical girder are shown in Fig. 34.

In addition to their service of stiffening the web, intermediate stiffeners may help to transmit concentrated loads to the web and distribute them over it. For example, the stiffeners in a deck plate girder, or any girder loaded along the top flange, help to carry the applied load down into the web and relieve the top flange rivets of much of the vertical component of their stress (see Art. **51**). They also to some extent lessen the deflection of the inner outstanding legs of the top

flange angles in deck railway girders carrying ties. If definite concentrated loads are applied to the top flange over stiffeners, or are applied to them anywhere within their length, they must be treated as concentrated load stiffeners, as described in Art. **53**.

Formerly, stiffeners were sometimes placed in a diagonal direction pointing downward towards the near support at an angle of 45 deg. with the neutral axis. The idea was virtually to construct columns in the line of maximum stress at a number of points. However, *between* the columns so constructed little support would be given the web, and so, for reasons of greater efficiency as well as simplicity, vertical stiffeners were employed. These as shown in Fig. 48 restrict the length of a laterally unsupported strip of web to the length of the diagonal of the rectangle bounded by consecutive stiffeners and by the flanges. Moreover, there is but one narrow strip of this length in each rectangle. All other strips are intercepted by stiffeners and prevented from buckling over their full length.

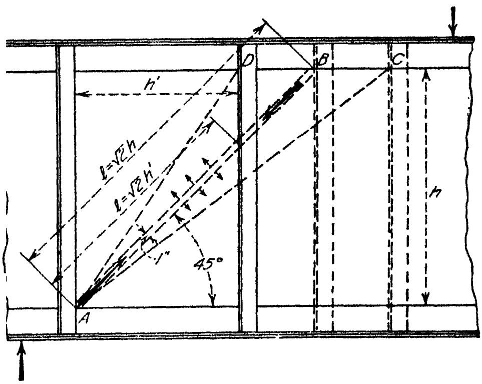


FIG. 48.—Spacing of indeterminate web stiffeners.

Whether stiffeners are to be used or not is usually determined by the application of the rule, stated in Art. **42,** that the clear or unsupported distance between the flange angles must not exceed 50 or 60 times the thickness of the web. Assuming the latter ratio, the slenderness ratio $\frac{l}{r}$ of a 45-deg. strip of web might thus be as high as 295. Such is permissible under the circumstances, however, since the strip receives important support from the adjacent material.

Even though a strict application of the above rule might not indicate the necessity of stiffeners, it is often considered desirable by designers to insert them to stiffen the web in fabrication, transportation, and erection.

The spacing of stiffeners has an important influence on the buckling stability of the web. Since at the sections where web crippling is an important factor—that is, where the shear is large—the direction of the maximum diagonal compressive stresses makes an angle of about 45 deg. with the neutral axis throughout the unsupported depth of the web (Art. **42**), the critical diagonal strips are those inclined at 45 deg. Along steeper ones, as AD, or along flatter ones as AC, Fig. 48, the intensity of the diagonal compressive stress is reduced. To determine, therefore, the spacing of stiffeners that will permit the employment of a certain thickness of web in a girder, it is necessary to find how long a 45-deg. strip of web, Fig. 48, may be without support in order that the safe buckling strength of the strip may not be exceeded.

Let AB be a strip of web 1 in. wide and t in. thick inclined at 45 deg. to the flanges or to the stiffeners, and let h' be the clear distance between stiffeners.

This strip may be regarded as a modified column of length $l = \sqrt{2} \cdot h'$ and radius of gyration at right angles to the web of $\frac{t}{\sqrt{12}}$. It is more favorably situated than a column, in that tensile stresses in the web at right angles to its length prevent it from buckling in the plane of the web, and the presence of the adjacent material hinders its buckling at right angles to the plane of the web. Reduction of the normal safe stress should, therefore, be at a much less rate than for ordinary columns. If the A.R.E.A. column formula

$$p = 15{,}000 - 50\frac{l}{r}$$

be applied to this situation, it is probable, considering available experimental evidence, that the reducing term should be somewhere between $20\frac{l}{r}$ and $30\frac{l}{r}$. Assuming the average value, and making it $l = \sqrt{2}.h'$ and $r = \frac{t}{\sqrt{12}}$, the formula becomes very nearly

$$p = 15{,}000 - 120\frac{h'}{t} \tag{1}$$

If now the diagonal compressive stress at the section in question, which at critical sections very nearly equals the average shearing stress be represented by v, then for complete realization of the buckling strength of the web

$$v = p = 15{,}000 - 120\frac{h'}{t}$$

If the existing stress, v, is greater than p, the section is unsafe, and the stiffeners require to be closer together, or the web should be thickened.

The necessary spacing of stiffeners to permit the use of a given web may be found by re-arranging the last equation, or

$$h' = \frac{t}{120}(15{,}000 - v) \tag{2}$$

If a more conservative reduction of the A.R.E.A. column formula be adopted —that is, $30\frac{l}{r}$—we have very nearly

$$p = 15{,}000 - 150\frac{h'}{t} \tag{3}$$

and for the safe clear spacing of stiffeners

$$h' = \frac{t}{150}(15{,}000 - v) \tag{4}$$

These latter formulas are seen to be in accord with the recommended conservative formula for the buckling strength of beam webs derived in Art. **17** and plotted in Fig. 11, p. 206. If $\frac{h'}{t}$ is over 60, they would give results that are too severe. Adopting Formula (7) of Art. **17** for values of $\frac{h'}{t}$ over 60, they would become respectively

$$p = 10{,}200 - 70\frac{h'}{t} \tag{5}$$

and

$$h' = \frac{t}{70}(10{,}200 - v) \tag{6}$$

The provisions of the A.R.E.A. (1920) Specification covering the spacing of intermediate stiffeners contain the following:

The webs of plate girders shall be stiffened by angles at intervals not greater than the distance given by the formula

$$d = \frac{t}{40}(12{,}000 - v)$$

where

d = distance between rivet lines of stiffeners in inches.
t = thickness of web in inches.
v = web shear at point considered, in lb. per sq. in.

This formula gives spacings in excess of those given by either Formulas (2), (4) or (6). The spacing is, of course, between rivet lines, but this accounts for only a small part of the difference. Actual tests of built-up girders with stiffeners show surprisingly large diagonal compressive strength, as will be seen from the three plotted points for such girders in Fig. 11, p. 206. The spacing formula of the A.R.E.A. specification appears to conform very well with these tests, although it does not conform closely to the tests of I-beams. More conservative formulas would, therefore, appear to be required in the investigation of I-beam webs.

In the determining of safe stiffener spacing by any of the formulas above, if the quantity h' should come out equal to or greater than h, the clear depth between flanges, the inference is that stiffeners are not required. If they were spaced in the clear at distances apart greater than the clear depth of the web, they would uncover diagonal strips of considerable width along which buckling could take place.

In recent years the practice has developed of spacing stiffeners practically uniformly from end to end of girders, thus disregarding the effect of intensity of diagonal compressive stresses on the situation. Where such practice is followed, the spacing is generally about equal to the depth of the web. If, however, such a spacing is adequate at points of maximum shear and diagonal compression it is more than adequate in regions of low diagonal compression. Of course, if the shearing stress is low at the section, or if the ratio of unsupported depth of the web to its thickness is low, the practice cannot be said to be objectionable. For example, the specifications of the Department of Railways and Canals, Canada, provide that for girders under 4 ft. in depth, stiffeners may be spaced 4 ft. apart. Such a rule should not be applied if the shearing stresses are over about 50 per cent of the maximum permissible shearing stress.

The effort to utilize a comparatively thin web for high diagonal compressive stresses by very close stiffener spacing is likely to prove inadvisable. Where the necessary stiffener spacing works out to be less than half the depth of the web, it is more economical to use a thicker web.

Most specifications prescribe an arbitrary maximum permissible spacing for stiffeners. For the A.R.E.A. specifications, the limit, center to center, is 6 ft. or the depth of the web. The specifications of the Canadian Engineering Standards Association fix it at the depth of the web, or 7 ft. These rules should not be interpreted to mean that such spacings may be adopted without question for the whole girder. The spacing should be regarded as dependent upon the actual web stresses and in the typical girder these require a gradually diminishing spacing near the ends.

Although calculation of the web stresses may indicate a low shearing stress, stiffeners must nevertheless be used if the ratio of unsupported depth to thickness is over the limit set down in the specification, say 50 or 60. An arbitrary rule such as this is open to question, but it is of the same kind as the one fixing the upper limit of the slenderness ratio for columns.

Proportioning of Stiffeners.—Experimental investigation has shown that under working conditions the deformation of an unstiffened web or of ordinary intermediate stiffeners along vertical lines is small. The function of stiffeners not carrying concentrated loads is, therefore, merely to stay the web laterally. They are to be regarded as vertical beams rather than columns. For this reason, the outstanding legs should be relatively wide, but the legs next to the web may be as narrow as the riveting will permit.

No recognized method for the scientific proportioning of intermediate stiffeners exists. A useful empirical rule given in the A.R.E.A. specifications is that the outstanding leg of each angle shall not be less than 2 in. plus $\frac{1}{30}$ of the depth of the girder, nor more than 16 times its thickness. The thickness for railway girders is usually $\frac{3}{8}$ in., while for highway bridges and building girders it is usually $\frac{5}{16}$ in.

The following sizes for intermediate stiffeners of railway girders are fixed in the specifications of the Department of Railways and Canals, Canada:

Depth of Web (Feet)	Size of Stiffener Angles (Inches)
3 ft. and under	$3 \times 2\frac{1}{2} \times \frac{3}{8}$
4	$3 \times 3 \times \frac{3}{8}$
5	$3\frac{1}{2} \times 3\frac{1}{2} \times \frac{3}{8}$
6	$4 \times 3\frac{1}{2} \times \frac{3}{8}$
7	$5 \times 3\frac{1}{2} \times \frac{3}{8}$

While so far as duty is concerned, intermediate stiffeners not carrying concentrated loads may be crimped over the flange angles, opinion differs considerably as to the economic advantages of crimping. For stiffeners shorter than 3 ft., it is doubtful if the saving in filler material will offset the cost of crimping, unless the fillers be very thick. For depths greater than 3 ft., most shops crimp the stiffeners.

In case forces are applied to the flange of a girder acting towards the neutral axis, the stiffeners should be carefully fitted to the flange to which the loads are applied. It is not theoretically necessary that the other end should be fitted to the flange, but it is found easier to make a tight fit at two ends than at one. If a tight fit is specified at one end, therefore, the fabricating shop would probably fit stiffeners tight at both ends.

Unless known concentrated loads are to be provided for, the rivet spacing in stiffeners is a matter of judgment. The maximum spacing permitted is usually 6 in.

Illustrative Problem.—Recommend a size for the intermediate stiffeners of a building girder 54 in. deep.

Based on the A.R.E.A. rule, the outstanding leg should be not less than

$$2 + \frac{54}{30} = 3.8 \text{ in.}$$

For building work the stiffeners would probably be two $4 \times 3 \times \frac{5}{16}$-in. angles, with the 4-in. legs outstanding.

53. Concentrated Load Stiffeners.—If at any point a concentrated load is applied to a girder flange or web, reinforcement of the web is usually required. Should the load be applied on the tension side of the neutral axis, the stresses due to the concentration will augment the existing diagonal tensile stresses in the web and it may be necessary to add distributing plates or angles to carry the load up into the web.

It is more usual, however, for concentrated loads to be applied to the compression side of the girder and very commonly to the top flange. If these would raise the existing diagonal compressive stresses in the web above the limit of safety, distributing angles or stiffeners, such as described for beams in Art. **18,** must be introduced. The principles of design of these stiffeners, as given in Art. **18,** apply when they are used for plate girders.

A very common occurrence of the concentrated load stiffener is at the end or reaction stiffeners for a girder. These consist usually of two or three pairs of vertical angles tightly fitted in between the outstanding legs of the flange angles and arranged in one or other of the ways shown in Fig. 49.

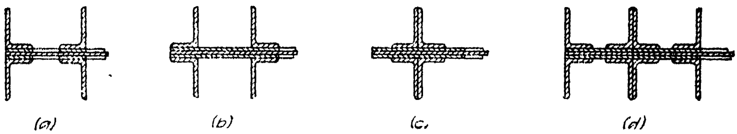


Fig. 49.—Typical arrangements of end stiffeners.

Arrangement (*a*) is very commonly adopted when the girder reaction is transmitted to the masonry through a shoe plate and a bed plate only. Type (*b*) is preferred by some designers for the same situation on the ground that the load is not applied to the edges of the shoe plate, as in (*a*), but is fairly well placed on the four quarters of the plate. If the girder rests on a bolster or rocker, arrangements (*c*) or (*d*) should be used, the former for light reactions and the latter for heavy ones.

End stiffeners have a dual function to perform: (1) They serve as vertical beams to stay the web against buckling under diagonal compression; and (2) they receive the concentrated load of the end reaction and distribute it to the web. In the latter role they act in some degree as columns, but since their loading varies gradually from a maximum at one end to zero at the other, they are commonly assumed to have a length as columns of only one-half the depth of the girder. In order that they may be well adapted to carry loads as columns, they should not be crimped, but should be kept straight by the use of fillers between them and the web.

Since the lower edge of the web cannot be counted upon as bearing on the shoe plate, and since the rivets passing through the flange angles and the web are already stressed to their safe limit for flange purposes, and also since the bearing of the ends of the fillers on the inner edges of the flange angles is uncertain, it is customary to assume that the entire end reaction is carried into the web by the stiffener angles. This load they receive in end bearing on the outstanding legs of the flange angles. No bearing value can be credited to the legs of the stiffeners in contact with the vertical legs of the flange angles by reason of the grinding or

bevelling necessary to clear the flange angle fillets, as shown in Fig. 50. If the outstanding legs of the stiffener angles should extend beyond the rounded corners of the flange angles, the projection cannot be counted, so the bearing length is restricted to b, shown in Fig. 50. Close fitting of the lower ends of the stiffener angles to the bottom flange angles is imperative for all girders and in the case of deck girders to the top flange angles as well. While theoretically it is unnecessary to have a close fit to the upper flange angles in through girders, it is found that the fitting at the bottom is facilitated if they are fitted tightly at both ends.

To give large bearing area as well as to increase the lateral stiffness of stiffeners, the outstanding legs are usually specified to be as wide as the flange angles will permit. The legs in contact with the fillers may be as narrow as riveting will allow, for although these legs have value in the column it is small as compared with the outstanding legs and as has been seen, their bearing value is zero.

By reason of the deflection of a girder resting on ordinary shoe and bed plate bearings, the stiffeners nearest the face of the support probably receive much more load n those farther back. Practice varies h respect to the assumed distribution of l amongst the pairs of angles, but if re are two pairs at the extreme inner and er edges of the shoe plate, it is probably

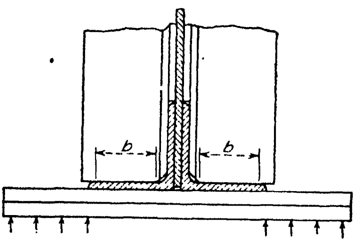


FIG. 50.—Bearing of end stiffener angles.

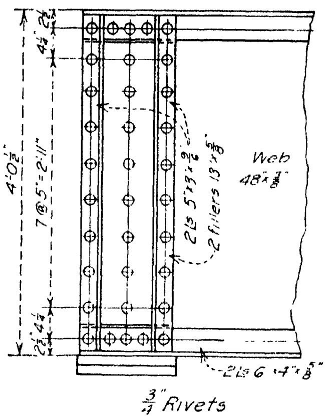


FIG. 51.—Design of end stiffeners.

fair to assume that the inner pair receives about twice as much load as the outer pair. If the pairs are placed closer together, or in somewhat from the edges of the bearing, the inequality of loading would be somewhat lessened.

Rivets connecting the stiffeners to the web plate must be sufficient in number to transmit safely to the web the maximum stress calculated to be borne by each pair. As has been pointed out, the flange rivets should not be counted. If loose fillers are used, the number of rivets required, figured in bearing on the web, would, under most specifications, need to be increased by 50 per cent.

Illustrative Problem.—The total end reaction of a plate girder is 120,000 lb., the end section consisting of a 48 × $\frac{3}{8}$-in. web and four 6 × 4 × $\frac{5}{8}$-in. angles, as shown in Fig. 51. Determine the size and arrangement of the end stiffeners. Permissible stresses in compression and bearing = $15{,}000 - 50\,\frac{l}{r}$ and 16,000 lb. per sq. in., respectively. Rivets $\frac{3}{4}$ in. diameter. Safe shearing and bearing stresses on rivets 12,000 and 24,000 lb. per sq. in., respectively. Rivets through loose fillers to be increased 25 per cent.

Load on inner pair of stiffener angles is approximately $(\frac{2}{3})(120{,}000) = 80{,}000$ lb.

Outstanding legs must extend at least to rounded corners of flange angles, which will require 5-in. legs. Length of bearing, b, for each angle is approximately 4.5 in.

Required total bearing area for two angles = 80,000/16,000 = 5.0 sq. in.

Required thickness of each stiffener angle $t = \frac{5.0}{(2)(4.5)} = 0.55$ in., say $\frac{9}{16}$ in.

Assuming 3-in. legs next the web, the angles would be $5 \times 3 \times \frac{9}{16}$ in.

Radius of gyration of two such angles about axis in plane of web

$$\sqrt{\frac{2[(10.4) + (4.18)(2.58^2)]}{8.36}} = 3.02 \text{ in.}$$

Assuming effective length of stiffener column of 48/2 = 24 in.,

$$p = 15,000 - (50)\left(\frac{24}{3.02}\right) = 14,600 \text{ lb.}$$

Required area for column action = 80,000/14,600 = 5.48 sq. in.

Area provided = (2)(4.18) = 8.36 sq. in. and hence section is adequate.

Least value of one rivet is in bearing on $\frac{3}{8}$-in. web = (0.75)(0.375)(24,000) = 6,750 lb.

Number of connecting rivets required = 80,000/6,750 = 12.

If loose fillers be employed, this must be 12 + 25 per cent = 15, exclusive of those in the flange angles.

This number cannot be driven without adopting practically the minimum permissible spacing and so it is best to use tight fillers These will be 13 in. wide and extend under both angles on each side of the web. The rivets driven in the fillers may be counted as effective for the stiffeners, since they increase the bearing area of the rivets through the stiffener angles. The same riveting arrangement will be adopted for the outer pair of angles as for the inner pair, as shown in Fig. 51. For the stress borne by the inner angles there are 8 rivets in the angles themselves plus one-half of those in the line along the center of the fillers, or 12 in all.

54. Girders Under Lateral Flexure.—Apart from the accidental lateral flexure to which girders are subjected, due to vibration, swaying of trains, wind, or flange buckling, there are sometimes definite lateral loads to be provided for in design. In the case of crane runway girders, for example, the end thrust of the crane due to sudden stopping of the loaded trolley on the crane bridge or due to lifting heavy loads by inclined pull is often definitely fixed in specifications. In Schneider's "General Specifications for the Structural Work of Buildings," it is required that the top flanges of crane girders shall withstand, in addition to the vertical load, a lateral loading based upon one-fifth the lifting capacity of the crane equally divided amongst the four wheels.

In proportioning the top flanges of such girders a section is first assumed which appears suitable for the combination of vertical and lateral moment. The stress due to vertical moment may then be found at the extreme fibers by the approximate Formula (2) of Art. **44** (using the subscript v to indicate quantities having to do with vertical moment),

$$f_v = \frac{M_v}{d\left(A_f' + \frac{A_w}{8}\right)}$$

or preferably by the exact flexure formula

$$f_v = \frac{M_v c}{I_v}$$

The stress due to lateral moment is then found by the flexure formula

$$f_l = \frac{M_l c'}{I_l}$$

the subscript l indicating quantities pertaining to lateral moment and c' being the half width of the flange.

If $f_v + f_l$ exceeds the permissible working stress, the section must be increased.

55. Torsion on Plate Girders.—If girders are curved horizontally or are of such slope in plan that vertical loads do not come on the line joining the supports, a condition of torsion on the cross-section is produced. The comments on the design of beams for torsion in Art. **23** will apply to built-up girders.

BEARING PLATES AND BASES FOR BEAMS AND GIRDERS

By C. R. Young

56. Types and Uses.—In order to transmit its loads to the masonry, a beam or girder must in general have bearing plates or bases placed under its ends. The bearing strength of the masonry is so much less than that of a steel beam flange that the flange must in effect be enlarged to prevent the masonry from being crushed.

The simplest bearing is the flat, rectangular plate, Fig. 52*a* or *b*, extended out on either side of the beam or girder flange, and of a length parallel to the beam

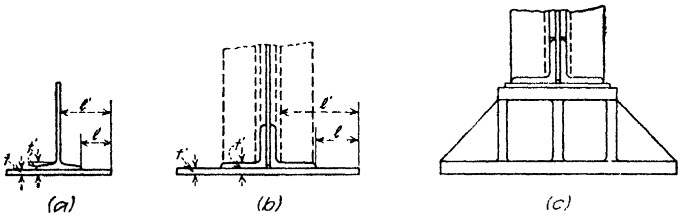


FIG. 52.—Simple fixed bearings for beams and girders.

equal to the bearing on the support. When the plate is of cast iron, it is possible to taper it out towards the ends and so vary the section in accordance with the bending requirements.

When the reaction becomes heavy, a much deeper bearing must be employed in order to transmit loading to a large area without exceeding the safe bending stresses or the allowable upward deflection. Ribbed bases of cast iron or cast steel, such as shown in Fig. 52*c*, can be made to distribute heavy loads over large areas. They are also incidentally useful to raise the end of the girder to a greater height above the masonry, thus permitting the more ready attachment of bracing and serving to keep the masonry at the fixed end of a long span girder at the same elevation as at the roller end. Cast steel is preferable to cast iron for such pedestals on account of its greater reliability, but it costs more.

If the span is longer than about 60 to 80 ft., it is desirable to use hinged bolsters so as to prevent the high intensification of stress near the inner edge of the bearing due to the deflection of the girder. Some engineers make this limit as low as 50 ft. These bolsters may be of cast steel, as shown in Fig. 53*b*, or may be built up of plates and angles, as in Fig. 53*c* and *d*. They consist of an upper and a lower part connected by a pin, pivot, or disc.

For all spans over 25 or 30 ft. it is necessary to provide for changes of length due to expansion and contraction. One end of a beam or girder must, therefore,

be allowed to slide freely. At the fixed end of a girder the base or bolsters described above are placed directly on the masonry, with perhaps a sheet of lead between, and are bolted or anchored down with usually two anchors, one on each side of the girder, so as to prevent movement.

At the sliding end it is necessary to place a bed plate or bed casting on the masonry to provide a smooth surface on which the adjustment may take place. For girders up to about 60- or 80-ft. span, adjustment is allowed to take place merely by the upper or sole or shoe plate sliding on the lower plate, as in Fig. 53*a*. If bolsters are used, rollers must be interposed between the sole plate of the bolster and the bed plate at the sliding end. These consist of a group, or "nest"

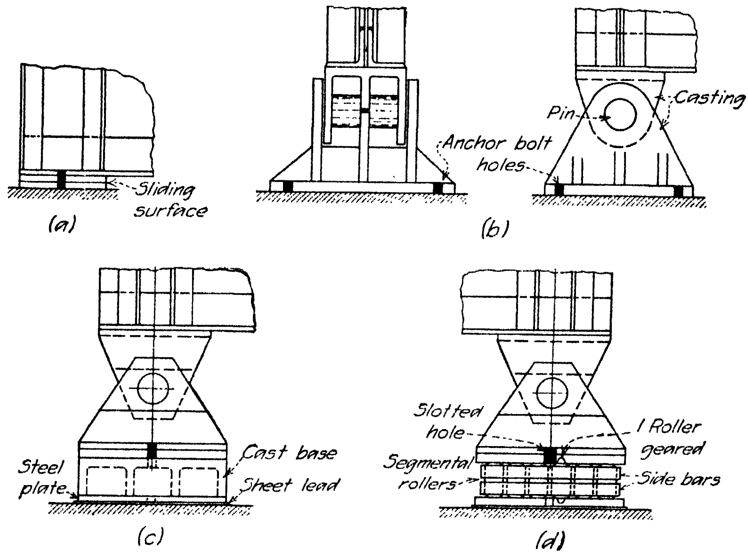


FIG. 53.—Sliding and hinged bearings.

of either round or segmental rollers held against shifting endwise by planed grooves in the shoe and bed plates or a ridge on them engaging a turned recess in the rollers, and kept at the proper distance apart by spacing bars attached to the ends, as shown in Fig. 53*d*. Segmental rollers are flattened on their vertical sides so as to take advantage of the higher strength of large rollers without having to provide an unduly large shoe or bed plate. Sometimes they are concaved on the sides, thus making a greater rotation possible for the same clear spacing. To prevent segmental rollers from tipping over, it is frequently required that at least one of them be geared to the upper and lower plates. With close spacing and substantial side bars this danger is minimized.

57. Design of Simple Bearing Plates.—In proportioning a bearing plate, Fig. 52*a* or *b*, the required area is first found by dividing the maximum reaction by the allowable pressure on masonry of the class employed. If there is no specification for this, an average value may be found in any of the civil engineering handbooks. The dimensions are then fixed so as to utilize readily-obtained

plates, one of them being equal to the bearing on the support. Recommended bearings and sizes of plates for building work are found in all of the steel handbooks.

To determine the thickness required, consider a strip of plate at right angles to the beam or girder, Fig. 52*a* or *b*. It is in effect a cantilever of span l subjected to an upward loading. This is commonly assumed as uniformly distributed, although the pressure at the outer edge is much less than near the beam flange. Let p be the uniformly distributed upward pressure in pounds per square inch, f the allowable flexural stress on steel plates, and t the thickness required. Then

$$t = 1.732l\sqrt{\frac{p}{f}} \tag{1}$$

The strength of the combined flange and bearing plate in bending should also be investigated. They form a cantilever beam of span l' and thickness t', Fig. 52*a* or *b*. It is not always safe to assume this cantilever as a single beam of depth t', since there may be insufficient connection between the flange and the bearing plate to resist the horizontal shear accompanying beam action for the whole depth. The moment of resistance of each should in such a case be found and the two added.

Tables may be prepared of the allowable projections of bearing plates based on Formula (1). They will vary in accordance with the allowable pressure and flexural stress assumed, but are usually from 3 to 5 times the thickness.

58. Design of Ribbed Bases.—The load applied to a ribbed base, such as shown in Fig. 52*c*, must be transmitted to the masonry by means of the ribs acting as short columns or prisms. These ribs should be so placed as to receive the load from the beam or girder in the most direct manner possible. One rib should be directly under the web and, when cast pedestals are used for plate girders, transverse ribs should be directly under the outstanding legs of the end stiffener angles. The rib under the pair of stiffeners nearest the edge of the support is much more highly stressed than the others by reason of the deflection of the girder. Some designers assume that the whole reaction is applied over this rib. If it be proportioned for the whole load and the other ribs made of the same thickness, the base will be amply strong. While it is desirable to have three ribs running longitudinally, as shown in Fig. 52*c*, only two need be run transversely if there be but two pairs of stiffener angles. With three pairs of angles there should be three ribs. The top plate of the pedestal must be fixed largely by judgment and experience, but should not be less than 1¼ in. The bottom plate is designed by assuming it as a cantilever projecting past the most heavily-loaded rib. The moment of the uniformly varying pressure on the projection taken about the edge of the rib is then found and the projection, including both base plate and extension of ribs, is figured in the same manner as a cast iron lintel section (Art. **30**). The ribs should not be thinner than 1 in. for important work and the bottom plate not thinner than 1½ in.

To equalize the pressure on the masonry, it is desirable to make the top plate of the casting as narrow as possible in a direction parallel to the length of the girder.

59. Design of Bolsters.—The design of the upper and lower parts of bolsters involves the same principles, although, by reason of its bearing on masonry, the lower part is usually the larger. Considering Fig. 53*b* or *c*, it is seen that the

ribs which transfer the load from the girder to the pin, and from the pin to the lower part of the bolster, must be of sufficient thickness to carry their loads without exceeding the safe bearing pressure on the ribs or on the pin. To determine their thickness, the size of pin must, therefore, be assumed first. If the rib thicknesses are satisfactorily fixed and the moment or shear on the pin is excessive, the pin size will need to be increased, which may make it possible to use thinner ribs. The bottom plate of the lower part of the bolster must be investigated as a beam continuous under the ribs and cantilevered past them. The upward pressure may be assumed as uniformly distributed over the lower parts of the bolsters.

Care must be taken not to make the ribs so thin as to render them likely to buckle laterally. To safeguard against this it may be necessary to introduce transverse diaphragms between them.

As both the upper and lower parts of the bolster are in effect beams resting on a single central support, they should be figured in bending at the vertical section through the pin hole. Due to the lateral deformation of the pin under pressure, causing it to bear against the right and left faces of the hole, no deduction need be made for the hole.

60. Design of Expansion Bearings.—If sliding is provided for merely by allowing a sole or shoe plate to slide on a bed plate, as in Fig. 53*a*, the shoe plate is designed in the same manner as an ordinary simple bearing plate (Art. **57**) and the bed plate is made about the same thickness. Each of these plates should either be planed or straightened to ensure true contact. They are rarely less than ⅞ in. thick.

Round holes for anchors are provided in both sole and bed plates at the fixed end, but slotted holes must be provided in the sole plate at the sliding end. These must be elongated about ⅛ in. for each 10 ft. of span.

Where the span is in excess of from 60 to 80 ft., rollers must be provided. Since round rollers take up a great deal of room for their pressure value, segmental ones are now very commonly used. Large rollers also tend to overcome the frictional difficulties incident to the use of small rollers. This is evident from the formula for the permissible pressure on cast steel rollers which in pounds per lineal inch is usually placed at 600 times the diameter in inches. Railway bridge specifications now frequently require that rollers be not less than 6 in. in diameter. The bearing length provided must be clear of all recesses engaging ribs on the shoe or bed plates.

To make the masonry of the same height at the two ends and at the same time keep the bolsters alike, a cast or built-up bed equal to the height of the rollers and the bed plate at the expansion end should be placed under the bolster at the fixed end, as shown in Fig. 53*c* and *d*. In the case illustrated, this bed is made up of a cellular casting resting on a steel plate.

To give true bearing on the masonry, beds are sometimes set ½ to 1 in. high and grouted underneath. Another method is to place a sheet of lead ⅛ in. thick under the bed, as shown in Fig. 53*c* and *d*.

61. Anchors.—Beams and girders are prevented from shifting horizontally on their supports by means of anchors. For ordinary rolled beams resting on loose bearing plates, a round rod is frequently used, preferably bent so as to engage the masonry beyond the end of the beam. Sometimes two angles are

bolted to the end of the beam, the outstanding legs securing the beam to the masonry. Rod anchors are usually $\frac{3}{4}$ in. in diameter, projecting about 9 in. on either side of the web. Angle anchors should have a 6-in. outstanding leg and be $\frac{3}{8}$ or $\frac{7}{16}$ in. thick. Such anchors are illustrated in the Carnegie Pocket Companion.

If beams or girders rest on sole plates riveted thereto, anchor bolts are inserted vertically in the masonry through holes left in it for this purpose or specially drilled after the erection of the steel, care being taken to so locate the holes that this can be done. Usually two anchors, one on each side of the beam or girder, are employed. For building work, they should not be smaller than $\frac{3}{4}$ in. diameter, while for railway bridge work the minimum should be $1\frac{1}{4}$ in. They should extend at least 12 in. into the masonry and be well grouted or otherwise secured thereto. To this end, split bolts with wedges in the ends, or hacked bolts, are sometimes used.

Holes for anchor bolts should be $\frac{1}{8}$ or $\frac{3}{8}$ in. larger than the bolts, to provide for errors in placing the bolts or for easier drilling of the holes.

STEEL TENSION MEMBERS

By C. R. Young

62. Forms and Uses.—Steel tension members vary in form with the magnitude and character of the stress carried by them, with the character and situation of the structure of which they form a part, and with the methods of construction adopted.

Round or square rods, single or multiple, made adjustable by end nuts, turnbuckles or sleeve nuts, Fig. 54*a*, *b* and *c*, are used in many cases where loads are light and cheapness is desired. In pin-connected bridges they are employed as counters, and sometimes, though inadvisedly, as laterals in both riveted and pin-connected bridges; in buildings they serve for lateral and sway bracing, for hangers and for tie rods in arch floors; in towers they are frequently used for bracing. The end connections are made by forging the bar into a simple or a forked loop, Fig. 54*b* and *c*, or, perhaps, by attaching a clevis thereto, as shown in Fig. 54*d*.

Eye bars, which consist essentially of rectangular bars of metal with a head, Fig. 54*e*, forged at each end, through which a pin hole is bored, have an extensive use as tension members, usually in multiples of two bars. While formerly used for bridge and roof trusses of almost all spans, their use is now confined very largely to the longer spans, where there is little likelihood of objectionable rattling or vibration. For such situations they afford a reliable, easily transported and readily erected tension member. In places where reversal of stress is possible, or where attachment of rivited work is necessary, eye bars are replaced by built-up tension members, as for the end panels of the bottom chord and for the hip hangers of railway bridge trusses.

Adjustible eye-bars are frequently used as counters in bridge trusses. The adjustment is made possible by inserting in the body of the bar at a conveniently accessible point, a turnbuckle or a sleeve nut, Fig. 54*f*.

Flats, or narrow plates, Fig. 54*g*, are little used, as they are flexible, easily become bent in transportation and erection and do not hold their length or

straightness well against drifting or reaming. They are sometimes used as hangers, but apart from such use are seldom employed in America.

Single angles, Fig. 54*h*, are extensively used as light tension members in trusses and as lateral and sway bracing.

Double angles, arranged as in Fig. 54*i* or *j*, are used as truss tension members of medium capacity.

Forms consisting of four angles arranged in H-shape and connected together by a single plane of latticing or tie plates, Fig. 54*k*, or by a web plate, are very economical and are advantageously used as bottom chords, tension diagonals, and hip hangers of riveted trusses.

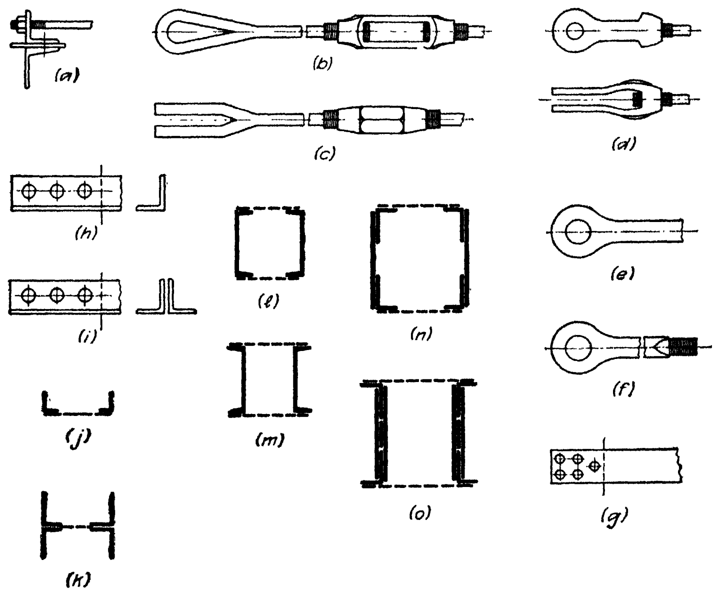


FIG. 54.—Typical steel tension members.

Two channels with flanges turned in, as in Fig. 54*l*, and connected together by latticing or tie plates, are well adapted for bottom chords of through spans, and for tension diagonals and hip hangers of both deck and through trusses. If the two channels are arranged with their flanges turned out, as in Fig. 54*m*, the section is well adapted for the bottom chords of deck spans.

Built channels, with flanges turned in, as in Fig. 54*n*, or with flanges turned out, as in Fig. 54*o*, and with or without added side plates, are the commonest forms for riveted tension members in both riveted and pin-connected trusses of both moderate and long span.

63. Theory of Design of Tension Members.—Assuming a uniform distribution of stress over that portion of the cross-section that may be considered

as effective, it follows that the effective area required in a member for the resistance of a given force is given by the formula

$$A = \frac{P}{p}$$

where A = effective area of section required in square inches.
P = force, or total stress, to be resisted, in pounds.
p = permissible stress in pounds per square inch.

If it should happen that the force to be resisted by the tension member is applied at some other point than the center of gravity of the net section, the member must be designed for a combination of direct and bending stress. The principles governing the design of such members have been thoroughly explained in Sec. 1. If the load is applied on a principal axis of the cross-section, then, neglecting the effect of the deflection arising from the eccentric application of the load, the maximum resultant stress at the most highly stressed fiber, distant c from the neutral axis, is given by the formula

$$f_1 = \frac{P}{A} + \frac{Pec}{I} \tag{1}$$

where e = the eccentricity, and I = moment of inertia of section about an axis normal to the plane of bending.

Let p be the permissible stress on the most highly stressed fiber, and r the radius of gyration of the section in the plane of bending. Then the total required area

$$A = \frac{P}{p} + \frac{Mc}{r^2 p} \tag{2}$$

Should the resultant axial load be applied eccentrically and not on a principal axis of the cross-section, the maximum resultant stress[1] on the most highly stressed fiber, as established by Professor C. Batho, is

$$f_1 = \frac{P}{A} + \frac{Px_p(y_1 - x_1 \tan a)}{J - I_y \tan a} \tag{3}$$

where x_1 and y_1 are co-ordinates of the most highly stressed fiber, the origin being at the center of gravity.

x_p = x-co-ordinate of point of application of load P.
a = angle which neutral axis makes with x-axis, found by equating f_1 to zero.
J = product of inertia of section.
I_y = moment of inertia of section about x-axis.

The necessary effective area can be found by a trial and error application of Formula (3).

64. Choice of Section.—Members should be composed, in so far as possible, of sections symmetrically placed and should be of forms such as to obviate large eccentricity in the end connections. In order to secure the maximum efficiency of the material employed, the form chosen should permit the direct connection at the joints of as large a proportion of the area as possible. Thus, in the case of single—or double—angle tension members, connected by one leg only, unequal-legged angles should be employed and the longer legs connected to the gusset plate. Even though both legs are connected to the gusset, it is preferable to use unequal-legged angles with the longer leg in contact with the gusset, for the

[1] See *Transactions Canadian Society of Civil Engineers*, Vol. 26, 1912, p. 249.

reason that the avoidance of dependence upon rivets where possible requires that the amount of stress delivered through attached details—such as a lug and its two sets of rivets—should be reduced to a minimum. There is, too, the further advantage that the use of a relatively narrow outstanding leg effects a saving in the material of the lugs.

For members of structures subject to vibration, rigid or stiff forms should be used. Open sections are in all cases preferable to closed ones. Limiting sections and thicknesses prescribed by the governing specifications must be observed.

65. Net Section.—The net section of a tension member at any right section is the gross area of the member less all rivet holes, pin holes or cuts, or fractions thereof, that diminish the resistance of the member at that section. It is determined in accordance with the form of the member. As explained in Art. **68**, when the stress is assumed as uniformly distributed, sometimes only a fraction of the net section is effective. In such cases it is this effective portion that must be considered in designing.

For rods, the net section is the section at the root of the thread, unless the rod is upset, when it is the section in the body of the rod.

In the case of eye bars the net section is the net area of the body, since considerable excess area is always provided in the eye. For adjustable eye bars it is also the area in the body.

For riveted or built tension members the net section depends on the types of body and end details adopted, and the arrangement and size of the rivets. Details involving the minimum practicable number of rivet holes on, or within certain critical distances of, any right section through the body of the members are desirable since they give the maximum net section. Other arrangements should be avoided if possible. Obviously the larger the rivet, the greater the loss of section. It is almost universal practice in calculating net areas to consider the diameter of the rivet hole as ⅛ in. greater than the diameter of the rivet before driving.

The number of rivet holes that must be deducted from the gross area of a right section, to give the net section, depends upon the number of gage lines and the stagger of the rivets. Obviously, the net area adopted should be such that nowhere on any diagonal or zig-zag section should the maximum stress due to the combination of normal and shearing stresses exceed that on a right section. This result will be attained if the number of rivet holes N deducted from the gross right section be taken as the maximum number given on any zig-zag section by the formula

$$N = 1 + x_1 + x_2 + x_3 + \ . \ . \ . \ . \tag{1}$$

where x = a fraction of a rivet hole

$$= \frac{g}{h} - \frac{2\,(g^2 + s^2 - h\sqrt{g^2 + s^2}}{h(g + \sqrt{g^2 + 4s^2})}$$

g = the gage, i.e. the distance between any two holes measured at right angles to the axis of the member.

s = stagger of these holes.

h = diameter of rivet hole as considered for deduction purposes = diameter of rivet + ⅛ in.

This formula is to be applied to alternative sections, the successive terms representing the deductions for successive holes considered in a chain across the member. The particular group of rivets to be considered is that which will give the greatest total deduction, whether the rivets lie on adjacent gage lines or not.

To obviate the large amount of work required in solving Formula (1), the diagrams of Fig. 55 have been prepared. These give the theoretically correct deductions for any assured ratio of stagger to gage. The curves have been drawn for 5/8-, 3/4- and 7/8-in. rivets, for staggers up to 9 in., and for gages up to 15 in.

66. Proportioning of Rod Members.—If a tension member consisting of one or more rods or bars of uniform section is threaded for end connections or for insertion of a turnbuckle or sleeve nut, its strength will depend on the net area at the root of the thread. If, however, the rod is upset before threading, the rods may be weaker through the body than at the root of the thread. Good practice requires that this be the case and hence the standard upsets for both round and square rods tabulated in the steel handbooks provide for an area at the root of thread usually between 20 and 30 per cent greater than through the body of the bar. If, then, standard upsets are to be employed, the designer need concern himself only as to the area to be provided in the body of the member.

It is not always economical to use upset ends. If the rods are short and of small area, the actual saving of material effected by upsetting may be more than offset by the labor cost of making the upsets. For example, tie rods for floors and sag rods for roofs are not upset.

Loops at ends are so proportioned as not to constitute a source of weakness to the member. If standard loops are to be provided, no attention need be given to the looped ends in design.

For adjustable rod members, the stress may, in most cases, be assumed as uniformly distributed. The different rods, if there are more than one, may be given equal tension by adjustment, and care should be taken to see that in service they are equally stressed. Only regular inspection can ensure this. If the rod is not at right angles to the pin to which it connects, the load will be applied at one edge and allowance would then need to be made for flexural stress in addition to the tensile stress.

Illustrative Problem.—A round rod tension member of soft steel with ends threaded but not upset, is to carry safely a load of 22,000 lb. If the permissible stress in tension is 15,000 lb. per sq. in., determine the necessary size of rod.

Required net area = 22,000/15,000 = 1.47 sq. in.

From Carnegie or Cambria, it is found that one 1 5/8-in. round rod with area of 1.52 sq. in. at root of thread, or two 1 1/4-in. round rods with combined area of 1.78 sq. in. at root of thread would do. One rod would be cheaper, if practicable for the situation.

Illustrative Problem.—A tension member is to consist of one or two round or square soft steel rods upset and threaded at the ends. Determine the required size. Load and permissible stress as in last problem.

Required net area = 1.47 sq. in.

Area of one 1 3/8-in. round rod upset = 1.49 sq. in. in body of bar and 1.74 sq. in. at root of thread.

Area of one 1 1/4-in. square rod upset = 1.56 sq. in. in body of bar, or 2.05 sq. in. at root of thread.

The round rod is the more economical.

Two 1-in. round rods, or two 7/8-in. square rods, would also be sufficient.

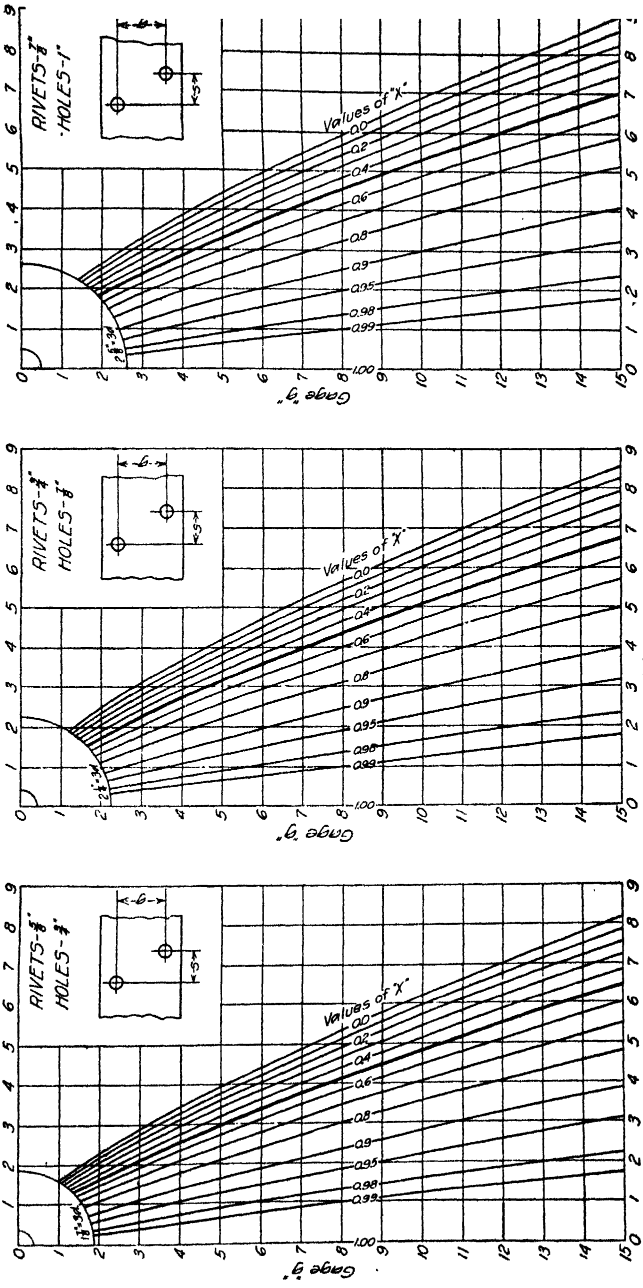


FIG. 55.—Theoretically-correct deductions of rivet holes for tension members.

67. Proportioning of Eye Bar Members.—For pin connected structures, the eye bar, Fig. 54e affords a very convenient unit from which to build up a tension member. The heads are now so well standardized that the designer need have no fear of the bar failing therein. Examination of the tables of proportions of heads in Carnegie or Cambria shows that the area through the eye is ordinarily from 35 to 40 per cent in excess of that through the body of the bar. The A.R.E.A. specifications require the excess to be 37½ per cent.

Experience has shown that there should be a certain minimum thickness for each width of bar, for reasons of manufacture and since very thin bars tend to fail by buckling in the head. This varies from ½ in. for bars, 2 in. wide, to 1¾ in. for bars 16 in. wide. On the other hand, the thinner bars simplify packing at joints and reduce pin moments. It is commonly specified that the thickness of the bars shall not be less than ⅛ the width.

The size of head to be employed will, of course, depend on the necessary size of pin. Ordinarily, it is about 2¼ times the width of the bar. In selecting a bar, care should be taken to ensure that the head is not too large to fit into any built-up member, such as a top chord, to which it may be required to connect. Bars should be selected with the size of pin in mind. It is now often required that the pin be not over ⅞ the width of the widest bar attached.

Ample basis for fixing safe working stresses on eye bars exists in the many full-sized tests that have been made. Their ultimate strength is on the average less than for small specimens of the same material, usually about 85 to 90 per cent as great, due to the less perfect working received by the thick metal of the bar and to the annealing of the heads after forging. However, since the eye bar member is subjected to low secondary stress in the structure, and also since the probable percentage loss by corrosion is small and the resilience is large, the permissible working stress may be quite as great as for riveted members.

In building up eye bar tension members, the constituent bars should be packed symmetrically about the plane of the truss with the inclination of any bar thereto as small as possible and in no case greater than $\frac{1}{16}$ in. per foot. By keeping the inclination or "cradling" down, the flexural stresses are thereby minimized. Bars should be secured against lateral shifting and so arranged that adjacent bars in the same panel will not be in contact with each other lest there be corrosion between them.

Illustrative Problem.— A tension member of a truss carrying a load of 285,000 lb. is to consist of two or four medium steel, non-adjustable eye bars connecting at each end to a pin estimated to be 4½ in. diameter. The thickness of the bars is not to be less than one-eighth the width, nor less than 1 in., and the width of the bars not over eight-sevenths of the diameter of the pins. Permissible tensile stress, $p = 16{,}000$ lb. per sq. in. Heads of American Bridge Co. standard.

Required area in body of bars = 285,000/16,000 = 17.82 sq. in.

Maximum permissible width of bars = $(4.5)(\frac{8}{7}) = 5.1$, say 5 in. Minimum thickness for 5-in. bars = 1 in.

Two bars, $5 \times 1\frac{13}{16}$ in. with a total area of 18.13 sq. in. would be sufficient. If four bars be used, they must each be 5×1 in. because of the rule respecting minimum thickness. This would give a considerable excess of area, although the use of the thinner bars is desirable for other reasons.

68. Proportioning of Riveted Tension Members.—Design of the simplest form of riveted tension member, the narrow plate, or flat, is vitally related to the

arrangement of rivets in the end connections. To develop the highest possible efficiency the rivets should be arranged in a triangular group, as shown in Fig. 54g, with the apex of the triangle formed by a single rivet, pointing towards the center of the member. With such a connection it is necessary to reduce the gross section by only one hole in proportioning for the full calculated stress, since at the second line of rivet holes the stress in the member is less than the full stress in the body by the amount of the stress taken out by the first rivet.

Practical objections to the arrangement of connections as shown in Fig. 54g usually result in a less efficient arrangement. The proper deduction to be made, however, can easily be determined by following the methods of Art. 65.

Although a flat as a tension member is deficient in lateral rigidity and may for that reason contribute to the vibration of a structure, the rigidity of the end connections as compared with those of the pin type is an advantageous feature.

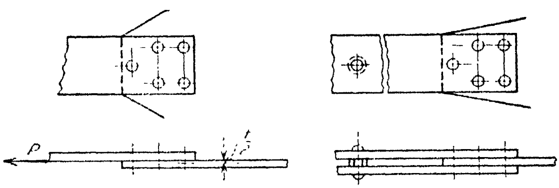


FIG. 56.—Effect of end connections on efficiency of flat plate tension members.

One inherent defect of a single flat as a tension member is that by reason of the end connections being made to a gusset in the form of a lap joint, as shown in Fig. 56a there is an eccentricity of one-half the thickness of the plate, assuming, as appears justifiable in the light of experiment, that the load is applied at the plane of contact of the member with the gusset. If there be two flats side by side forming a single-member, and possibly stitch riveted together, with the end connections in the form of a butt joint, Fig. 56b, the effect of eccentricity is not wholly overcome, for each component part of such a tension member tends to bend in its own way, due to the eccentric application of the part of the applied load that goes to it. The bending is somewhat restrained, however, by the body details.

With the single angle, or more complicated forms of tension members built up of angles or other shapes and plates, it is desirable to have as large a portion as possible of the cross-section directly connected to the end gussets. By so doing, the length of the end connections and size of gusset plates is thereby reduced and often a more equable distribution of stress over the cross-section is brought about, thus improving the efficiency of the member. If possible, single angles should have unequal legs and, if only one leg is connected, it should be the longer one.

In most riveted tension members there is an unavoidable eccentric application of the load, by reason of the fact that the component parts lack perfect symmetry in themselves or are connected to the gussets in the manner of a lap joint. Typical cases of this kind are shown in Fig. 57.

The load is applied to a single angle tension member, Fig. 57a, along the line of connecting rivets, and slightly inside the gusset. If G is the center of

gravity of the angle, there is consequently an eccentricity $KG = e$. The true maximum stress resulting from this combination of direct and bending stress can only be calculated by employing the theory of unsymmetrical bending, as explained in Art. 22.

The main component parts of the members shown in Fig. 57*b* and *c* will receive their loads eccentrically. Each one should, therefore, properly be designed for a combination of direct and bending stress, unsymmetrical bending being considered for the case of Fig. 57*b*. The presence of connecting stitch rivets, tie plates, or battens does not entirely prevent this bending, but, according to Professor C. Batho[1] each component part tends to bend in its own way.

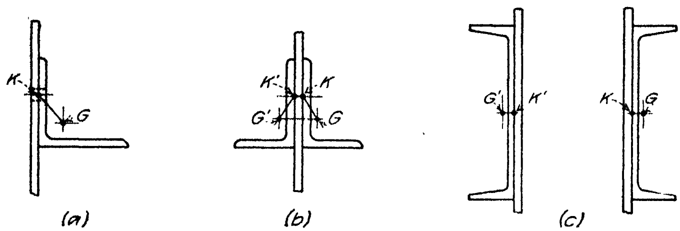


FIG. 57.—Eccentric application of load on tension members.

The relation of the maximum to the mean stress is dependent on the amount of initial eccentricity and the restraining effect of the end connections. By utilizing heavy, wide gusset plates, the deflection of the member due to eccentricity is lessened, which in turn tends to equalization of stress over the cross-section. While the effect of the restraint in a direction normal to the plate is small, it is important in the plane of the plate. The average decrease of the ratio of maximum to mean stress for single angles due to the stiffness of the end plate in its own plane, was found by Prof. Batho to be about 35 per cent at the highest load applied. It, therefore, appears most desirable, in order to improve the efficiency of members connected unsymmetrically, to fix the direction of the ends as far as possible.

The effort to equalize the stress in an angle member by connecting it by both legs is shown by the tests of Prof. Batho to bring comparatively little advantage. In the most favorable cases it decreased the ratio of maximum to mean stress at working loads by about 4 per cent. The earlier tests by Prof. F. P. McKibben[2] showed an improvement in efficiency of under 15 per cent in the most favorable case and in most cases under 10 per cent. For single angles connected by one leg, the efficiency ranged from 75 to 83 per cent, while for single angles connected by both legs, it ranged from 86 to 96 per cent. The practice of permitting only the connected leg to be counted for angles connected by only one leg, while allowing both legs to be counted if lug angles are used, is thus seen to be unduly favorable to the use of lug angles.

Experimental investigation shows that a considerable change may be made in the position of the line of pull of the gussets with respect to the gravity line of the angles connected, without greatly affecting the stress distribution over the

[1] *Transactions Canadian Society of Civil Engineers*, vol. 26, Part I, 1912.
[2] *Engineering News*, July 5, 1906, and Aug. 22, 1907.

angles or the efficiency of the member. From Prof. McKibben's tests, it appears that changing the line of pull from the gage line of a single angle connected by only one leg to the projection of the gravity line, improved the efficiency only 5.5 per cent in the most favorable series of tests. In another series it was improved by only 2 per cent. No great sacrifice in efficiency is thus brought about by placing the gage line of an angle member on the skeleton line of a truss.

Double angle members, such as shown in Fig. 57*b*, have the merit of somewhat hindering each other from bending perpendicularly to the gusset plate, with the result that the ratio of maximum to minimum stress over the cross-section is thereby reduced. In Prof. McKibben's tests, no particular advantage appeared to attach to this form of member, but Prof. Batho found them to give better results than single angle members.

In the practical design of tension members composed of either single or double angles, it is desirable to proportion on the assumption that the stress is uniformly distributed over the cross-section. Allowance can be made for the loss of efficiency arising from eccentricity by reducing the net area by a percentage to give the true effective area. This percentage will vary with the amount of eccentricity and the ratio of the legs of the angle, and will depend on whether lug angles are used or not.

Although the tests cited indicate that only from 75 to 83 per cent of the net area of representative angles connected by one leg is effective, the adopted efficiency in design should be *higher* than this. Built tension members with an average efficiency of 87 per cent are regarded as 100 per cent effective, and consequently for consistency, single and double angles should be considered as having an efficiency about 15 per cent higher than that shown by actual test. To be on the safe side, however, it would seem desirable to limit this excess to 10 per cent. The percentage efficiency of single and double angles connected by one leg only has been found to be expressed fairly well by the formula

$$p = 100 - 30 \cdot \frac{sg}{c^2} \tag{1}$$

where p = percentage of net area effective.

s = length of outstanding leg of angle in inches.

g = gage in inches of angle if one gage line only is used, or two-thirds of the sum of the gages if two gage lines are used.

c = length of connected leg in inches.

Efficiencies of angles connected by both legs may with safety be taken as 5 per cent higher than those given for angles connected by only one leg.

These effective percentages are to be applied to the net sectional area as determined by the principles of Art. **65**. The reduced net area then becomes the true effective area.

Illustrative Problem.—Let it be required to find the effective percentage of the net section of a $3\frac{1}{2} \times 2\frac{1}{2}$-in. angle connected by the $3\frac{1}{2}$-in. leg, with rivets on a 2-in. gage.

$$p = 100 - 30sg/c^2 = 100 - (30)(2.5)(2)/(3.5)^2 = 87.8 \text{ per cent.}$$

Find the effective percentage of the net sectional area of a 6×4-in. angle, connected by the 6-in. leg, with two lines of rivets driven on gages of $2\frac{1}{4}$ and $2\frac{1}{4}$ in.

$$p = 100 - (30)(4)(0.67)(2.25 + 2.25)/(6)^2 = 90 \text{ per cent.}$$

Although no records of tests of single channels in tension are available, it is probable that single channels connected by their webs only, would, because of the relatively high ratio of eccentricity to corresponding section modulus as compared with angles of the same area, show an efficiency somewhat less than that of single or double angles. The relative areas of the flanges and the web, or the weight of a channel for a given depth, should materially affect the efficiency. It is probable that for single channels the effective area is about equal to the net area of the web plus 70 per cent of the net area of the flanges.

Due to eccentricities and imperfections of material and workmanship, built up tension members will not develop under test the same strength in pounds per square inch of net area as would be given by a small specimen. Tests reported by J. E. Greiner[1] on (1) members of H-shape made up of four angles connected by latticing, battens, and solid web plates, and (2) on members composed of two built up channels connected by latticing and battens, showed that the efficiency ranged from 80.4 to 96 per cent. Some of the lower figures were for specimens with a highly eccentric pin-plate connection to the outstanding legs of the angles of the member. The lowest were, strangely enough, for box-shaped members with both legs of the four main angles connected to the end pin plates. This lack of strength was probably due to the fact that the pin plates did not extend to the near ends of the end batten plates.

In fixing working stresses, the efficiency likely to be attained should be borne in mind. In view of the fact that for tests on eye bars, the average efficiency is over 90 per cent, the working stresses on built up tension members may safely be as great as on eye bars.

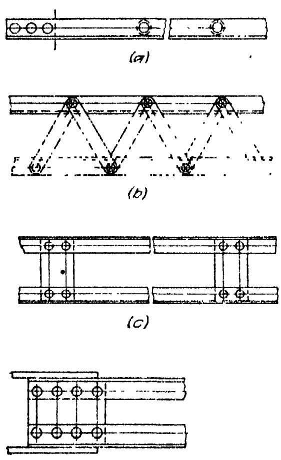


FIG. 58.—Body details of steel tension members.

Body Details.—When a tension member is built up of an assemblage of rolled sections, it is desirable to connect them together at certain intervals depending on the character of the member.

If angles or channels are used without web plates, the connection may be in the form of stitch rivets, Fig. 58*a*, or latticing, Fig. 58*b*, or battens, Fig. 58*c*, or *d*. Stitch rivets are used where the two connected parts are sufficiently close together to make it practicable to insert washers between them through which the rivets may be driven. Latticing, formerly much used for the larger tension members, is now chiefly employed for compression members, or for members subject to reversal of stress. Battens are found to be cheaper and practically as satisfactory as latticing for tension members. If web plates are used, the connection of angles or channels thereto is made by lines of rivets spaced as described below.

Body details such as shown in Fig. 58 serve several purposes. They lessen the transverse vibration by so connecting the parts that they will act practically as one unit with a width or depth equal to the overall lateral dimension. This

[1] *Transactions American Society of Civil Engineers*, vol. 38, 1897.

reduces general vibration in the structure and minimizes the likelihood of loosening of the rivets in the end connections. Web or batten plates near the end of the member, Fig. 58*d*, serve to lessen the effect of the eccentric application of load by restraining the end of the member against bending. Solid webs or closely spaced battens or latticing help to equalize the stress over the cross-section by transferring stress from a heavily loaded to a lightly loaded part.

Where stitch rivets are used, they are spaced from 2 to 4 ft. apart in angle members and, where they connect shapes to web plates or various plates together, they are spaced in the line of stress a distance not over 16 times the thickness of the thinnest outside metal, nor over 6 in.

Latticing, if used, is designed and arranged in the same manner as for compression members.

Battens are spaced center to center about 3 or 4 ft. apart.

The thickness of battens for tension members may be determined by the rule applied to single lattice bars—that is, not less than $\frac{1}{40}$ of the distance between the rivet lines. In no case should it be less than the minimum prescribed by the specification for secondary material. For light members the length of the batten plates parallel to the axis of the member need not be greater than is required to accommodate two rivets in each line. At the ends of members, in order to contribute to the restraint of the individual parts of the member and lessen bending due to eccentricity, the battens should be as long as the member is wide and be placed inside the end gusset plates.

Illustrative Problem.—In the following problems, let the permissible stress in tension be 16,000 lb. per sq. in. of effective area, and the rivets $\frac{7}{8}$ in. dia. with holes 1 in. dia.

(*a*) Find the size of a double flat plate tension member to carry 47,000 lb., if a line of stitch rivets runs along its axis and if provision must be made for two rivets opposite each other at the inner edge of the end connections.

Required net area $= 47{,}000/16{,}000 = 2.94$ sq. in.

Net section will be at the end connection, and if w = width of plates and t the thickness of each plate, it $= (w - 2)2t$.

Hence $(w - 2)2t = 2.94$ sq. in., from which assuming $w = 6$ in., t required $= 0.368$ in. Two $6 \times \frac{3}{8}$-in. plates will be used.

(*b*) A single angle with one leg only connected is to carry a load of 28,000 lb. Find the required size.

Effective area required $= 28{,}000/16{,}000 = 1.75$ sq. in.

Assume a $3\frac{1}{2} \times 3 \times \frac{3}{8}$-in. angle connected by the $3\frac{1}{2}$-in. leg on a 2-in. gage.

Net section $= 2.30 - (1)(0.38) = 1.92$ sq. in.

Effective percentage of net section from Formula (1) is

$$p = 100 - (30)\frac{(3)(2)}{(3.5)^2} = 85.4 \text{ per cent}$$

and effective area of angle $= (0.854)(1.92) = 1.64$ sq. in.

The angle is not large enough. A $3\frac{1}{2} \times 3 \times \frac{7}{16}$-in angle gives an effective area of 1.89 sq. in., and hence would be satisfactory.

(c) Find the required size of angle for problem (*b*) if both legs were connected and the stagger of the inner rivets were 2 in.

Assuming a $3\frac{1}{2} \times 3$-in. angle with gages of 2 and $1\frac{3}{4}$ in., respectively, in the $3\frac{1}{2}$- and 3-in. legs, the distances of the rivet lines apart, or g, assuming a thickness of $\frac{7}{16}$ in., is 3.31 in. The deduction, from Fig. 55, is $1 + 0.7 = 1.7$ holes, and the net area $= 2.65 - (1.7)(1)(0.44) = 1.90$ sq. in.

Assuming the efficiency as 5 per cent greater than for an angle connected by one leg, or say 90 per cent, the effective area $= (0.90)(1.90) = 1.71$ sq. in. This is slightly below the requirement but would in most cases be accepted.

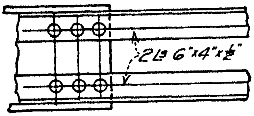
(*d*) A truss member carrying 235,000 lb. is to be of H-shape, consisting of two pairs of angles 12½ n. back to back with connecting battens, as shown in Fig. 59. Determine the size, assuming the rivet arrangement as shown. Assume the full net area as effective, because of the restraining effect of the end battens.

Required effective area of member = 235,000/16,000 = 14.7 sq. in.

Assume four 6 × 4 × ½-in. angles. Considering one angle developed as in Fig. 59*a*, it is found by consulting Fig. 55 that the least net section is *S-S*, the deduction from each angle being 2 holes.

Net section = (4)(4.75) − (8)(1)(½) = 15.0 sq. in. which is adequate.

(*e*) A truss member carrying 370,000 lb. is to be made up of two channels, 12 in. deep, with flanges turned out, reinforced by two 11-in. plates on the backs of the channels, as shown in Fig. 60. Battens connect the flanges of the channels, the end batten being *outside* the gussets. Determine the necessary section if only 70 per cent of the net area of the channel flanges is considered effective.



Required effective area = 370,000/16,000 = 23.2 sq. in.

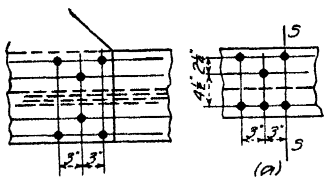


FIG. 59.—Design of four-angle tension member.

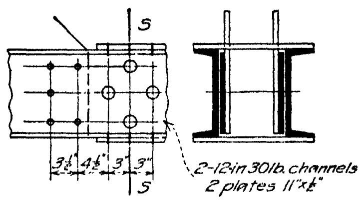


FIG. 60.—Design of a double channel tension member.

Assume two 12-in., 30-lb. channels and two 11 × ½-in. plates with the riveting at the inner edge of the connection as shown.

Critical section in *S* − *S*, cutting four web holes and four flange holes.

Gross area of section = (2)(8.79) + (2)(11)(0.5) = 28.58 sq. in.

Area of 4 holes through webs of channels and plates (section *S-S*) = (4)(1)(0.51 + 0.50) = 4.04 sq. in.

Gross area of flanges of channels = total area − area of webs = (2)(8.79) − (2)(12)(0.51) = 5.34 sq. in.

Net area of flanges, the grip being ½ in., = 5.34 − (4)(1)(0.5) = 3.34 sq. in.

Reduction of flange area = (0.30)(3.34) = 1.00 sq. in.

Effective area of member = 28.58 − (4.04 + 1.00) = 23.54 sq. in. which is adequate.

69. Tension Members Subject to Cross Bending.—It frequently happens that tension members are subjected to transverse as well as to axial loading. This produces a combination of direct and bending stress for which the member must be designed. The problem is essentially the same as for a tension member subjected to eccentric axial loading and consequently a moment of eccentricity.

Common cases of cross bending in bending members are a tension member subjected to its own weight or to a directly applied load. The bottom chords of roof trusses in mill buildings are frequently loaded with trolleys, piping, wires, etc. Another case is a tension chord subjected to the thrust of a cross strut of a lateral system.

If the effect of the deflection in augmenting the moment is neglected, the maximum fiber stress is

$$f_1 + f_2 = \frac{P}{A} + \frac{Mc}{I} \tag{1}$$

If the deflection be considered, assuming the ends free to turn, the maximum stress becomes

$$f_1 + f_2 = \frac{P}{A} + \frac{Mc}{I - \frac{Pl^2}{10E}} \tag{2}$$

where f_1 = uniformly distributed stress due to axial load P.
f_2 = flexural stress on extreme fiber.
P = axial load.
A = area of member.
M = bending moment.
c = distance from neutral axis to most highly stressed fiber.
l = length of member.
I = moment of inertia in direction of bending.
E = modulus of elasticity.

Illustrative Problem.—The bottom chord of a truss is 15 ft. long between panel points l consists of two 4 × 4 × $\frac{5}{16}$-in. angles with vertical legs back to back and separated by a space of $\frac{3}{8}$ in. Washers and stitch rivets are inserted 2 ft. apart. The axial load is 40,000 lb. If a wind strut carries a load of 3,000 lb. into the chord at right angles to it and $1\frac{1}{2}$ ft. from one end, as shown in Fig. 61, find the maximum resulting stress, neglecting the effect of the deflection. Rivets, $\frac{3}{4}$-in. dia.

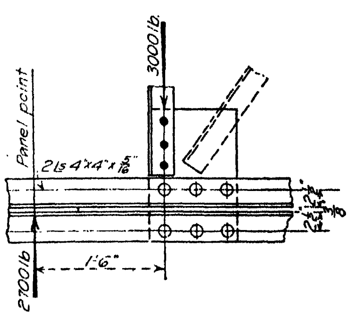


Fig. 61.—Tension member subjected to bending.

As the angles are fairly closely stitch riveted, it will be assumed that they act as a single section against transverse loading.

Net area of member = (2)(2.40) − (2)(0.875)(0.3125) = 4.25 sq. in.

Uniformly distributed stress

$$f_1 = \frac{40,000}{4.25} = 9,400 \text{ lb. per sq. in.}$$

Wind reaction at end of chord nearest the strut connection = (3,000)(13.5)/15 = 2,700 lb.

Wind moment at strut connection = (2,700)(1.5)(12) = 48,600 in.-lb.

Gross moment of inertia of chord about a vertical axis,

$$I_g = 2\,[3.7 + (2.40)(1.31)^2] = 15.6$$

Moment of inertia of 2 holes in outstanding legs assuming a $2\frac{1}{2}$-in. gage, approximately, = (2)(0.875)(0.3125)(2.69)² = 3.9.

Net moment of inertia = 15.6 − 3.9 = 11.7.

Extreme fiber stress due to wind moment

$$f_2 = \frac{(48,600)(4.19)}{11.7} = 17,400 \text{ lb. per sq. in.}$$

Total extreme fiber stress, $f_1 + f_2$ = 9,400 + 17,400 = 26,800 lb. per sq. in., an excessive stress even considering the usual increase permitted for a combination of dead load, live load and wind stresses.

CAST-IRON COLUMNS

By H. S. Rogers

70. Use of Cast-iron Columns.—Cast-iron columns are suitable only for small buildings of non-fireproof construction. They offer somewhat greater resistance to fire than unprotected steel columns and occupy a minimum of space

in the building, but cast iron is by no means as reliable as steel and the bolted connections of cast-iron columns allow more or less lateral movement which is serious in high buildings.

Columns of this material should not be used with fabricated steel in skeleton construction or under conditions which produce flexural stresses of any magnitude, other than those due to concentrically-loaded column action. The unreliability of cast-iron columns is due to the variation in quality of the material, defects likely to occur in casting, and the difficulty of thorough inspection.

71. Properties of Cast-iron.—Cast-iron has a very high unit compressive strength—usually considered to be about 80,000 lb. per sq. in. This material, however, is not strong in shear or tension, the average ultimate shearing stress being 18,000 lb. per sq. in., and the average ultimate tensile stress 15,000 lb. per sq. in. The ultimate intensity of stress which can be developed in a piece of cast-iron varies with its fineness of grain, and depends largely upon its thickness and the rate of cooling, as well as its composition. The high compressive stresses make it a very desirable material to use in compression, but because of the somewhat treacherous nature of cast-iron, the high compressive stresses found are often misleading. Also, the low shearing and tensile values preclude its use under any condition other than that of direct compression. It does not rust as quickly as steel and resists fire somewhat better, but may, however, be subjected to serious strains because of sudden cooling with water from a fire stream. It is very hard and brittle, and fractures suddenly without warning. No riveted connections should be made to cast-iron. All connections of girders to columns, or column to column, must therefore be made by bolts which impair the rigidity of a structure by the allowance for clearance.

72. Manufacture of Cast-iron Columns.—Cast-iron columns may be cast in sand molds either upon the side or on end. In either case a baked core molded to the dimensions of the inside of the column must be made of sand, flour, and water, and supported within the sand mold. There are practical conditions surrounding every part of the work which will determine the quality of the column produced. Many pronounced defects found in columns are due to the method of pouring used in their manufacture.

If the column is cast on its side, the core will be buoyed up within the mold because of the great difference in density between it and the molten metal. Provision must, therefore, be made to prevent the core from rising toward the top side of the mold, or from being sprung from line so that the mid-portion of the top side of the casting will be thinner than the desired thickness. This defect produced by "floating cores" is one which is frequently found in cast-iron columns. The molten metal rising in the mold carries dirt and air above, in which will form "honeycomb" and "blowholes" along the top side of the column, unless provision is made by vents for the escape of the air. This provision can be made by forcing a wire rod through the mold at intervals. When these difficulties have been overcome, there are still others which may arise due to unequal cooling produced by the manner or speed of pouring, by the condition of part of the mold, or by the unequal radiation in the molds. The last may be due to an unequal uncovering of the mold. Unequal cooling may produce stresses which will crack the column before any load is placed upon it.

The end method of casting avoids some of these difficulties if the molten metal is introduced at the bottom of the mold. The dirt, sand, and air that collect will thus be borne to the top of the mold so that they can be removed, but the pressure produced by the head of molten metal will often be greater than the mold can withstand, if the column is of any considerable length. The defects found in columns cast on end will not, however, be so numerous as those found in columns cast on the side. These defects can be eliminated to some extent by careful foundry work. If not eliminated, they should be caught at the time of inspection.

73. Inspection of Cast-iron Columns.—Cast-iron columns may have defects either in the surface, or within the metal, or may have insufficient strength due to variation in the section of the metal due to displacement of the core. Defects in the surface can be found by a careful examination of the column. Defects within the metal can be discovered by a careful tapping of the column with a hammer, as the honeycomb or sand spots will sound dead. In hollow square or round columns, variation in thickness of the metal can be determined by drilling two or three ¼-in. holes through the column. If this variation is more than ¼-in., the column should be rejected. The H-section affords easy access to the surface for inspection and painting, and opportunity to measure the section. Columns with brackets should be carefully inspected at these details, especially if the column has been poured on its side through the bracket.

74. Tests of Cast-iron Columns.—The Department of Buildings of New York City made a series of tests upon cast-iron columns some years ago at the works of the Phoenix Bridge Co. Nine columns were tested to destruction and a tenth to the capacity of the testing machine. Six of the ten columns had a diameter of 15 in., a length of 15 ft. 10 in., and a thickness of shell of 1 in.; two had a diameter of 8 in., a ratio of L/d equal to 20, and a shell thickness of 1 in.; two had a diameter of 6 in., a ratio of L/d equal to 20, and a shell thickness of 1 in.

The columns broke at loads varying from 22,700 lb. per sq. in. to over 40,400 lb. per sq. in., the latter being the intensity of stress in one of the 15-in. columns which withstood the total capacity of the machine. The other five 15-in. columns all exhibited foundry dirt, honeycomb, cinderpockets, or blowholes.

75. Cast-iron Column Formulas.—The following are some of the formulas for cast-iron columns used by different authorities:

New York Building Law . . . . $P/A = 9{,}000 - 40l/r$

Cambria Steel Handbook (Round Columns) . . . . . $$P/A = \frac{10{,}000}{1 + \dfrac{l^2}{800d^2}}$$

Cambria Steel Handbook (Rectangular Columns) . . . $$P/A = \frac{10{,}000}{1 + \dfrac{l^2}{1{,}067w^2}}$$

Chicago Building Law . . . . $P/A = 10{,}000 - 60l/r$

Boston Building Law . . . . $P/A = 11{,}300 - 30l/r$

Watertown Arsenal Tests . . . . $P/A = 34{,}000 - 88l/r$

In these formulas r is the radius of gyration, d is the outside diameter, and w is the least lateral dimension. The formulas are all for flat ends, and all but one are for working loads. The Watertown Arsenal Tests formula, which is for ultimate loads, is quoted by J. B. Johnson, who says it fits very well the results obtained on certain tests made on full sized cast-iron columns.

In order to compare the above column formulas, the allowable unit load has been calculated for a column 15 ft. long, outside diameter 10 in., and inside diameter 8 in. These dimensions give a radius of gyration of 3.2 and a slenderness ratio of 56.2. The results are as follows:

Formula	P/A
New York Building Law	6,750
Cambria Steel Handbook	9,620
Chicago Building Law	6,630
Boston Building Law	9,615
Watertown Arsenal Tests	7,250

A factor of safety of 4 was used with the Watertown Arsenal Tests formula.

The results indicate that the Cambria Steel Handbook and the Boston Building Law formulas give results which are probably somewhat too high. Any one of the other three formulas would be a safer one to use in design.

76. Design of Cast-iron Columns.[1]—The sections of cast-iron columns in general use are shown in Fig. 62. The hollow cylindrical section gives the best distribution of metal in a column, but the connection details do not work as nicely as those for the hollow square section, which is almost as efficient in distribution of material. The hollow square section, on the other hand, has disadvantages which are not found in the hollow cylindrical section. The corners of the square section are very liable to crack, due to the cooling of the column; but this can be obviated by an outside curved corner and an inside fillet. The H-section, though not affording a distribution of material as efficient as the hollow cylindrical or hollow square column, has the advantages of being open to inspection, of being cast without a core, and of being easily built into a brick wall. It meets the greatest favor as a wall column.

Fig. 62.—Cast-iron column sections.

In all the formulas given in the preceding article it will be noted that the area, A, and the radius of gyration, r, both appear in the formula. Therefore, for the ordinary column, in a design problem, the designer is confronted with two unknowns, neither of which can usually be expressed in terms of the other. For this reason it is necessary to design columns by trial and error methods. The procedure involves choosing a size of column which is assumed to be satisfactory, and then calculating the load which it can carry. If the column is too small or too large, then the dimensions must be increased or decreased, respectively, and a second trial calculation must be made.

The following specifications should be observed in the design of the shafts of cast-iron columns:

The minimum thickness of the shell should not be less than ¾ in.; the maximum thickness should not be greater than 1¾ to 2⅛ in.

The maximum diameter should not be greater than 16 in.; the minimum diameter should not be less than 5 or 6 in.

The slenderness ratio, L/r, should not exceed 70; the unsupported length of the column should not exceed 20 times the least diameter.

All corners should be filleted with a radius of ¼ to ⅜ in.

[1] By J. B. Kommers.

No inside offset nor any sudden change in the thickness of shaft should be made.

Illustrative Problem.—A hollow, round cast-iron column is 16 ft. long, flat ended, and is to carry safely a load of 200,000 lb. The New York Building Law formula is to be used.

One rough method of making a first guess as to the size required is to assume that the column is a short compression member with l/r equal to zero. According to the formula this would mean that the unit load could be 9,000 lb. per sq. in. The area required on this basis is $\frac{200,000}{9,000} = 22$ sq. in. This is known to be too small, so that an area of 30 sq. in. will be chosen for a first trial. Assuming an outside diameter of 10 in., $\frac{\pi}{4}(10^2 - d_2^2) = 30$, $d_2 = 7.85$ in., and an inside diameter of 8 in. will be used. For the diameters of 10 and 8, the radius of gyration is 3.2 in. and the area is 28.3 sq. in. From the formula the load

$$P = 28.3\left[9,000 - \frac{(40)(16)(12)}{3.2}\right] = 187,000 \text{ lb.}$$

which is a little too small. If the thickness of the column is increased to $1\frac{1}{8}$ in., the inside diameter will be 7.75 in., the radius of gyration will be 3.16 in. and the area will be 31.4 sq. in. Substituting in the formula,

$$P = 31.4\left[9,000 - \frac{(40)(16)(12)}{3.16}\right] = 206,000 \text{ lb.}$$

which is satisfactory.

The handbooks published by the steel companies contain tables of safe loads for various sized columns, so that the work of computation may be considerably reduced by using these tables.

77. Column Caps and Bases.—Hollow cylindrical and square cast-iron columns are generally fastened together by a simple flanged base and cap as shown in Fig. 63*a* and 63*b*. The flanges should not be thinner than the shaft of the column and should be at least 3 in. wide; which width will be sufficient for hexagonal nuts on ¾-in. bolts. These flanges should be faced at right angles to the axis of the column. The bolt holes in the flanges should be drilled to a templet so that the columns can be fitted together in proper alignment and the flanges should be spot-faced at bolt holes so that they will give a square firm bearing to bolts and nuts. If the ends of cast-iron columns must be left rough, sheets of lead or copper should be placed between flanges of columns bolted together, so that an even bearing will be obtained by the soft metal taking up the inequalities of the surface. In no case should shims be used to wedge up one side of a column.

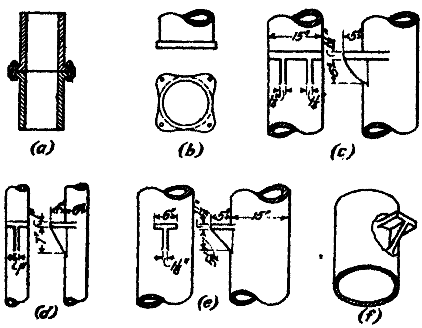


FIG. 63.—Cast-iron column details.

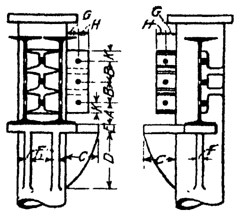
If it is desired to give any architectural pretentions to the caps or bases of cast-iron columns, the design of such should be made so as not to weaken the shaft section of the column by change of dimensions or offsets that will throw transverse stresses into the column. Ornamental caps or bases of large size should be cast separate from the column.

78. Bracket Connections.—The usual forms for the connections of beams and girders of cast-iron columns are shown in Fig. 63*c*, *d*, and *e* and in the table

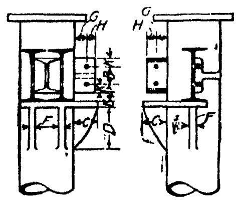
of "Manufacturers' Standard Cast-iron Column Connections." The beam rests upon the bracket shelf and is bolted to the lug on the column through the web. The holes in the web of the beam for bolting to the lugs should be drilled in the field in order to match the cored holes of the lug.

Connections should be designed with a bracket directly below the web of a single girder or below each web of a box girder so that no transverse bending strains will be thrown into the bracket shelf. The bracket shelf should be given a slope of ⅛ in. to the foot away from the column so that the load cannot be applied at the end of the shelf. A bracket will bear only about one-half as great a load applied eccentrically at the edge of the shelf as one distributed over the shelf. A bracket shelf may fail in one of three ways, (1) by shearing through shelf and bracket next to the column, (2) by transverse bending, or (3) by tearing out a section of the column as shown in Fig. 63*f*.

MANUFACTURERS' STANDARD CAST-IRON COLUMN CONNECTIONS
Dimensions in Inches



Depth of beam	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*K*	Thickness of lugs	
20	5	5	6	10½	1½	1½	2	1½	2	1	Holes cored for ¾-in. bolts
18	4	5	6	10½	1½	1½	2	1½	2	1	
15	4	3½	5½	9½	1½	1¼	2	1½	1¾	1	
12	3	3	4½	7¾	1¼	1¼	2	1½	1½	1	



Depth of beam	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*K*	Thickness of lugs	
10	3¼	3½	4	7	1¼	1	2	1½	1½	1	Holes cored for ¾-in. bolts
9	3	3	4	7	1	1	2	1½	1½	1	
8	2½	3	4	7	1	1	2	1½	1½	¾	
7	2¼	2½	4	7	1	1	2	1½	1½	¾	

Tests by the Building Department of New York City have shown that brackets will not fail by shear or transverse bending on columns of more than 6-in. diameter if designed according to standard practice. Of 22 brackets tested, those on 8- or 15-in. columns failed by tearing holes in the body of the column, and 4 on 6-in. columns failed by shearing or transverse stress.

The design of bracket shelves by any rigorous analytical method is impossible. Some of the factors which complicate it are the rate of cooling, variations in the thickness of metal, and imperfections. The design should, however, be checked against failure due to shear or transverse bending.

STEEL COLUMNS

By J. B. Kommers

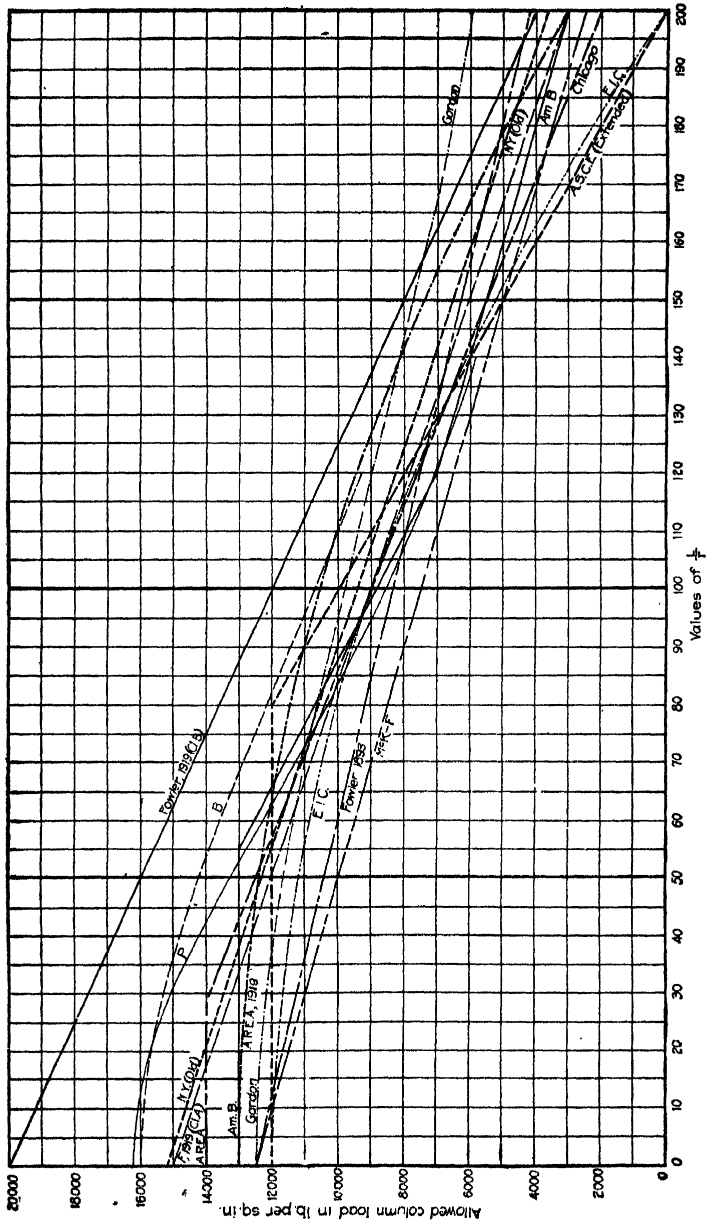
79. Steel Column Formulas.— A diagram of the allowed unit stresses for structural-steel columns as given by the principal column formulas which have received sanction among engineers is shown in Fig. 64, given by C. E. Fowler, *Eng. News-Rec.*, Feb. 13, 1919. The formulas graphically represented are as follows:

Am. B. . .	Am. Bridge Co.	$19,000 - 100L/r$
A.R.E.A.	Am. Ry. Eng. Assn. . .	$16,000 - 70L/r$
A.R.E.A.,1919 . . .	Am. Ry. Eng. Assn. proposed. . . .	$13,000 - 0.25(L/r)^2$
E.I.C . . .	Eng. Inst. Canada.	$12,000 - 0.3(L/r)^2$
F., 1893. .	Fowler's Spec., 1893. .	$12,500 - 41\frac{3}{8}L/r$
F., 1919 (Cl. A.).	Fowler's Spec., 1919.	$15,000 - 60L/r$
F., 1919 (Cl. B.). . . .	Fowler's Spec., 1919. . .	$20,000 - 80L/r$
McK-F.	Fowler, mod. by McKibben.	$12,500 - 50L/r$
N. Y. (Old). . . .	New York Bldg. Code (Old).	$15,200 - 58L/r$
B.	Boston Bldg. Code.	$16,000/1 + L^2/20,000r^2$
G.	Gordon Formula.	$12,500/1 + L^2/36,000r^2$
P.	Philadelphia.	$16,250/1 + L^2/11,000r^2$

The limitations of the formulas as to maximum unit stresses and maximum values of L/r are shown by the diagram. All of the formulas lie in a diagonal zone, the upper limit of which is $18,000 - 60L/r$ and the lower limit of which is $12,000 - 60L/r$ with the exception of Fowler's 1919 (Cl.B.). The average of the zone would be $15,000 - 60L/r$ which is the formula that has been adopted in a 1919 edition of "General Specifications for Steel Roofs and Buildings" by C. E. Fowler. The A.R.E.A. formula, $16,000 - 70L/r$, with a maximum stress of 14,000 lb. per sq. in. and maximum limit of L/r at 120 has received very wide sanction in building codes, being found in the codes of New York, Detroit, Chicago, St. Louis, and Seattle.

80. Forms of Cross-section.[1]—For economy, the radius of gyration of the section should be as large as possible. This makes it desirable to place as much of the material as possible as far from the axis of the column as is consistent with good design. The hollow cylinder is theoretically the most economical form of column cross-section, for in this form all of the material is at a maximum distance from the axis.

[1] By Clyde T. Morris.



Allowed column load in lb. per sq. in.
Values of $\frac{l}{r}$
Fowler 1919 (C.B.)
N.Y. (Old)
Am. B.
Gordon
Chicago
E.I.C.
A.S.C.E. (Extended)
AREA, 1919
Fowler 1893

FIG. 64.

Steel pipe columns are frequently used for light loads where the loads are quiescent and there is no probability of a lateral component to the forces acting on the column. The caps and bases of these are usually cast-iron and the use of this form of column has the same limitations as that of cast-iron columns.

Figure 65 shows the more common forms of cross-section for steel columns and struts.

Struts of 2 angles (Fig. 65*a*) are commonly used for light lateral bracing. The section is unsymmetrical and for this reason is undesirable for main compression members. Columns composed of 2 channels laced (Fig. 65*g*, *h*, and *k*) or 2 pairs of angles laced (Fig. 65*b*) are not as rigid in the plane of the lacing as those in which the parts are connected by plates. Care should be used in proportioning the lacing in such columns. Types *i* and *l* are forms which are commonly used for top chords and end posts of bridges. The lattice on the lower side permits access for cleaning and painting. The Bethlehem H-section (Fig. 65*e* and *f*) is a form much used in building work. Type *e* without cover plates is very economical on account of the small amount of fabrication necessary. Type *f* is much more expensive as it is necessary to drill the holes in the heavy flanges of the H-section for riveting on the cover plates. These flanges are too thick to punch. Z-bar columns (Fig. 65*q* and *r*) are seldom used in modern structures. The Grey column (Fig. 65*s*) and the 4-angle column (Fig. 65*t*) are frequently used in combined steel and concrete columns.

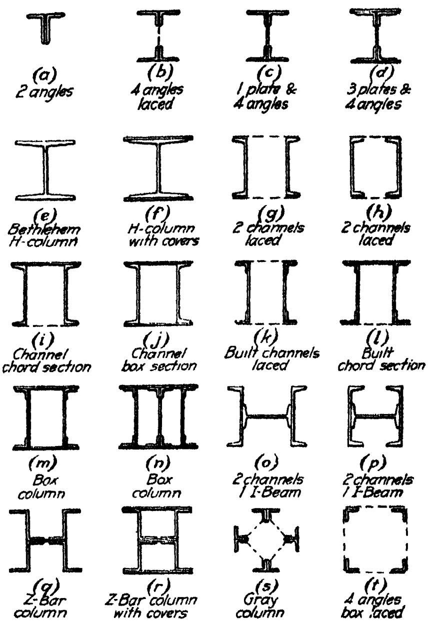


FIG. 65.

81. Design of Cross-section.—The method of designing steel columns is quite similar to that used for cast-iron columns. Here also the radius of gyration and the area both appear in the formulas, and it is usually not possible to express one in terms of the other. This means that a column size must be chosen and tried out to determine whether it is satisfactory.

The nature and size of the work will determine whether a single structural shape may be used as a column, or whether several parts must be riveted together to obtain sufficient area. The method used in design will be illustrated by working a problem using an I-beam, and a second problem using a section built up of two channels and two plates.

Illustrative Problem.—A steel column is 8 ft. long, flat ended, and is to carry safely a load of 100,000 lb. The American Railway Engineering formula, $P/A = 16{,}000 - 70l/r$ is to be used.

If the column were a short compression member with slenderness ratio zero, it could carry a unit load of 16,000 lb. per sq. in. On this basis an area of $\frac{100,000}{16,000} = 6.25$ sq. in. would be required. This is known to be too small. Therefore, a size will be chosen from the tables of standard I-beams giving a larger area. A 9-in. 30-lb. I-beam has an area of 8.82 sq. in. and a least radius of gyration of 0.85. Substituting in the formula

$$= 8.82\left[16,000 - \frac{(70)(8)(12)}{0.85}\right] = 71,400 \text{ lb.}$$

which means that the I-beam is too small. To carry 100,000 lb. the area of the above column would have to be increased about 40 per cent, therefore the next choice will be a 12-in. 40-lb. I-beam with an area of 11.76 sq. in. and a least radius of gyration of 0.90. Substituting in the formula

$$P = 11.76\left[16,000 - \frac{(70)(8)(12)}{0.90}\right] = 100,000 \text{ lb.}$$

which is satisfactory.

In designing columns it is good practice to calculate the slenderness ratio after choosing a size for a tentative design, in order to determine whether the column is a long column or not. In the above problem, if the column had been 15 ft. long the slenderness ratio for the first tentative design would have been 212. This means that the column is a long column and that therefore the straight line formula cannot be used. In such a case Rankine's or Euler's formula may be employed.

Illustrative Problem.—A steel plate and channel column is to be designed to carry a load of 300,000 lb. The column has flat ends and is 20 ft. long. The Milwaukee Building Law formula $P/A = 17,100 - 57l/r$ is to be used.

The standard plate and channel columns, taken from the steel handbooks, will be used, because the work of computation is greatly simplified when the area and least radius of gyration can be found in the handbook tables. For $l/r = 0$, a unit load of 17,100 lb. per sq. in. could be carried, so that on this basis an area of $\frac{300,00}{17,100} = 17.6$ sq. in. will be required. Since this is known to be small, the first choice will be a column of 19.93 sq. in. area, made up of two plates $\frac{5}{8}$- × 9-in. and two 7-in. channels, each weighing 14.75 lb. per ft. For this column the least radius of gyration is 2.53 in. Substituting in the formula,

$$P = 19.93\left[17,100 - \frac{(57)(20)(12)}{2.53}\right] = 233,000 \text{ lb.}$$

which means that the column is too small. To carry 300,000 lb. the above column area should be increased about 30 per cent. Therefore, the next choice will be a column made up of two plates $\frac{5}{8}$- × 10-in. and two 8-in. channels, each weighing 21.25 lb. per ft. The area in this case is 25 and the least radius of gyration is 2.80. Substituting in the formula

$$25\left[17,100 - \frac{(57)(20)(12)}{2.8}\right] = 305,000 \text{ lb.}$$

which is satisfactory.

82. Eccentrically Loaded Columns.—When a column carrying direct loading is also subjected to bending moment due to the column load, or any part of it, being applied away from the axis of the column, the resulting fiber stresses may be determined by the formulas given in the chapter on Bending and Direct Stress in Sec. 1. The fiber stress may also be determined from the equation in Sec. 1, Art. **80,** p. 132, which may be written in the form

$$f = \frac{P}{A} + \frac{Mc}{KI} \tag{1}$$

Values of K for pin-ended columns are given in Fig. 66. For columns with fixed ends use one-half the column length in determining values of l/r for use in Fig.

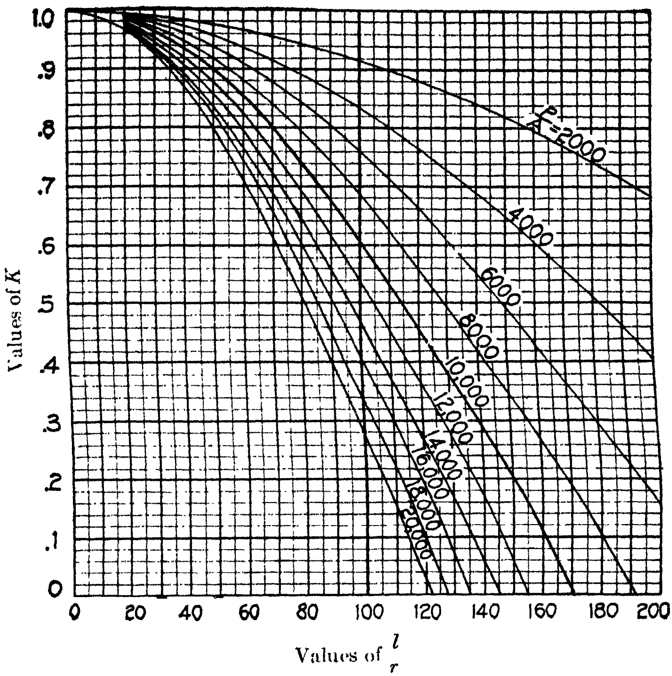


FIG. 66.—Use for eccentrically loaded columns with pin ends. For columns with fixed ends use $\frac{1}{2}l$ in determining $\frac{l}{r}$.

66. In all cases the radius of gyration should be taken about an axis normal to the plane of bending. Note that this value of r may not give the greatest value of l/r which should be used in the column formula.

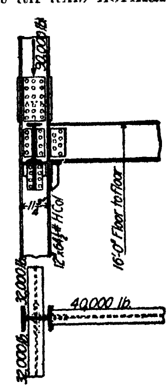


FIG. 67

Illustrative Problem.—Figure 67 shows a building column to which floor beams are connected unsymmetrically, causing an eccentric load on the column. Determine the fiber stress in the column section. Solve by means of eq. (1), and also by means of the formulas given in the chapter on Bending and Direct Stress in Sec. 1.

If the beams are riveted to the column in addition to resting on shelf angles, it is safe to assume that the load is applied at the face of the column. The deflection of the shelf angles would probably be sufficient to bring the center of pressure very near to the face of the column in any case.

The total load, $P = 90{,}000 + 32{,}000 + 32{,}000 + 40{,}000 = 194{,}000$ lb.

The bending moment, $M = (40{,}000)(5\frac{7}{8}) = 235{,}000$ in.-lb.

Solution by eq. (1):

For the given conditions $\frac{l}{r} = \frac{192}{5.13} = 38$. From Fig. 66 with $\frac{l}{r} = 38$ and $\frac{P}{A} = \frac{194,000}{19} = 10,210$, we find $K = 0.935$. Then from eq. (1),

$$f = \frac{194,000}{19} + \frac{(235,000)(5.875)}{(0.935)(499.0)}$$
$$= 10,210 + 2,960 = 13,170 \text{ lb. per sq. in.}$$

Solution by eq. (5), *p.* 139:

$$f = \frac{194,000}{19} + \frac{(235,000)(5.875)}{499}$$
$$= 10,210 + 2,760 = 12,970 \text{ lb. per sq. in.}$$

Solution by eq. (14), *p.* 143:
Since the ends of the column are probably partially fixed, use $C = \frac{1}{10}$.

$$M = \frac{235,000}{1 - \frac{(194,000)(16)^2(12)^2}{(10)(30,000,000)(499)}} = 246,000 \text{ in.-lb.}$$

Then

$$f = \frac{194,000}{19} + \frac{(246,000)(5.875)}{499}$$
$$= 10,210 + 2,890 = 13,100 \text{ lb. per sq. in.}$$

Since the results given by these three solutions are practically identical, we conclude that the second solution is preferable because it is more simple than the others.

83. Column Details.—No element of a column should be left in a condition which will make it possible for this element to fail locally. A column made up of several parts must be so designed that no element can fail as a column between the rivets attaching it to the adjacent column parts. It is evident from Fig. 64 that if the slenderness ratio for any element is made less than about 40 to 50, this possibility of failure will be provided against. Specifications cover this matter by prescribing rules for the pitch of rivets for lacing, the pitch of rivets for attaching plates, the thickness of side plates and cover plates, and the maximum pitch of rivets which attach plates to other shapes.

84. Shear in Column.—Because a column fails partly by direct stress and partly by bending, it is necessary to make provision for the shear which is produced in a column because of the bending. In a column made up of two channels latticed together it is necessary to design the details so that the column may act as a unit. These details must be designed so that the column will have the necessary stiffness as well as the necessary strength.

It may be well to recall that when two wooden beams, each 4 × 4 in. in cross-section, are placed one on top of the other to form a beam, the strength is proportional to the section modulus of two 4- × 4-in. beams, or $\frac{(2)(4)^3}{6} = 21.3$. If, however, instead of separate beams a solid beam 4 × 8 in. is used, then the strength is proportional to $\frac{(4)(8)^2}{6} = 42.7$. In the second case the beam is twice as strong as the two separate beams because of the horizontal shearing stresses which it can resist. The same effect could have been produced by fastening the two 4- × 4-in. beams together in some other way so that they would act as a unit.

In a latticed column the lacing bars must provide the material for taking care of the shearing stresses which are developed because of the bending which occurs. If the column is made up of two 15-in. 40-lb. channels, it will have a strength equal only to twice the strength of a single channel unless they are properly fastened together. Two such single channels 12 ft. long could carry a load of 114,600 lb.

according to the Carnegie Steel Company handbook. If, however, the channels are laced together they will develop a strength of 377,000 lb.

Shear in columns may occur in the case of a rather long column bent into a single loop (Fig. 68), in the case of a short column bent in double curvature due to the fact that the load is applied with opposite eccentricity at the two ends (Fig. 69), or in the case of a short column subjected to secondary bending moments.

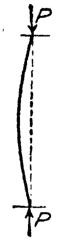


FIG. 68.

FIG. 69.

One method of estimating the shear in a column is that specified by the specifications of the American Railway Engineering Association. These require that the shear be calculated as equal to that produced by a uniformly distributed load, assuming that the column is loaded as a beam, and that the bending stress produced is equal to that assumed in the column formula. The American Railway formula is $P/A = 16{,}000 - 70\,l/r$. Using the form $P/A + 70\,l/r = 16{,}000$, it is evident that the direct stress, P/A, plus the bending stress, $70\,l/r$, is limited to 16,000 lb. per sq. in. The bending moment produced by a uniformly distributed load, W, is $\frac{Wl}{8}\cdot$ Therefore,

$$\frac{Wl}{8} = \frac{70\,l/r\,I}{c}$$

From which

$$W = \frac{560\,I}{rc} = \frac{560\,Ar}{c}$$

For such a beam the maximum shear at the ends is $V = \frac{W}{2}\cdot$ Hence

$$V = \frac{280\,Ar}{c} \tag{2}$$

In which

A = area of cross-section, in square inches.
r = radius of gyration of cross-section, in inches.
c = distance from bending axis to extreme fiber, in inches.

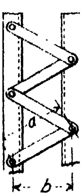


FIG. 70.

Since this formula was developed from a safe load formula, it may be used for working conditions.

85. Design of Latticing.—When the shear which must be provided for in a column is known the lattice bars may be so designed as to take care of this shear.

In a column with single lacing on each side, Fig. 70, the total stress in each lattice bar is

$$F = \frac{V}{2}\cdot\frac{a}{b} \tag{3}$$

in which

a = length of the lattice bar.
b = width between rows of rivets.

This follows from the fact that if F is the total stress in the lattice bar it will have a horizontal component of $F\frac{b}{a}$, and the equilibrium equation is

$$2F\frac{b}{a} = V$$

An alternative method makes use of the formula developed for the horizontal shearing unit stress in a beam, which was

$$v = \frac{VQ}{It}$$

in which

v = shearing unit stress, horizontal or vertical.
V = total shear at the section.
Q = statical moment of one-half the area of the cross-section with respect to the neutral axis.
I = moment of inertia of the cross-section.
t = thickness of beam at neutral axis.

If the unit shear is uniform, then the total shear for 1 in. along the beam, either horizontally or vertically, would be (v) (1) (t). From this it is evident that the shear per lineal inch is $\frac{VQ}{I}$. If the lacing is the same on both sides, then the total shear carried by a section covered by one lacing bar is $\frac{VQb}{2I}$. If the force in the bar is F, then again, for single lacing

$$F\frac{b}{a} = \frac{VQb}{2I}$$

and

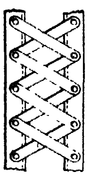


FIG. 71.

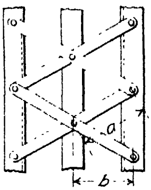


FIG. 72.

$$F = \frac{VQa}{2I} \tag{4}$$

If the lacing is double, as in Fig. 71, then

$$F = \frac{Va}{4b} \tag{5}$$

or

$$F = \frac{VQa}{4I} \tag{6}$$

If a column has three webs, as in Fig. 72, then again the shear per lineal inch is $\frac{VQ}{I}$, and, if the lacing is the same on both sides, then the total shear carried by a section covered by one lacing bar is $\frac{VQb}{2I}$. Here Q = statical moment of the outer rib section with respect to the column center.

If the force in the bar is F, then for horizontal equilibrium,

$$\frac{Fb}{a} = \frac{VQb}{2I}$$

and

$$F = \frac{VQa}{2I} \tag{7}$$

Illustrative Problem.—A column 15 ft. long is composed of two 15-in. 45-lb. channels placed as shown in Fig. 73. Determine the size of the lacing bars required.

Equation (2) may be used to estimate the shear carried by the lacing bars. For the given column A = 26.48 sq. in., r = 5.48 in., and c = 8. Hence

$$V = \frac{(280)(26.48)(5.48)}{8} = 5{,}100 \text{ lb.}$$

Since the column has single lacing, eq. (4) is to be used. For the given column $b = 14\frac{3}{4}$ in., and $a = 17$ in. The stress in a lacing bar at the end of the column is then

$$F = \frac{(5{,}100)(17)}{(2)(14.75)} = 2{,}940 \text{ lb.}$$

This stress may be either tension or compression.

For the given channels the usual specifications require a $\frac{7}{8}$-in. rivet. The minimum size of lacing bar is generally taken as $2\frac{1}{2} \times \frac{3}{8}$ in. Assuming a working stress in tension of 16,000 lb. per sq. in., the net area required for the lacing bar is

$$\frac{2{,}940}{16{,}000} = 0.184 \text{ sq. in.}$$

The net area provided, allowing for a 1-in. rivet hole, is

$$(2\tfrac{1}{2} - 1)(\tfrac{3}{8}) = 0.562 \text{ sq. in.}$$

Since the ends of the bars are rigidly fastened, the length used in the column formula may be taken as half the length of the bar. The allowable working stress for the $2\frac{1}{2} \times \frac{3}{8}$-in. bar is then

$$16{,}000 - 70\frac{l}{r} = 16{,}000 - \frac{(70)(8.5)}{0.1082} = 10{,}500 \text{ lb. per sq. in.}$$

and the area required is

$$\frac{2{,}940}{10{,}500} = 0.28 \text{ sq. in.}$$

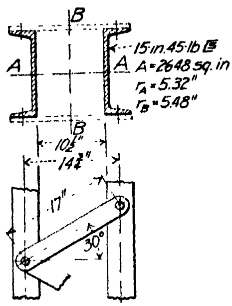


Fig. 73.

The area furnished is $(2\frac{1}{2})(\frac{3}{8}) = 0.937$ sq. in. Hence the assumed bar is larger than required, but since it is the minimum allowed by good practice it will be adopted.

86. Design of Tie-plates and Forked Ends.—At the ends of compression members the lacing is generally replaced by *tie-plates*, as shown in Fig. 74. These plates act as lacing and in addition they hold the segments of the member rigidly in line, and assist in transmitting the stress uniformly over the cross-section of the column.

The end connections for compression members, whether riveted or pin connected, are generally so constructed that moments are set up due to eccentricity of application of the applied load. Thus in Fig. 74*a*, the usual type of connection is so arranged that the stress at the end of the member is transmitted to the web of the channel at lines *b-b*. In the body of the member the load may be considered as applied at the center of gravity of the segments, as shown by the lines *a-a*. The moment due to eccentricity is then the load on that segment times the distance between lines *a-a* and *b-b*. This moment must be resisted by the tie-plate and the rivets connecting the tie-plate to the segments of the member.

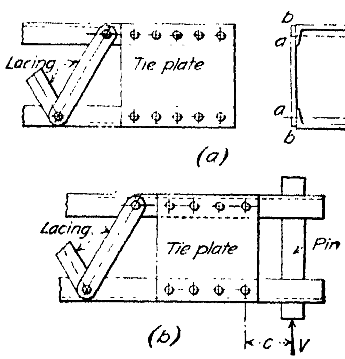


Fig. 74.

Tie-plates should be placed as near the end of the member as possible. The loads should be transferred from the member to the joint or bearing plates as outlined in the chapter on Column Bases which follows. In pin-connected structures

it often happens that the tie-plates cannot be placed at the ends of the members due to interference with other members at the joint. This makes necessary the use of a *forked end*, as shown in Fig. 74*b*. These forks must be designed to carry the shear to the tie-plate and lacing. The total moment carried by both forks is Vc, where V = shear determined as in Art. **84**. The distribution of the shear V between the two forks is indeterminate. It will probably be best to design each fork for a moment equal to $\frac{3}{4}$ Vc. If the forks are overstressed, they may be strengthened by side plates placed on the webs of the channels and extending beyond the edge of the tie-plate.

COLUMN BASES

By C. R. Young

87. Types and Uses.—To transmit the load of a column to the masonry without exceeding the safe bearing pressure on the latter, the lower end of the column must be enlarged by constructing a base for it. This may be either an independent construction or an enlargement of the column itself.

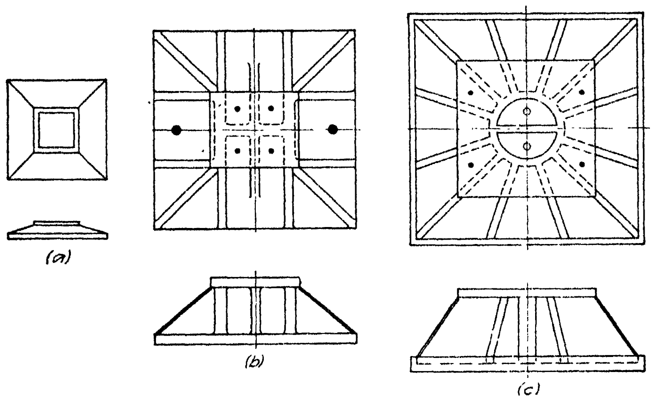


Fig. 75.—Cast bases for columns.

If the former, it may be a separate steel plate or slab; it may be an iron or steel casting, as shown in Fig. 75*a*, *b* or *c*; or it may be a built up steel bolster or grillage, such as shown under the casting in Fig. 15. The solid tapered cast plate, shown in Fig. 75*a*, can be used for only light loads, and, if its thickness would exceed about 4 in., it should be replaced by a ribbed pedestal. This may be either rectangular or circular, usually the former. The advantage of the separate base is that is can be placed and levelled much more easily than can a column with a base riveted to it. Steel grillages are preferred by some engineers as being more reliable than cast-iron bases, and cheaper than cast steel ones. They lend themselves well to situations where long narrow bases must be provided and where the bending moment on them is very large.

Steel columns resting on separate plates, slabs, or cast bases require at most only side connection angles to the base merely to hold them in position. For

light columns not subjected to lateral forces or uplift, no connection between the column and the separate base is required.

If the base is riveted to the bottom of the column, forming an enlargement of it, the spread must be large, and projecting side angles must be used, supplemented perhaps by side plates and by stiffener angles. The type shown in Fig. 76*a* is the simplest of these, consisting of a base plate and two pairs of side angles. Type *b* shows the addition of distributing gussets or side plates; type *c* shows the further addition of stiffener angles to assist in the distribution of load; while the type illustrated in Fig. 78 shows a base with stiffener angles arranged to transfer the pull of anchor bolts to the column shaft. This latter type is used only where an uplift on the column is likely to occur.

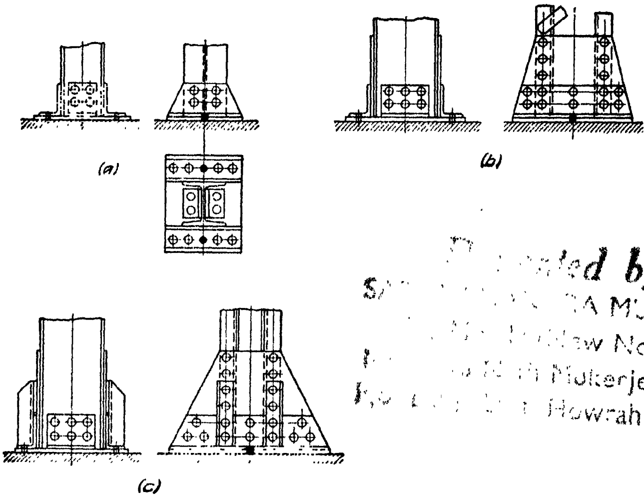


FIG. 76.—Built-up bases for columns.

88. Design of Plain Bases.—To find the size of any base, the load to be transferred by it must be divided by the allowable pressure on the masonry as fixed by the specification. If the base consists of a plain steel plate or slab of rectangular shape, the thickness may be determined by figuring the maximum moment on the plate, or slab, and applying the common flexure formula to the dangerous section. If it is assumed that the moment is a maximum at the center of the base—an assumption on the side of severity—the moment in the direction of the length is

$$M = \tfrac{1}{8}W(L - l) \tag{1}$$

and in the direction of the width

$$M = \tfrac{1}{8}W(B - b) \tag{2}$$

where W = total upward reaction on base.
L = length of slab in inches.
B = breadth of slab in inches.
l = outside dimension of column parallel to L.
b = outside dimension of column parallel to b.

If the moment be taken at the edge of the column shaft, corresponding formulas may be readily written. The whole width of the plate or slab is assumed to be effective in calculating the resistance.

Knowing the bending moment on the base, the thickness may then be found.

89. Design of Ribbed Cast Bases.—Having found the required size of the base in plan, and having fixed the dimensions of the top plate so as to provide the necessary area to receive the column and to accommodate the connecting bolts, the height of the base, the number and arrangement of ribs, and the thicknesses of all parts must be determined.

It has been found that the height of cast bases is best made between one-third and one-half the side of the bottom plate. This enables the upper edges of the ribs to be sloped down at approximately an angle of 45 deg. If the slope is appreciably flatter than this to the horizontal, the flexural stresses in the part of the base projecting past the upper plate become high and lessen the efficiency of the base. For the same reason it is found that as the reacting pressure in pounds per square inch increases, the economical ratio of height to side of the base also increases.

The arrangement of the ribs underneath the column shaft should correspond as closely as possible to the shape of the column, so that the pressure may be taken directly down to the bottom plate without putting much flexure in the top plate. The plan views of the bases shown in Fig. 75*b* and *c* show how two bases were arranged to suit an H-column. Frequently, a circular hub is provided at the center of the casting, as in Fig. 75*c*, with ribs radiating to the edges. A partion rib across the space within this hub is provided, if the column has one central web, so as to receive the load from the web. The number of ribs to be provided will depend on the size of base and the load to be carried. There may be as many as 16 radiating from the center to the edges. The sectional area of the ribs should be such that the portions under the load will take the vertical load safely as short columns or prisms. It is best to limit their ratio of clear height to thickness to about 15 and proportion for a compressive stress of not over 8,000 lb. per sq. in. if they are of cast-iron. The heaviest ribs are, of course, under the load. Those radiating to the edges are the thinnest. In no case should ribs be thinner than 1 in.

The bottom plate is proportioned as a beam continuous under the various ribs. Between ribs, it should be calculated as a restrained beam at the allowable tensile flexural stress for cast iron, or about 3,000 lb. per sq. in., if the base be of this material. The projecting portions should be calculated as cantilevers and similarly proportioned. To strengthen the edges of the bottom plate, a flange is frequently provided around the outer edge, as shown in Fig. 75*c*. This is commonly from 3 to 5 in. deep overall. The bottom plate itself may be from 1 to 3 in. thick.

To test the sufficiency of both the ribs and the bottom plate, the moment on the projection past the edge of the top plate on one of the four sides should be calculated, and the moment of resistance of the section cut by a vertical plane passing through this edge should be computed and compared with the bending moment. The moment of resistance is found in the same manner as that of a cast-iron lintel, Art. **30**, p. 231.

To ensure that the pressure is uniformly applied to the top of the base, it should be planed. If it rest on a steel grillage, both the latter and the bottom of the base should be planed.

Holes should be left through the bottom plate to enable grout to be poured in under the base plate after it is brought to the required height and levelled.

90. Design of Built-up Bases.—The size of the base plate for a built-up base is found by dividing the total load by the permissible bearing on the masonry. Its thickness should be sufficient to withstand the upward uniform pressure without exceeding the allowable flexural stress on steel, or without undue deflection. The portions that project farthest past the column shaft, or span the greatest distances between column flanges or side plates, should be investigated as cantilever or continuous beams, as the case may be. The thickness of base may vary from $\frac{3}{8}$ in. for light angle columns to $1\frac{1}{2}$ in. for very heavy columns. It is frequently somewhat less in practice than a strict calculation of bending stresses would warrant.

To attach the base plate to the column shaft, one or two pairs of angles may be used, two pairs being used for the larger columns. These transfer pressure to the base plate up to the limit of capacity of the rivets that attach them to the column shaft. The strength of the outstanding leg in flexure may need to be investigated to discover if the angle can transfer outward at right angles to its length, the load that its connecting rivets would warrant. The thickness of angles commonly used varies from $\frac{3}{8}$ to $\frac{3}{4}$ in. The length of vertical leg is commonly 6 in., but the horizontal leg is usually $3\frac{1}{2}$ or 4 in.

Side plates are from $\frac{5}{16}$ to $\frac{1}{2}$ in. thick and should be attached by sufficient rivets to the column shaft to ensure that the load which they are supposed to transmit to the base plate may be developed safely. It should be remembered that the rivets through both the base angles and the side plates have to do double duty. The upper edges of the side plates and ends of the base angles riveted over them are usually cut to one slope, as shown in Fig. 76*b* and *c*. This is not usually less than 45 deg. with the horizontal.

Stiffeners, where used, are $\frac{3}{8}$ or $\frac{1}{2}$ in. thick with outstanding legs wide enough to cover the outstanding legs of the base angles on which they bear. Their attachment must be sufficient to devlope the load they are supposed to transmit.

The proportion of the total column load to be taken by the side plates, base angles, and stiffeners, will depend on how much is assumed as transmitted to the base plate directly by the faced end of the column shaft. This is commonly taken at only 40 or 50 per cent of the total load, so that the side details must account for the other 50 or 60 per cent.

Illustrative Problem.—Design a riveted steel plate and angle base for a 10-in. 49-lb. Bethlehem H-column of the type shown in Fig. 77. The vertical centric load is 170,000 lb. Consider 40 per cent of the total axial load as carried directly to the base plate by the faced end of the column shaft. Rivets, $\frac{3}{4}$ in. Anchor bolt holes $\frac{1}{8}$ in. larger than the bolts. Permissible stresses:

Bending = 16,000 lb. per sq. in.
Bearing on end of column = 16,000 lb. per sq. in.
Shearing on shop rivets = 12,000 lb. per sq. in.
Bearing on shop rivets = 24,000 lb. per sq. in.
Bearing on concrete = 500 lb. per sq. in.

Base Plate.—Required area of plate, $A = 170{,}000/500 = 340$ sq. in.

To facilitate details, adopt a plate 18 × 19 in., giving an area of 342 sq. in. The 18-in. dimension is made parallel to the web to accommodate two base angles with 3½-in. horizontal legs and two ⅜-in. side plates.

Calculations of the thickness required, assuming the base plate as an overhanging or continuous beam, gives results in excess of the thickness found satisfactory by experience. For a base of this character, the base plate is usually about ¾ in. This thickness will be adopted.

Side Plates and Base Angles.—Since only 40 per cent of the total axial load is assumed to be transferred to the base plate by the faced end bearing of the column shaft, the remaining 60 per cent, or (170,000)(0.60) = 102,000 lb., must be delivered to the base plates by the side plates and base angles. To make this possible, enough rivets must be placed through the column shaft to develop 102,000 lb. Assume 16 rivets through the flanges, for which the least value (single shear) is (0.44)(12,000) = 5,280 lb., and 4 rivets through the web, for which the least value (bearing on 0.36-in. web) is (0.36)(0.75)(24,000) = 6,480 lb. The total safe resistance of these two groups of rivets, therefore, = (16)(5,280) + (4)(6,480) = 110,500 lb., which is adequate for the load.

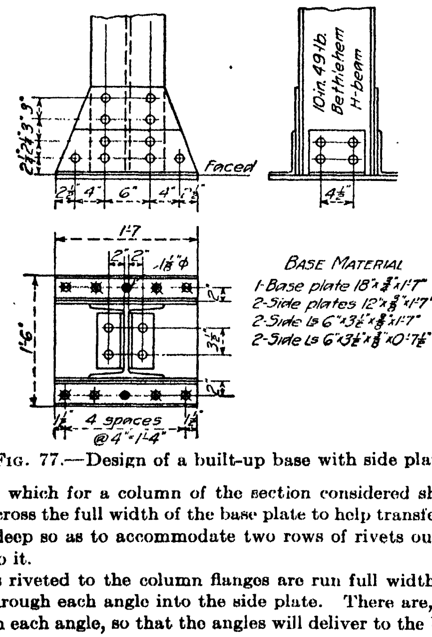


FIG. 77.—Design of a built-up base with side plates.

The side plates, which for a column of the section considered should be about ⅜ in. thick, will extend across the full width of the base plate to help transfer load out to its edges, and will be 12 in. deep so as to accommodate two rows of rivets outside the base angles, which are riveted to it.

The base angles riveted to the column flanges are run full width of the base and two rivets are driven through each angle into the side plate. There are, therefore, 6 rivets in single shear through each angle, so that the angles will deliver to the base plate (12)(5,280) = 63,400 lb., or 37 per cent of the total column load, leaving 23 per cent to be delivered by the side plates and the base angles on the web.

Four rivets through the base angles in the column web will develop (4)(6,480) = 25,900 lb., or 15 per cent of the total column load. This leaves only 8 per cent of the total column load to be delivered to the base plate by the two side plates.

All vertical rivets through base angles must be countersunk on the under side. Only sufficient rivets are employed to hold the angles and plate tightly together.

Anchor bolt holes are provided ⅛ in. larger than the anchors, which will be 1 in. diam.

91. Anchorage.—If there be no appreciable lateral force or uplift exerted on columns, the bases do not really need to be anchored down to the masonry. The frictional resistance of the base on the top of the pier, once the column has received as full dead load, is sufficient to prevent it being displaced by blows or shock.

In case there is considerable lateral force exerted on the base, anchor bolts will need to be provided. To resist sliding, their shear value should be equal to the difference between the lateral force and the frictional resistance of the base. As a safeguard against overturning of the column, the anchor bolts should be embedded far enough in the masonry, or sufficiently anchored thereto, to develop the maximum tension likely to come on them. The mass of masonry engaged should weigh at least 1½ times the tension on the bolt.

In order to develop high resistance to overturning, the bolts should be placed as far apart as possible in the direction of the moment.

Illustrative Problem.—A column consisting of a 24 × ⅜-in. web, two 5 × 3½ × ½-in. angles and two 5 × 3½ × ⅜-in. angles with the 5-in. legs outstanding, as shown in

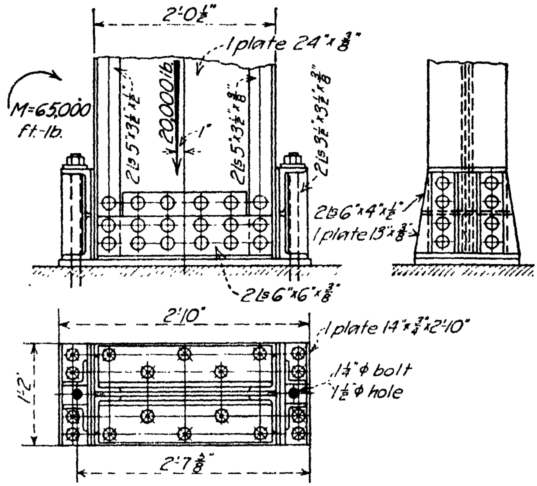


FIG. 78.—Design of an anchorage for column.

Fig. 78, is subjected to an overturning wind moment of 65,000 ft.-lb. The minimum axial load is 20,000 lb. applied 1 in. off center on the side towards the wind. Assuming the base plate, side angles, and side plates shown as already fixed, design the anchor bolts and an attachment for them to develop the required tension.

Net overturning moment about leeward edge of base plate, M = (65,000)(12) − (20,000)(18) = 420,000 in.-lb.

If the bolts pass through the outstanding legs of the base angles on the column flange. their distance from the far edge of the plate would be 31⅝ in.

Tension in windward bolt = 420,000/31.63 = 13,300 lb.

Required area of one bolt at root of thread = 13,300/16,000 = 0.83 sq. in.

One 1¼-in. diam. bolt with a net area of 0.89 sq. in. at root of thread will be adopted. Shelf angles, 6 × 4 × ½ in., riveted to the sides of the column, as shown, will take the anchor bolt tension into the column shaft. Two stiffeners, 3½ × 3½ × ⅜, will be employed under each shelf angle. The rivets through the shelf angles and the stiffeners are ample at any ordinary working stresses to carry the stress into the column.

The embedment of the anchors in the masonry must be such as to develop the tension in them. Each should engage a mass of masonry weighing at least 1½ times the amount of the uplift in it. The pressure on the masonry under the leeward side of the base plate should also be investigated to ensure that the safe bearing pressure on it at the leeward edge of the base plate is not exceeded.

SECTION 3

SPLICES AND CONNECTIONS FOR STEEL MEMBERS

By C. A. Willson

1. Kinds of Connections.—Three different kinds of connections are used in steel construction, as follows: (1) Riveted connections; (2) bolted connections; and (3) pin connections. Rivets are used for fastening together the elements of a built-up structural member and for connecting the members themselves in the finished structure. Bolts are used for holding the parts together while rivets are being driven and in certain cases are used for permanent connections in place of rivets. Where several members must be connected in such a way that they will be free to turn with respect to each other at a joint, a pin connection must be used. Since the stresses involved in riveted connections and bolted connections are alike, and since the design of pin connections involves several distinctly different features, the first two will be discussed together while the subject of pin connections will be treated separately (see Art. **18**).

2. Kinds of Rivets and Conventional Signs for Riveting.—When classified with regard to the method of driving, there are two kinds of rivets, namely: (1) Those driven at the fabricating shop, called *shop rivets*, and (2) those driven at the place of erection called *field rivets*. At the fabricating shop most of the work is done by means of heavy hydraulic or pneumatic riveters and facilities are provided for properly supporting the members while the rivets are being driven. At the place of erection the riveting must be done by a much lighter portable device, called a pneumatic riveting hammer or air gun, or it must be done by hammering the rivet set with a sledgehammer. Except on very small jobs, this latter method is not used. While the blows delivered by the pneumatic hammer are relatively light, yet they are delivered rapidly and very satisfactory work can be produced by this method.

Shop rivets are used whenever practicable since higher working stresses may be allowed for them than for field rivets. They are used entirely for connecting the various parts of built-up members and also for connecting members so as to make larger parts which may be handled and transported conveniently as units. Necessarily, field rivets must be used to join parts of the structure which are transported to the place of erection separately.

Rivets may be classified as follows, depending on the way they are made: (1) Full button heads; (2) countersunk and chipped; (3) countersunk and not chipped; and (4) flattened. Rivets having full button heads are used almost entirely in structural work. In special cases where sufficient clearance cannot be obtained by the use of button head rivets, it may be necessary to flatten the heads. Where a smooth surface is desired the rivets must be countersunk and clipped.

Of course, rivets with countersunk and flattened heads are not as strong as those with full button heads. The practice of R. Fleming of the American Bridge Company is as follows.[1]

Shear and bearing values of countersunk rivets are assumed at three-fourths the values of rivets with full heads when the metal is not thinner than one-half the diameter of the rivet. As a rule, countersunk rivets are not used in thinner metal, but when unavoidable, they are rated at three-eighths the values of full-headed rivets. Flattened heads are assumed equal to full heads when their height is not less than ⅜ in., or ½ the diameter of the rivet for ⅝-in. rivets and less. Rivets flattened to less than these heights are regarded as countersunk rivets.

In order that the designer in the office might be able to indicate to the man in the shop or out on the job just what kind of rivets are desired at different points, it has been found necessary to adopt a system of indications or signs which would convey this information. The system of conventional rivet signs shown in Fig. 1, known as the Osborne System, is the one which is used almost entirely in this country. Bolts are usually indicated by notes.

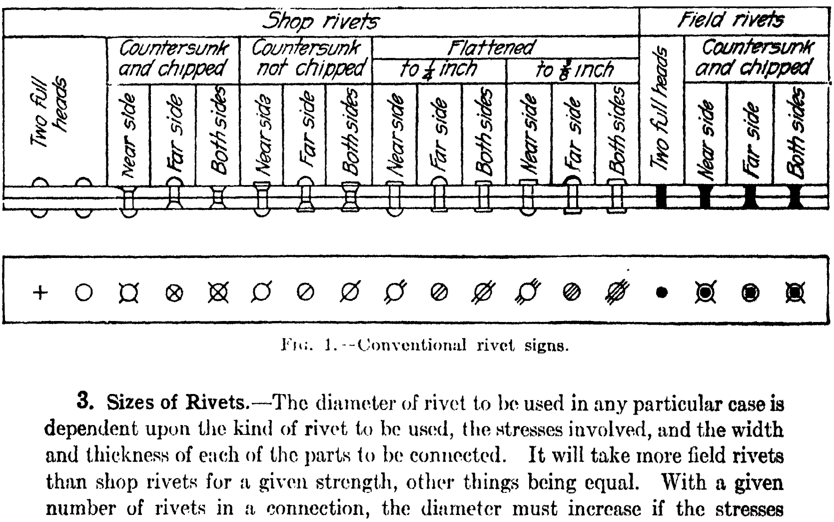


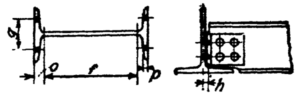
FIG. 1.—Conventional rivet signs.

3. Sizes of Rivets.—The diameter of rivet to be used in any particular case is dependent upon the kind of rivet to be used, the stresses involved, and the width and thickness of each of the parts to be connected. It will take more field rivets than shop rivets for a given strength, other things being equal. With a given number of rivets in a connection, the diameter must increase if the stresses increase. A rivet of large diameter cannot be used in a member or portion of a member of relatively small width, because of the reduction of the area of the cross-section at the rivet. The maximum sizes of rivets which can be used in angles and in the flanges of beams and channels are given in Tables 1, 2, and 3.

The sizes of rivets used most in structural steel work are ⅝ in., ¾ in., and ⅞ in. Rivets ½ in. in diameter are sometimes used in very light work, while in very heavy work rivets 1 in. in diameter are employed, and occasionally larger sizes are necessary. In the Hell Gate arch bridge there are about 840,000 shop rivets and 334,000 field rivets, of which 400,000 are 1¼ in. in diameter.[2]

[1] *Engineering News-Record*, Feb. 24, 1921, p. 336.
[2] *Engineering and Contracting*, vol. 50, p. 394.

TABLE 1.[1]—STANDARD GAGES AND DIMENSIONS FOR BEAMS



Nominal dimensions are: flange width and "o" in eighths, web thickness in sixteenths. Gages for connection angles are determined by ½ web thickness. Standard gages may be varied if conditions require.

Depth beam	Weight per foot	Flange width	Web thickness	½ web thickness	Gage g	Grip p	Distance f	Distance o	Distance h	Max. rivet in flange
In.	Lb.	In.	In.	In.	In.	In.	In.	In.	In.	In.
27	90.0	9	½	¼	4	¾	22½	2¼	5/16	⅞
24	115.0	8	¾	⅜	4	1⅛	20¼	1⅞	7/16	
	110.0	8	11/16	⅜	4	1⅛	20¼	1⅞	7/16	⅞
	105.9	7⅞	⅝	5/16	4	1⅛	20¼	1⅞	⅜	
24	100.0	7¼	¾	⅜	4	⅞	20¾	1⅝	7/16	
	95.0	7¼	11/16	⅜	4	⅞	20¾	1⅝	7/16	
	90.0	7⅛	⅝	5/16	4	⅞	20¾	1⅝	⅜	⅞
	85.0	7⅛	9/16	5/16	4	⅞	20¾	1⅝	⅜	
	79.9	7	½	¼	4	⅞	20¾	1⅝	5/16	
24	74.2	9	½	¼	4	⅝	20	2	5/16	⅞
21	60.4	8¼	7/16	3/16	4	9/16	17½	1¾	¼	⅞

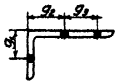
Depth of beam	Weight per foot	Flange width	Web thickness	½ web thickness	Gage g	Grip p	Distance f	Distance o	Distance h	Max. rivet in flange
In.	Lb.	In.	In.	In.	In.	In.	In.	In.	In.	In.
12	55.0	5⅝	13/16	7/16	3½	¾	9¼	1⅜	½	
	50.0	5½	11/16	⅜	3½	⅜	9¼	1⅜	7/16	
	45.0	5⅜	9/16	5/16	3	¾	9¼	1⅜	⅜	¾
	40.8	5¼	½	¼	3	¾	9¼	1⅜	5/16	
12	35.0	5⅛	7/16	¼	3	9/16	9¾	1⅛	5/16	
	31.8	5	⅜	3/16	3	9/16	9¾	1⅛	¼	¾
12	27.9	6	5/16	⅛	3	7/16	9½	1¼	3/16	¾
10	40.0	5⅛	¾	⅜	2¾	½	8	1	7/16	
	35.0	5	⅝	5/16	2¾	½	8	1	⅜	¾
	30.0	4⅞	½	¼	2¾	½	8	1	5/16	
	25.4	4⅝	5/16	3/16	2¾	½	8	1	¼	

20	100.0	7¼	7/8	7/16	4	1	16½	1¾	½	7/8
	95.0	7¼	13/16	7/16	4	1	16½	1¾	½	
	90.0	7⅛	¾	⅜	4	1	16½	1¾	7/16	
	85.0	7⅛	11/16	⅜	4	1	16½	1¾	⅜	
	81.4	7	⅝	5/16	4	1	16½	1¾	⅜	
20	75.0	6⅜	11/16	5/16	4	¾	17	1½	⅜	7/8
	70.0	6⅜	9/16	5/16	4	¾	17	1½	⅜	
	65.4	6¼	½	¼	4	¾	17	1½	5/16	
18	90.0	7¼	13/16	7/16	4	1	14½	1¾	½	7/8
	85.0	7⅛	¾	⅜	4	1	14½	1¾	7/16	
	80.0	7 1/16	⅝	5/16	4	1	14½	1¾	⅜	
	75.6	7	9/16	5/16	4	1	14½	1¾	⅜	
18	70.0	6¼	¾	⅜	3¾	¾	15¼	1⅜	7/16	7/8
	65.0	6¼	⅝	5/16	3¾	¾	15¼	1⅜	⅜	
	60.0	6⅛	9/16	5/16	3¾	¾	15¼	1⅜	⅜	
	54.7	6	½	¼	3¾	¾	15¼	1⅜	5/16	
18	48.2	7½	⅜	3/16	3¾	½	14¾	1⅝	¼	7/8
15	75.0	6⅜	7/8	7/16	3½	7/8	11¾	1⅝	½	¾
	70.0	6¼	13/16	⅜	3½	7/8	11¾	1⅝	7/16	
	65.0	6⅛	11/16	⅜	3½	7/8	11¾	1⅝	7/16	
	60.8	6	⅝	5/16	3½	7/8	11¾	1⅝	⅜	
15	55.0	5¾	11/16	5/16	3½	⅝	12½	1¼	⅜	¾
	50.0	5⅝	9/16	5/16	3½	⅝	12½	1¼	⅜	
	45.0	5⅝	½	¼	3½	⅝	12½	1¼	5/16	
	42.9	5½	7/16	3/16	3½	⅝	12½	1¼	¼	
15	37.3	6¾	5/16	3/16	3½	7/16	12¼	1⅜	¼	¾

10	22.4	5½	¼	⅛	2¾	⅜	7¾	1⅛	3/16	¾
9	35.0	4¾	¾	⅜	2½	½	7	1	7/16	¾
	30.0	4⅝	9/16	5/16	2½	½	7	1	⅜	
	25.0	4½	7/16	3/16	2½	½	7	1	¼	
	21.8	4⅜	5/16	⅛	2½	½	7	1	3/16	
8	25.5	4¼	9/16	5/16	2¼	½	6¼	7/8	5/16	¾
	23.0	4¼	7/16	¼	2¼	7/16	6¼	7/8	5/16	
	20.5	4⅛	⅜	3/16	2¼	7/16	6¼	7/8	¼	
	18.4	4	¼	⅛	2¼	7/16	6¼	7/8	3/16	
8	17.5	5	¼	⅛	2¼	⅜	6	1	3/16	¾
7	20.0	3⅞	½	¼	2¼	⅜	5¼	7/8	5/16	⅝
	17.5	3¾	⅜	3/16	2¼	⅜	5¼	7/8	¼	
	15.3	3⅝	¼	⅛	2¼	⅜	5¼	7/8	3/16	
6	17.25	3⅝	½	¼	2	⅜	4½	¾	5/16	⅝
	14.75	3½	⅜	3/16	2	⅜	4½	¾	¼	
	12.5	3⅜	¼	⅛	2	⅜	4½	¾	3/16	
5	14.75	3⅜	½	¼	1¾	⅜	3½	¾	5/16	½
	12.25	3¼	⅜	3/16	1¾	⅜	3½	¾	¼	
	10.0	3	¼	⅛	1¾	⅜	3½	¾	3/16	
4	10.5	2⅞	7/16	¼	1½	5/16	2¾	⅝	¼	½
	9.5	2⅞	⅜	3/16	1½	5/16	2¾	⅝	¼	
	8.5	2¾	¼	⅛	1½	5/16	2¾	⅝	3/16	
	7.7	2⅝	3/16	⅛	1½	5/16	2¾	⅝	3/16	
3	7.5	2½	⅜	3/16	1½	5/16	1¾	⅝	¼	⅜
	6.5	2½	¼	⅛	1½	5/16	1¾	⅝	3/16	
	5.7	2⅜	5/16	⅛	1½	5/16	1¾	⅝	⅛	

[1] From Pocket Companion, 22nd edition, Carnegie Steel Co., Pittsburgh, Pa.

TABLE 2.[1]—GAGES FOR ANGLES



Leg	8	7	6	5	4	3½	3	2½	2	1¾	1½	1⅜	1¼	1	¾
g_1	4½	4	3½	3	2½	2	1¾	1⅜	1⅛	1	⅞	⅞	¾	⅝	½
g_2	3	2½	2½	2											
g_3	3	3	2¼	1¾											
Max. rivet	1⅛	1	⅞	⅞	⅞	⅞	⅞	¾	⅝	½	⅜	⅜	⅜	¼	¼

[1] From Pocket Companion, 20th edition, Carnegie Steel Co., Pittsburgh, Pa.

For column details, 6-in. leg (½ in. thick or less) against column shaft, g_2 = 1¾ in., g_3 = 3 in.
For diagonal angles, etc., gage in middle, where riveted leg equals or exceeds 3 in. for ¾ in. rivets 3½ in. for ⅞-in. rivets.
Use special gages to adapt work to multiple punch, or to secure desirale details.

4. Spacing of Rivets.— Rivets are located on lines running parallel to the edges of the members. These lines are called *gage lines* (see Fig. 2). The distance between gage lines, or the distance from a gage line to some surface, is known as *gage*. Standard gages for angles, beams, and channels are given in Tables 1, 2, and 3. The distance from the center of a rivet to the edge of a member is called the *edge distance*. There must be enough distance between the rivet and the edge of the member so that there will be no tendency to cause bulging of the material and consequent failure. The distance from a sheared edge should always be greater than the distance from a rolled edge since the metal near a sheared edge is injured to a certain extent in the shearing process. The minimum distance from the center of any rivet to a sheared edge should not be less than the diameter of the rivet plus ½ in. The distance center to center of rivets measured along the gage lines is called the *pitch*. In extreme cases the distance between centers of rivet holes may be made three times the diameter of the rivet, but a minimum distance of 3 in. for ⅞-in. rivets and 2½ in. for ¾-in. rivets is preferable. For members composed of plates and shapes the maximum pitch in the line of stress should be 6 in. for ⅞-in. rivets and 5 in. for ¾-in. rivets. Where two or more plates are used in contact, or where two angles in contact are used as tension members, a maximum pitch of 12 in. may be allowed. A rivet cannot be placed close against the web of an I-beam or close to the leg of an angle because space is necessary for the die which forms the head of the rivet. This space is called the *clearance* and is another of the factors controlling the spacing of rivets. Data representing standard practice in regard to the pitch and clearance for various sizes of rivets are given in Table 4.

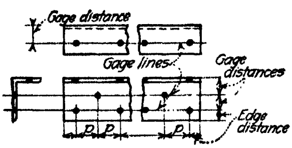
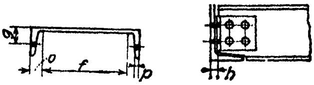


FIG. 2.

TABLE 3.[1]—STANDARD GAGES AND DIMENSIONS FOR CHANNELS



Nominal dimensions are: flange width and "o" in eighths, web thickness in sixteenths. Gages for connection angles are determined by ½ web thickness. Standard gages may be varied if conditions require.

Depth of channel (in.)	Weight per foot (lb.)	Flange width (in.)	Web thickness (in.)	½ web thickness (in.)	Gage g (in.)	Grip p (in.)	Distance f (in.)	Distance o (in.)	Distance h (in.)	Max. rivet in flange (in.)
15	55.0	3 7/8	13/16	7/16	2 1/2	11/16	12 1/4	1 3/8	7/8	7/8
	50.0	3 3/4	3/4	3/8	2 1/2	11/16	12 1/4	1 3/8	13/16	
	45.0	3 5/8	5/8	5/16	2	5/8	12 1/4	1 3/8	11/16	
	40.0	3 1/2	1/2	1/4	2	5/8	12 1/4	1 3/8	9/16	
	35.0	3 1/2	7/16	1/4	2	5/8	12 1/4	1 3/8	1/2	
	33.9	3 3/8	7/16	3/16	2	5/8	12 1/4	1 3/8	1/2	
13	50.0	4 3/8	13/16	3/8	3	9/16	10 1/4	1 3/8	7/16	7/8
	45.0	4 1/4	11/16	5/16	2 3/4	9/16	10 1/4	1 3/8	3/8	
	40.0	4 1/4	9/16	5/16	2 3/4	9/16	10 1/4	1 3/8	3/8	
	37.0	4 1/8	1/2	1/4	2 1/2	9/16	10 1/4	1 3/8	5/16	
	35.0	4 1/8	7/16	1/4	2 1/2	9/16	10 1/4	1 3/8	5/16	
	31.8	4	3/8	3/16	2 1/2	9/16	10 1/4	1 3/8	1/4	
12	40.0	3 1/2	3/4	3/8	2	5/8	10	1	13/16	7/8
	35.0	3 3/8	5/8	5/16	2	5/8	10	1	11/16	
	30.0	3 1/4	1/2	1/4	1 3/4	1/2	10	1	9/16	
	25.0	3 1/8	3/8	3/16	1 3/4	1/2	10	1	7/16	
	20.7	3	5/16	1/8	1 3/4	1/2	10	1	3/8	
10	35.0	3 1/4	13/16	7/16	1 3/4	1/2	8 1/4	7/8	7/8	3/4
	30.0	3 1/8	11/16	3/8	1 3/4	1/2	8 1/4	7/8	3/4	
	25.0	2 7/8	1/2	1/4	1 3/4	1/2	8 1/4	7/8	9/16	
	20.0	2 3/4	3/8	3/16	1 1/2	7/16	8 1/4	7/8	7/16	
	15.3	2 5/8	1/4	1/8	1 1/2	7/16	8 1/4	7/8	5/16	
9	25.0	2 7/8	5/8	5/16	1 1/2	1/2	7 1/4	7/8	11/16	3/4
	20.0	2 5/8	7/16	1/4	1 1/2	1/2	7 1/4	7/8	1/2	
	15.0	2 1/2	5/16	3/16	1 3/8	7/16	7 1/4	7/8	3/8	
	13.4	2 1/2	1/4	1/8	1 3/8	7/16	7 1/4	7/8	5/16	
8	21.25	2 5/8	5/8	5/16	1 1/2	7/16	6 1/4	7/8	11/16	3/4
	18.75	2 1/2	1/2	1/4	1 1/2	7/16	6 1/4	7/8	9/16	
	16.25	2 1/2	7/16	3/16	1 1/2	7/16	6 1/4	7/8	1/2	
	13.75	2 3/8	5/16	3/16	1 3/8	3/8	6 1/4	7/8	3/8	
	11.5	2 1/4	1/4	1/8	1 3/8	3/8	6 1/4	7/8	5/16	
7	19.75	2 1/2	5/8	5/16	1 1/2	7/16	5 1/2	3/4	11/16	5/8
	17.25	2 1/2	1/2	1/4	1 1/2	7/16	5 1/2	3/4	9/16	
	14.75	2 3/8	7/16	1/4	1 1/4	7/16	5 1/2	3/4	1/2	
	12.25	2 1/4	5/16	3/16	1 1/4	3/8	5 1/2	3/4	3/8	
	9.8	2 1/8	1/4	1/8	1 1/4	3/8	5 1/2	3/4	5/16	
6	15.5	2 1/4	9/16	5/16	1 3/8	3/8	4 1/2	3/4	5/8	5/8
	13.0	2 1/4	7/16	1/4	1 3/8	3/8	4 1/2	3/4	1/2	
	10.5	2 1/8	5/16	3/16	1 1/8	3/8	4 1/2	3/4	3/8	
	8.2	2	3/16	1/8	1 1/8	5/16	4 1/2	3/4	1/4	
5	11.5	2 1/8	1/2	1/4	1 1/8	5/16	3 3/4	5/8	9/16	1/2
	9.0	1 7/8	5/16	3/16	1 1/8	5/16	3 3/4	5/8	3/8	
	6.7	1 3/4	3/16	1/8	1 1/8	5/16	3 3/4	5/8	1/4	
4	7.25	1 3/4	5/16	3/16	1	5/16	2 3/4	5/8	3/8	1/2
	6.25	1 5/8	1/4	1/8	1	5/16	2 3/4	5/8	5/16	
	5.4	1 5/8	3/16	1/8	1	5/16	2 3/4	5/8	1/4	
3	6.0	1 5/8	3/8	3/16	7/8	1/4	1 3/4	5/8	7/16	1/2
	5.0	1 1/2	1/4	1/8	7/8	1/4	1 3/4	5/8	5/16	
	4.1	1 1/2	3/16	1/8	7/8	1/4	1 3/4	5/8	1/4	

[1] From Pocket Companion, 22nd edition, Carnegie Steel Co., Pittsburgh, Pa.

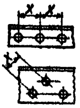
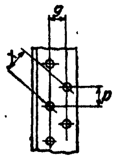


TABLE 4.[1]—RIVET SPACING AND CLEARANCES

Minimum Rivet Spacing

All dimensions in inches

Diameter of rivet	1/2	5/8	3/4	7/8
"x" minimum	1 1/4	1 1/2	1 7/8	2 1/4
"x" preferable	1 3/4	2	2 1/2	3

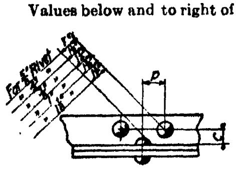


Distance Center to Center of Staggered Rivets

Values of x for varying value of g and p

p, in.	g, inches													
	7/8	1	1 1/8	1 1/4	1 3/8	1 1/2	1 5/8	1 3/4	1 7/8	2	2 1/8	2 1/4	2 3/8	2 1/2
1 1/8	1 7/16	1 1/2	1 9/16	1 11/16	1 3/4	1 7/8	2	2 1/16	2 3/16	2 5/16	2 3/8	2 1/2	2 5/8	2 3/4
1 1/4	1 9/16	1 5/8	1 11/16	1 3/4	1 7/8	1 15/16	2 1/16	2 1/8	2 1/4	2 3/8	2 7/16	2 9/16	2 11/16	2 13/16
1 3/8	1 5/8	1 11/16	1 3/4	1 7/8	1 15/16	2	2 1/8	2 3/16	2 5/16	2 7/16	2 1/2	2 5/8	2 3/4	2 7/8
1 1/2	1 3/4	1 13/16	1 7/8	1 15/16	2	2 1/8	2 3/16	2 5/16	2 3/8	2 1/2	2 5/8	2 11/16	2 13/16	2 15/16
1 5/8	1 7/8	1 7/8	2	2 1/16	2 1/8	2 3/16	2 5/16	2 3/8	2 1/2	2 9/16	2 11/16	2 3/4	2 7/8	3
1 3/4	1 15/16	2	2 1/16	2 1/8	2 3/16	2 5/16	2 3/8	2 7/16	2 9/16	2 5/8	2 3/4	2 7/8	2 15/16	3 1/16
1 7/8	2 1/16	2 1/8	2 3/16	2 1/4	2 5/16	2 3/8	2 1/2	2 9/16	2 5/8	2 3/4	2 13/16	2 15/16	3	3 1/8
2	2 3/16	2 1/4	2 5/16	2 3/8	2 7/16	2 1/2	2 9/16	2 5/8	2 3/4	2 13/16	2 15/16	3	3 1/8	3 3/16
2 1/8	2 5/16	2 5/16	2 3/8	2 7/16	2 1/2	2 5/8	2 11/16	2 3/4	2 13/16	2 15/16	3	3 1/16	3 3/16	3 1/4
2 1/4	2 7/16	2 7/16	2 1/2	2 9/16	2 5/8	2 11/16	2 3/4	2 7/8	2 15/16	3	3 1/16	3 3/16	3 1/4	3 3/8
2 3/8	2 1/2	2 9/16	2 5/8	2 11/16	2 3/4	2 13/16	2 7/8	2 15/16	3	3 1/8	3 3/16	3 1/4	3 3/8	3 7/16
2 1/2	2 5/8	2 11/16	2 3/4	2 13/16	2 7/8	2 15/16	3	3 1/16	3 1/8	3 3/16	3 1/4	3 3/8	3 7/16	3 9/16

Values below and to right of upper zigzag line are large enough for 3/4 in. rivets.
Values below and to right of lower zigzag line are large enough for 7/8-in. rivets.



Minimum Stagger for Rivets

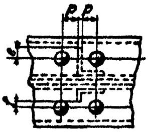
All dimensions in inches

Dia. of rivet	Minimum stagger, d																
	1 1/8	1 3/16	1 1/4	1 5/16	1 3/8	1 7/16	1 1/2	1 9/16	1 5/8	1 11/16	1 3/4	1 13/16	1 7/8	1 15/16	2 1/16	2 3/16	2 5/16
5/8	15/16	7/8	13/16	11/16	1/2	5/16	0										
3/4	1 1/4	1 3/16	1 1/8	1 1/16	15/16	7/8	3/4	9/16	3/8	0							
7/8	1 1/2	1 7/16	1 3/8	1 5/16	1 1/4	1 3/16	1 1/8	1	15/16	13/16	5/8	7/16	0				
1	1 13/16	1 3/4	1 11/16	1 5/8	1 9/16	1 1/2	1 7/16	1 3/8	1 5/16	1 9/16	1 1/8	1	7/8	3/4	0		
1 1/8	2 1/16	2	1 15/16	1 15/16	1 7/8	1 13/16	1 3/4	1 11/16	1 5/8	1 9/16	1 1/2	1 3/8	1 5/16	1 1/4	1	1 1/16	0

[1] From Pocket Companion 20th edition, Carnegie Steel Co., Pittsburgh, Pa.

Clearance for Cover Plate Riveting

Dimensions in inches



e	½	1	1½	2	2½	3	3½	4	4½	5	5½	6
p	2½	2⅝	2¾	2¾	2⅞	2⅞	3	3⅛	3⅛	3¼	3¼	3⅜
f	0	½	1	1½	2	2½						
p	2½	2¼	2⅛	2	1½	0						

5. Rivet Holes.—Depending upon the class of the work, the rivet holes are made in three different ways as follows: (1) Punched to the final size in a single operation; (2) punched under size or sub-punched and then increased in size by reaming; or (3) drilled. In order that the heated rivet may enter the hole easily the hole is made $\frac{1}{16}$ in. larger than the nominal diameter of the rivet in each of these three cases.

If a reasonable amount of care is taken in laying out and punching the holes, results sufficiently good for all ordinary work may be obtained in the single operation of punching to the required diameter. However, since there is an unavoidable injury to the metal surrounding the hole, in the better classes of work, sub-punching and reaming are often required. The punch used in this case should have a diameter not less than $\frac{3}{16}$ in. smaller than the nominal diameter of the rivet and then the hole should be increased in size by reaming until the diameter of the hole is $\frac{1}{16}$ in. larger than that of the rivet.

Tapered rods known as drift pins are used in assembling but their use in lining up rivet holes which do not match should not be tolerated because of the injurious effect upon the material surrounding the hole. Instead, the metal causing the difficulty should be reamed out. Such work cannot be considered as regular reamed work, however, since only a part of the metal surrounding part of the holes is removed.

Where the very highest class of work is desired, the rivet holes are drilled. Holes so made are truly cylindrical in form, are accurately centered, and the metal surrounding the hole is damaged less than in punching or in sub-punching and reaming.

6. Lap and Butt Joints.—Two kinds of joints are used in structural work—the lap joint and the butt joint (see Fig. 3). In the lap joint, shown in Fig. 3*a*,

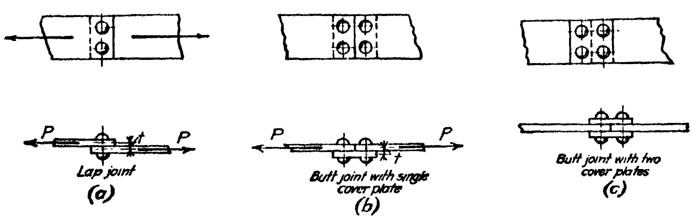


Fig. 3.

shearing stresses in the rivets along the plane of contact of the two plates are induced and in addition the pressures which the rivets and plates exert upon each other induce high *bearing stresses*. Since the plate is usually harder than the rivet, the resistance of the connection to this latter kind of action is limited by the bearing strength of the rivet. The rivets tend to shear along only one plane

and therefore are said to be in *single shear.* This is also true of the butt joint of Fig. 3*b*. Shearing stresses along two planes are induced in the butt joint shown in Fig. 3*c*, hence in this case the rivets are said to be in *double shear.*

The strength of the rivets in single shear is usually the factor which determines how many rivets shall be used in joints such as those shown in Figs. 3*a* and 3*b*. Ordinarily in such a joint as the one shown in Fig. 3*c* the bearing pressure will determine the number of rivets necessary. However, if this plate is comparatively thick, the strength of the rivets in double shear may control the

FIG. 4.

FIG. 5.

number required. The joint of Fig. 3*c* is the best one of the three, since the other two joints will deform as shown in Figs. 4 and 5 thereby causing some direct tension on the rivets.

7. Rivets vs. Bolts in Direct Tension.—In many specifications there is the statement that rivets shall not be subjected to direct tension but that turned bolts may be used in tension.

R. Fleming, Engineer for American Bridge Company, New York City, states that he does not think it is necessary to substitute bolts for rivets in tension in all cases—the kneebrace, for instance. Neither does he believe that it is necessary to use turned bolts, but that ordinary bolts will answer. He recommends the use of a unit stress of 7,000 lb. per sq. in. for rivets in tension and a unit stress of 9,000 lb. per sq. in. for bolts in tension.[1] Mr. Fleming states the value of 9,000 for bolts is specified in the present New York building code but that a value of 14,000 was given in the previous code which was copied into many other building codes.

In the building code recommended by the National Board of Fire Underwriters, fourth edition, 1920, the following recommendation is made:

When bolts are used in tension, the working stresses shall be reduced to 7,000 lb. per sq. in. of net area for steel, and to 5,000 lb. per sq. in. for wrought iron, and the load shall be transmitted into the head or nut by washers distributing the pressure evenly over the entire surface of the same.

8. Distribution of Stress in Riveted Joints.—In the design of riveted joints it is usually assumed that the stress is uniformly distributed over the rivets.

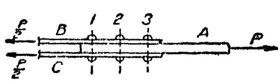


FIG. 6.

However, in a great many cases, the end rivets of a connection are subjected to a unit stress which is greater than that of any of the other rivets of the connection. This fact may be illustrated by a consideration of the joint shown in Fig. 6. Since plates B and C are each one-half as thick as A, the unit stress in all three plates will be the same and will be represented by f. If we assume, as is usually done, that the rivets are equally stressed, then each rivet carries a load equal to $\frac{F}{3}$. Then the unit stress in plate A between rivets 1 and 2 is equal to $\frac{1}{3}f$, and between rivets 2 and 3 it is equal to $\frac{2}{3}f$. The unit stress in plates B and C between rivets 1 and 2 is equal to $\frac{2}{3}f$ and between rivets 2 and 3 it is equal to $\frac{1}{3}f$. If we let l

[1] *Eng. News-Record*, Feb. 24, 1921, p. 336

represent the distance between rivets, then the total deformation in the length l is δl or $\frac{fl}{E}$. Therefore the deformation of plate A between rivets 1 and 2 is equal to $\frac{1}{3}\frac{fl}{E}$ and between rivets 2 and 3 it is equal to $\frac{2}{3}\frac{fl}{E}$. Likewise the deformation of plates B and C between rivets 1 and 2 is equal to $\frac{2}{3}\frac{fl}{E}$ and between rivets 2 and 3 it is equal to $\frac{1}{3}\frac{fl}{E}$. Then if we let k represent the distortion of rivet 1, the distortion at rivet 2 will be equal to k plus the deformation of plate A between rivets 1 and 2, minus the deformation of plates B and C in this same length. This is equal to

$$k + \frac{1}{3}\frac{fl}{E} - \frac{2}{3}\frac{fl}{E} = k - \frac{1}{3}\frac{fl}{E}$$

The distortion at rivet 3 will be equal to that at rivet 2, plus the stretch in plate A, minus the stretch in plates B and C. This is equal to

$$k - \frac{1}{3}\frac{fl}{E} + \frac{2}{3}\frac{fl}{E} - \frac{1}{3}\frac{fl}{E} = k$$

Therefore, the stresses in the first and last rivets are equal while the stress in the middle rivet is less. In general it will be found that the end rivets receive the most stress while those near the center receive the least stress. The results of tests confirm this conclusion.

Where more than two rows of rivets are used, an approach to uniformity of stress may be made by varying the cross-section of the cover plates as shown in Fig. 7.

In the above analysis no allowance is made for the frictional resistance which is developed by the clamping action of the rivets. In reality the load will reach a considerable amount before this frictional resistance is overcome. Until this occurs, there is no slipping of the plates and only a very slight distortion of the rivets. After the frictional resistance is overcome and slipping of the plate begins, an appreciable movement takes place before all the rivets come into action due to the fact that the rivets do not fit perfectly. From the point where all the rivets come into action to the yield point of the rivets or the members in tension, the deformation is about proportional to the load. The frictional strength of riveted joints varies from 7,000 to 12,000 lb. per sq. in. as given by tests. A value of about 10,000 lb. per sq. in. of rivet shear area was obtained in tests of both ordinary and nickel steel riveted joints made at the University of Illinois.

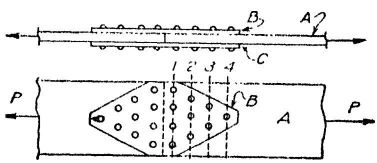


FIG. 7.

9. Splices in Tension Members.—In the design of splices for tension members allowance must be made for the reduction of area caused by rivet holes. Since the metal around a rivet hole is injured in the process of punching or drilling, it is customary to deduct for a hole ⅛ in. larger in diameter than the rivet. The areas to be deducted from plates of various thicknesses for rivet holes of different

diameters are given in Table 5. Where the rivets are staggered, it is necessary to consider the net area not only on a section normal to the axis of the member but also on an inclined section through a greater number of rivets. While adequate experimental data on this subject is lacking, the usual custom is to make either the inclined net section equal to the normal net section, or else to let the section giving the smaller net area govern the strength of the connection. The pitch necessary to give a diagonal net section equal to the normal net section is shown in Table 6.

Tests of angles connected by only one leg, where free bending of the angles has been allowed, have shown results which are only 75 to 80 per cent of the strength of the material. These results have been increased 5 to 10 per cent by connecting both legs. However, other tests in which the stress was applied along the gravity axis of the member and in which bending was prevented, have shown there was no advantage from connecting both legs of angles. Therefore we may conclude that in a case in which the angles are more or less free to bend, an advantage will be gained by the use of lug or clip angles, as shown in Fig. 8, whereas if the connection is such that bending of the angles is prevented, then little or nothing is gained by the use of lug angles. In any case the use of a lug will help distribute the stress over the member to which the angle is connected.

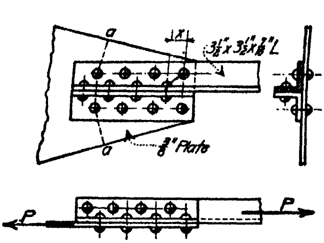


FIG. 8.

In the General Specifications for Steel Railway Bridges adopted in 1920 by the American Railway Engineering Association, the following unit stresses are recommended:

Shear in power-driven rivets and pins............... 12,000 lb. per sq. in.
Bearing on power-driven rivets, pins, outstanding legs
of stiffener angles, and other steel parts in contact... 24,000 lb. per sq. in.

The above-mentioned values for shear and bearing shall be reduced 25 per cent for countersunk rivets, hand-driven rivets, floor-connection rivets, and turned bolts.

Thus for rivets generally known as shop rivets we have unit stresses of 12,000 and 24,000 lb. per sq. in. and for rivets generally known as field rivets we have unit stresses of 9,000 and 18,000 lb. per sq. in. Values of 10,000, 11,000 and 12,000 lb. per sq. in. for rivets in shear, and values of 20,000, 22,000, and 24,000 lb. per sq. in. for rivets in bearing are generally used.

These three sets of values will be used in the examples in this section. In Table 7 are given the shearing and bearing values for rivets of various sizes and for plates of different thicknesses with these working stresses. In this table the single shear value is equal to the area of the rivet times the allowable unit shearing stress, and the bearing value is equal to the diameter of the rivet, times the thickness of the plate, times the allowable unit bearing stress.

If it is desired to use unit working stresses other than those given in the table, values similar to those tabulated therein may be obtained by proportion. For instance, suppose single shear is to be taken at 7,500 lb. per sq. in. and bearing at

Table 5.—Reduction of Area for Rivet Holes

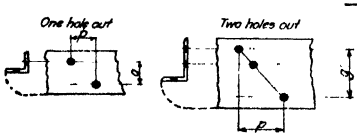
Area in square inches = diameter of hole × thickness of metal

Thickness of metal, in.	Diameter of hole, in. ¼	½	9/16	5/8	11/16	¾	13/16	7/8	15/16	1	1 1/16	1 1/8
3/16	0.05	0.09	0.11	0.12	0.13	0.14	0.15	0.16	0.18	0.19	0.20	0.21
¼	0.06	0.13	0.14	0.16	0.17	0.19	0.20	0.22	0.23	0.25	0.27	0.28
5/16	0.08	0.16	0.18	0.20	0.21	0.23	0.25	0.27	0.29	0.31	0.33	0.35
3/8	0.09	0.19	0.21	0.23	0.26	0.28	0.30	0.33	0.35	0.38	0.40	0.42
7/16	0.11	0.22	0.25	0.27	0.30	0.33	0.36	0.38	0.41	0.44	0.46	0.49
½	0.13	0.25	0.28	0.31	0.34	0.38	0.41	0.44	0.47	0.50	0.53	0.56
9/16	0.14	0.28	0.32	0.35	0.39	0.42	0.46	0.49	0.53	0.56	0.60	0.63
5/8	0.16	0.31	0.35	0.39	0.43	0.47	0.51	0.55	0.59	0.63	0.66	0.70
11/16	0.17	0.34	0.39	0.43	0.47	0.52	0.56	0.60	0.64	0.69	0.73	0.77
¾	0.19	0.38	0.42	0.47	0.52	0.56	0.61	0.66	0.70	0.75	0.80	0.84
13/16	0.20	0.41	0.46	0.51	0.56	0.61	0.66	0.71	0.76	0.81	0.86	0.91
7/8	0.22	0.44	0.49	0.55	0.60	0.66	0.71	0.77	0.82	0.88	0.93	0.98
15/16	0.23	0.47	0.53	0.59	0.64	0.70	0.76	0.82	0.88	0.94	1.00	1.05
1	0.25	0.50	0.56	0.63	0.69	0.75	0.81	0.88	0.94	1.00	1.06	1.13
1 1/16	0.27	0.53	0.60	0.66	0.73	0.80	0.86	0.93	1.00	1.06	1.13	1.20
1 1/8	0.28	0.56	0.63	0.70	0.77	0.84	0.91	0.98	1.05	1.13	1.20	1.27
1 3/16	0.30	0.59	0.67	0.74	0.82	0.89	0.96	1.04	1.11	1.19	1.26	1.34
1 ¼	0.31	0.63	0.70	0.78	0.86	0.94	0.02	1.09	1.17	1.25	1.33	1.41
1 5/16	0.33	0.66	0.74	0.82	0.90	0.98	1.07	1.15	1.23	1.31	1.39	1.48
1 3/8	0.34	0.69	0.77	0.86	0.95	1.03	1.12	1.20	1.29	1.38	1.46	1.55
1 7/16	0.36	0.72	0.81	0.90	0.99	1.08	1.17	1.26	1.35	1.44	1.53	1.62
1 ½	0.38	0.75	0.84	0.94	1.03	1.13	1.22	1.31	1.41	1.50	1.59	1.69

Table 6.—Stagger of Rivets to Maintain Net Section

American Bridge Company Standard

Dimensions in inches



d = diameter of rivet + ⅛ in.

$g - d = \sqrt{g^2 + p^2} - 2d \quad g' - 2d = \sqrt{(g')^2 + p^2} - 3d$

$p = \sqrt{2gd + d^2} \quad p = \sqrt{2g'd + d^2}$

g = sum of gages minus thickness of angle.

⅝-in. rivets, can be taken at ⅛ in. less than for ¾-in. rivets.

1-in. rivets, can be taken at ⅛ in. more than for ⅞-in. rivers

g	¾-in. rivet p	⅞-in. rivet p	g'	¾-in. rivet p	⅞-in. rivet p
1	1 ⅝	1 ¾	5	3 1/16	3 5/16
1½	1 ⅞	2	5½	3¼	3½
2	2 1/16	2 ¼	6	3⅜	3⅝
2½	2 ¼	2 7/16	6½	3½	3¾
3	2 7/16	2 ⅝	7	3⅝	3⅞
3½	2 9/16	2 13/16	7½	3¾	4
4	2 13/16	3	8	3⅞	4⅛
4½	2 15/16	3 3/16	8½	4	4¼

15,000 lb. per sq. in. The desired values may be obtained by multiplying those given in the first part of Table 7 by the ratio 0.75.

Illustrative Problem.—Find the number of $\frac{5}{8}$-in. rivets required to make a lap joint between two $2\frac{1}{2} \times \frac{3}{8}$-in. plates carrying a stress of 10,000 lb. Assume working stresses of 11,000 and 22,000 lb. per sq. in.

From Table 7 the resistance of each rivet in single shear is 3,370 lb. per sq. in. and in bearing is 5,160 lb. per sq. in. Therefore single shear governs and the number of rivets required is $\frac{10{,}000}{3{,}370} = 2.97$ or 3 rivets. On the basis of $\frac{3}{4}$-in. rivet holes, the area to be deducted for each rivet hole, as given in Table 5, is 0.28 sq. in. Then the net area of each plate at a rivet is $2\frac{1}{2} \times \frac{3}{8} - 0.28 = 0.94 - 0.28 = 0.66$ sq. in. Therefore the unit stress is $\frac{10{,}000}{0.66} = 15{,}150$ lb. per sq. in. Since this is less than the safe working stress of 16,000 lb. per sq. in., the design is satisfactory.

10. Splices in Compression Members.—Assuming that the rivet holes are completely filled by the rivets, compressive stress can be transmitted through the rivets and hence no deduction need be made and the total or gross area may be regarded as effective in transmitting stress. When a connection is made between two members carrying compressive stress, as at a splice in the top chord of a truss, the usual practice is to mill the abutting ends. Then splice plates with only a couple of rows of rivets each side of the joint are needed to hold the members in line. No reliance should be allowed on the abutting ends if they are not milled but enough rivets must be placed in the splice plates to transmit all the stress across the connection.

Columns used in building construction are made in one, two, or three story lengths. When columns one story in length are used, the designer may vary the size of the column with the load. In the case of columns two or three stories in length the section is constant throughout and sufficiently large to carry the load at the lower end. Too many connections are required when columns one story in length are used while columns three stories in length are difficult to erect. Hence the two story column is the one most frequently encountered in practice.

The column splice must be placed above the floor line far enough so that the splice plates will not interfere with the connections of beams to the column. Ordinarily the splice plates are not assumed to carry much stress but the ends of the abutting column sections are milled. Where a radical change of size or shape of section is made a bearing plate must be introduced.

11. Connection Angles.—Connections of beams to beams, of beams to girders, and of beams to columns are made by means of short lengths of angles, called *connection angles,* which are riveted to the members joined. There are two general types of angle connection—web connections and seat connections. In the former case the connection angles are riveted to the web of the member which is to be supported, as shown in Fig. 9. In the latter case the connection angle forms a seat upon which the beam to be supported may rest. This form of connection is shown in Fig. 10.

It has been found from experience that angles of a certain size and with a certain number of rivets may be used for the average conditions of loading for beams of a certain size. Therefore it has been found convenient to make a list of these connections and their limitations and to use them wherever possible. Such lists

TABLE 7.—SHEARING AND BEARING VALUES OF RIVETS

Values above or to the right of upper zigzag lines are greater than double shear. Values below or to the left of lower zigzag lines are less than single shear.

Diam. of rivet	Area in sq in.	Single shear at 10,000	Bearing values for different thicknesses of plate at 20,000 lb. per sq. in.												
			1/4	5/16	3/8	7/16	1/2	9/16	5/8	11/16	3/4	13/16	7/8	15/16	1
3/8	0.1104	1,100	1,880	2,340	2,810	3,280									
1/2	0.1963	1,960	2,500	3,130	3,750	4,380	5,000								
5/8	0.3068	3,070	3,130	3,910	4,690	5,470	6,250	7,030	7,810						
3/4	0.4418	4,420	3,750	4,690	5,630	6,560	7,500	8,440	9,380	10,310	11,250				
7/8	0.6013	6,010	4,380	5,470	6,560	7,660	8,750	9,840	10,940	12,030	13,130	14,220	15,310	16,410	
1	0.7854	7,850	5,000	6,250	7,500	8,750	10,000	11,250	12,500	13,750	15,000	16,250	17,500	18,750	20,000

Diam. of rivet	Area in. sq. in.	Single shear at 11,000	Bearing values for different thicknesses of plate at 22,000 lb. per sq. in.												
			1/4	5/16	3/8	7/16	1/2	9/16	5/8	11/16	3/4	13/16	7/8	15/16	1
3/8	0.1104	1,210	2,070	2,570	3,090	3,610									
1/2	0.1963	2,160	2,750	3,440	4,130	4,820	5,500								
5/8	0.3068	3,370	3,440	4,300	5,160	6,020	6,880	7,740	8,600						
3/4	0.4418	4,860	4,130	5,160	6,190	7,220	8,250	9,280	10,320	11,340	12,380				
7/8	0.6013	6,610	4,810	6,020	7,220	8,430	9,630	10,840	12,040	13,240	14,440	15,640	16,840	18,050	
1	0.7854	8,640	5,500	6,880	8,250	9,630	11,000	12,380	13,750	15,130	16,500	17,880	19,250	20,630	22,000

Diam. of rivet	Area in sq. in.	Single shear at 12,000	Bearing values for different thicknesses of plate at 24,000 lb. per sq. in.												
			1/4	5/16	3/8	7/16	1/2	9/16	5/8	11/16	3/4	13/16	7/8	15/16	1
3/8	0.1104	1,320	2,250	2,810	3,380	3,940									
1/2	0.1963	2,360	3,000	3,750	4,500	5,250	6,000								
5/8	0.3068	3,680	3,750	4,690	5,630	6,560	7,500	8,440	9,380						
3/4	0.4418	5,300	4,500	5,630	6,750	7,880	9,000	10,130	11,250	12,380	13,500				
7/8	0.6013	7,220	5,250	6,560	7,880	9,190	10,500	11,810	13,130	14,440	15,750	17,060	18,380		
1	0.7854	9,430	6,000	7,500	9,000	10,500	12,000	13,500	15,000	16,500	18,000	19,500	21,000	22,500	24,000

of connections are called *standard connections*. A set of standard beam connections is shown in Fig. 11. The limiting values of these beam connections are given in Table 8. The use of standard connections reduces the number of different connections which must be designed and fabricated, and hence simplifies the work in the office and in the shop. Of course, these standard connections cannot be used in all cases, such as for very short beams or very heavy concentrations of load. Wherever there is any doubt as to the strength of a connection for

TABLE 8.[1]—LIMITING VALUES OF BEAM CONNECTIONS

I-beams		Value of web connection	Values of outstanding legs of connection angles: Field rivets			Values of outstanding legs of connection angles: Field bolts		
Inches depth	Weight, lb. per ft.	Shop rivets in enclosed bearing, lb.	3/4-in. rivets or turned bolts, single shear, lb.	Minimum allowable span in feet, uniform load	t, in.	3/4-in. rough bolts, single shear, lb.	Minimum allowable span in feet, uniform load	t, in.
27	90	82,530	61,900	18.9	5/8	49,500	23.6	5/8
24	79.9	67,500	53,000	17.5	5/8	42,400	21.9	5/8
	74.2	64,260	53,000	16.4	5/8	42,400	20.4	5/8
21	60.4	48,150	44,200	14.2	5/8	35,300	17.8	5/8
20	65.4	45,000	35,300	17.6	5/8	28,300	22.1	5/8
18	54.7	41,400	35,300	13.3	5/8	28,300	16.7	5/8
	48.2	34,200	35,300	12.8	9/16	28,300	15.4	5/8
15	42.9	36,900	35,300	8.9	5/8	28,300	11.1	5/8
	37.3	29,880	35,300	9.7	1/2	28,300	10.2	9/16
12	31.8	23,600	26,500	8.1	9/16	21,200	9.0	5/8
	27.9	19,170	26,500	9.2	7/16	21,200	9.2	1/2
10	25.4	27,900	17,700	7.4	5/8	14,100	9.2	5/8
	22.4	22,680	17,700	6.8	5/8	14,100	8.6	5/8
9	21.8	26,100	17,700	5.7	5/8	14,100	7.1	5/8
8	18.4	24,300	17,700	4.3	5/8	14,100	5.4	5/8
	17.5	19,800	17,700	4.4	5/8	14,100	5.5	5/8
7	15.3	11,300	8,800	6.2	5/8	7,100	7.8	5/8
6	12.5	10,400	8,800	4.4	5/8	7,100	5.5	5/8
5	10.0	9,500	8,800	2.9	5/8	7,100	3.6	5/8
4	7.7	8,600	8,800	2.2	9/16	7,100	2.7	5/8
3	5.7	7,700	8,800	1.3	1/2	7,100	1.4	5/8

ALLOWABLE UNIT STRESS IN POUNDS PER SQUARE INCH

Single shear	Rivets	Shop 12,000	Bearing	Rivets—enclosed	Shop 30,000
	Rivets and turned bolts	Field 10,000		Rivets—one side	Shop 24,000
	Rough bolts	Field 8,000		Rivets and turned bolts	Field 20,000
				Rough bolts	Field 16,000

t = Web thickness, in bearing, to develop max. allowable reactions, when beams frame opposite.

Connections are figured for bearing and shear (no moment considered).

The above values agree with tests made on beams under ordinary conditions of use.

Where web is enclosed between connection angles (enclosed bearing), values are greater because of the increased efficiency due to friction and grip.

Special connections shall be used when any of the limiting conditions given above are exceeded—such as end reaction from loaded beam being greater than value of connection; shorter span with beam fully loaded; or a less thickness of web when maximum allowable reactions are used.

[1] From Pocket Companion, 22nd edition, Carnegie Steel Co., Pittsburgh, Pa.

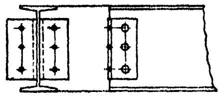


FIG. 9.

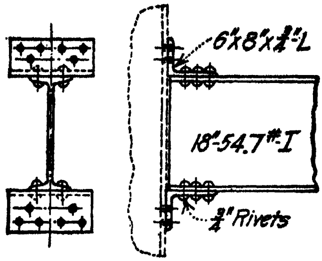
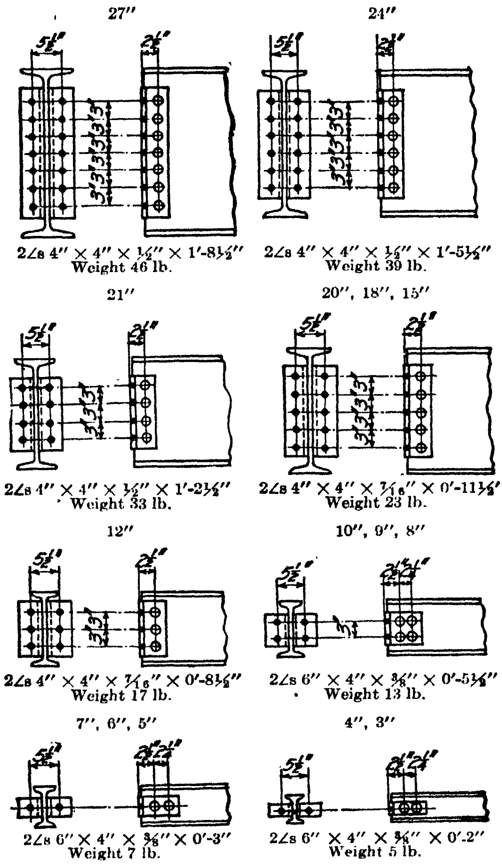


FIG. 10.



Rivets and bolts ¾-in. diameter.

Weights given are for ¾-in. shop rivets and angle connections; about 20 per cent should be added for field rivets or bolts.

FIG. 11.—Beam connections.

a given set of conditions, an investigation should be made. Draftsmen who have much of this sort of work to do, soon become familiar with the limitations of the various standard connections, and since they may be used in the majority of cases, a considerable saving of time and expense is effected by their adoption.

Seat angles are often used in erection for supporting beams to be carried by web connections. Since the two connections cannot be assumed to act simultaneously, the web connection is designed for the full load and no allowance is made for the strength of the seat angle. When a beam, however, is to be *supported* by a seat connection, it must be held rigidly against lateral displacement either by web or top angles. As in the preceding case, no allowance is made for the strength of these auxiliary angles, but the seat connection is designed to carry the whole load.

12. Eccentric Connections.—In order that the stress at the connection of a member shall be uniformly distributed, the line of action of the stress in the member must pass through the center of gravity of the group of rivets. If this is not the case, a bending moment will be produced at the joint, and instead of the stress on the rivets being uniform it will be variable and will be made up of two components, one due to the direct stress and the other due to the moment stress. The stress in the rivets due to the direct force will be uniform while the stress in a given rivet due to the bending moment will vary with the distance of the rivet from the center of gravity of the group. Therefore, one of the outside rivets will receive the greatest resultant of direct stress and moment stress. The strength of the connection will be governed by the fact that this resultant stress must not exceed the allowable working stresses in shear, bearing, or tension.

Let P = magnitude of the total force.
e = the distance from the line of action of P to the center of gravity of the group of rivets.
n = number of rivets in the group.
x, y = coordinates of any rivet referred to the center of gravity of the group of rivets as an origin.
z = distance of any rivet from origin = $\sqrt{x^2 + y^2}$.
X, Y = coordinates of rivet receiving maximum resultant stress.
Z = distance from rivet receiving maximum resultant stress to origin = $\sqrt{(X)^2 + (Y)^2}$
f_p = direct stress on each rivet.
f_o = moment stress on a rivet at a unit distance from the origin.
f_m = moment stress on any rivet = f_oz

The moment of resistance of the moment stress on any rivet is

$$f_mz = f_oz^2 = f_o(x^2 + y^2)$$

The total resisting moment must equal the total turning moment or

$$f_o(\Sigma x^2 + \Sigma y^2) = Pe, \quad f_o = \frac{Pe}{\Sigma x^2 + \Sigma y^2}$$

The moment stress on the rivet receiving the maximum resultant stress is

$$f_m = f_oZ = \frac{PeZ}{\Sigma x^2 + \Sigma y^2} = \frac{Pe}{\dfrac{\Sigma x^2 + \Sigma y^2}{Z}} \tag{1}$$

and the direct stress on this rivet is

$$f_p = \frac{P}{n}$$

Then the resultant stress in shear or bearing f_r on the rivet receiving maximum stress is the resultant of f_m and f_p. Let X and Y be the coordinates of such a rivet with the center of gravity of the group regarded as the origin, and let Z be the slant distance from the origin to the rivet. From Fig. 12 the horizontal component of f_r equals that of f_m, or is equal to $\frac{Y}{Z} f_m$. The vertical component of f_r equals f_p plus the vertical component of f_m, or is equal to $f_p + \frac{X}{Z} f_m$. Then

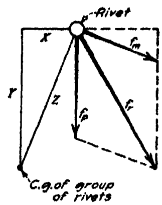


Fig. 12.

$$f_r = \sqrt{\left(f_p + \frac{X}{Z} f_m\right)^2 + \left(\frac{Y}{Z} f_m\right)^2}$$

Since $X^2 + Y^2 = Z^2$, we find upon reducing that

$$f_r = \sqrt{(f_p)^2 + 2\frac{X}{Z} f_p f_m + (f_m)^2} \tag{3}$$

The maximum value of f_r occurs when f_p and f_m are colinear. Then

$$f_r = f_p + f_m \tag{4}$$

Since the value of $\frac{X}{Z}$ is not known until after the number and arrangement of rivets have been determined, it will be found convenient to assume that $\frac{X}{Z} = 1$ (hence $f_r = f_p + f_m$), and to make a preliminary design based upon this assumption. Substituting $\frac{P}{n}$ for f_p and $\frac{Pe}{\frac{\Sigma x^2 + \Sigma y^2}{Z}}$ for f_m in eq. (4), we get

$$f_r = \frac{P}{n} + \frac{Pe}{\frac{\Sigma x^2 + \Sigma y^2}{Z}}$$

or

$$n = \frac{1}{\frac{f_r}{P} - \frac{e}{\frac{\Sigma x^2 + \Sigma y^2}{Z}}}$$

Neglecting the quantity

$$\frac{e}{\frac{\Sigma x^2 + \Sigma y^2}{Z}}$$

we have

$$n = \frac{P}{f_r} \tag{5}$$

The value of f_r may be obtained from Table 7 for the given rivet size and working stresses.

Values of the quantity $\frac{\Sigma x^2 + \Sigma y^2}{Z}$ of eq. (1) have been calculated for fifteen different typical arrangements of rivets and for rivet spacings varying in both

directions from 2 in. to 6 in. and have been plotted in Diagram 1. In this diagram h represents the number of rows of rivets in one direction, v represents the number of rows of rivets in the other direction, while g and p represent the distances in inches between the rows of rivets in the first and second directions respectively as indicated in the typical groups of rivets given above the curves of the diagram. Therefore, in any case, the coordinates of the rivet receiving maximum stress are

$$X = \frac{(v - 1)g}{2} \tag{6}$$

$$Y = \frac{(h - 1)p}{2} \tag{7}$$

and

$$Z = \sqrt{X^2 + Y^2} \tag{8}$$

In the design of connections, the approximate number of rivets is found by substituting in eq. (5). In this equation f_r is the value of the allowable stress in shear or bearing for the size of rivets and the thickness of metal under consideration. When the arrangement of the rivets has been assumed, the value of the resultant stress f_r can be found by substituting in eq. (3). If this value of f_r is within a few per cent of the value of the allowable stress, the assumed design is satisfactory. If this is not the case the assumed design may be altered until a satisfactory design is obtained.

Illustrative Problem.—Design a connection for a direct stress of 90,000 lb. and a bending moment of 60,000 in.-lb. on the assumption that the governing stress f_r is 6,010 lb. given by ⅞-in. rivets in single shear at a unit stress of 10,000 lb. per sq. in.

From eq. (5) $n = \frac{90,000}{6,010} = 15$ rivets. Let us assume that $n = 15$ rivets, $v = 3$ rows, $h = 5$ rows, $p = 4$ in. and $g = 4$ in. From Diagram 1

$$\frac{\Sigma x^2 + \Sigma y^2}{Z} = 72 \text{ in.}$$

Substituting in eqs. (1) and (2)

$$f_m = \frac{60,000}{72} = 833 \text{ lb.}$$

$$f_p = \frac{90,000}{15} = 6,000 \text{ lb.}$$

From eqs. (6), (7) and (8)

$$X = \frac{(3 - 1)4}{2} = 4 \text{ in.}$$

$$Y = \frac{(5 - 1)4}{2} = 8 \text{ in.}$$

$$Z = \sqrt{(4)^2 + (8)^2} = 8.95 \text{ in.}$$

Substituting the above values in eq. (3)

$$f_r = \sqrt{(6,000)^2 + 2\left(\frac{4}{8.95}\right)(6,000)(833) + (833)^2}$$

$$= \sqrt{41,164,000} = 6,410 \text{ lb.}$$

This value is greater than the allowable stress. If we change from $p = 4$ in. and $g = 4$ in. to $p = 6$ in. and $g = 6$ in., the maximum spacing, $\frac{\Sigma x^2 + \Sigma y^2}{Z} = 107$ in. from Diagram 1. and $f_m = 560$ lb., $f_p = 6,000$ lb., $X = 6$ in., $Y = 12$ in., $Z = 13.4$ in., and $f_r = 6,270$ lb. This value also is greater than the allowable stress. Therefore, let us assume that $n = 16$ rivets, $v = 4$ rows, $h = 4$ rows, $p = 4$ in., and $g = 4$ in. From Diagram 1 $\frac{\Sigma x^2 + \Sigma y^2}{Z} =$

75 in. Then f_m = 880 lb., f_p = 5,625 lb., X = 6 in., Y = 6 in., Z = 8.49 in. and f_r = 6,220 lb. Since this value also is too large, we shall try p = 6 in., g = 6 in. In this case $\frac{\Sigma x^2 + \Sigma y^2}{Z}$ = 113 in., and consequently f_m = 531 lb. Also f_p = 5,625 lb., X = 9 in., Y = 9 in., Z = 12.73 in., and f_r = 6,010 lb. This design is satisfactory.

The quantity $\frac{\Sigma x^2 + \Sigma y^2}{Z}$ for a group of two or more rows of rivets becomes $\frac{\Sigma y^2}{Y}$ for a single row of rivets. Values of this quantity for a row of rivets varying from 2 rivets to 24 rivets in length, and for rivet spacings varying from 2 in. to 6 in. are given in Diagram 2. The lower left-hand corner of Diagram 2 is shown to an enlarged scale in Diagram 3.

In either of the above cases, namely, one row of rivets, and two or more rows of rivets, it is assumed that the line of action of f_p is parallel to a row of rivets. If this is not true the direct stress may be resolved into components parallel to the rows of rivets and the moment stress f_m. Then the design may be made according to the methods which have been given.

In addition to causing moment stresses in the rivets in a connection, eccentricity causes bending stresses in the members connected. If a relatively flexible member is connected to a rigid member, then the greater proportion of the bending moment due to eccentricity is resisted by the more rigid member and its connection. The bending stresses in the more flexible members are occasionally very high and therefore eccentric connections should be avoided whenever practicable. Of course, in many cases it is necessary to use eccentric connections but sometimes they are used in places, where, with slight modifications, concentric connections might be obtained.

13. Bracket Connections.—In the preceding article methods have been given for the design of connections subjected to direct stress and a bending moment acting in the plane of the connection. Quite frequently it becomes necessary to design a connection which will withstand a shearing force acting in the plane of the connection and a bending moment acting in a plane perpendicular to that of the connection. In such a connection all of the rivets are stressed in shear and bearing by the shearing force and most of the rivets are stressed in direct tension due to the bending moment. The ordinary bracket connections are of this type. Many connections of beams to girders and columns are also of this type and may be designed according to the methods given in this article.

In the cases where the rivets are subjected to shearing or bearing stresses we can assume that the center of gravity of a group of rivets is the center of rotation and Diagrams 1, 2 and 3 are based on this assumption. However, in the case of a connection in which the rivets are in direct tension the center of rotation will move toward the portion of the connection which is in compression due to the fact that the tensile deformation is greater than the compressive deformation. This movement of the center of rotation will continue until, in the limiting case, the end rivet on the compressive portion of the connection becomes the center of rotation. Diagrams 4 and 5 have been prepared for use in this case.

Since the shearing stress and the tensile stress in the rivets produced by the shear and the bending moment respectively are principal stresses (see Art. **53**, Sec. 1), the presence or absence of one has no effect on the other. Hence it is

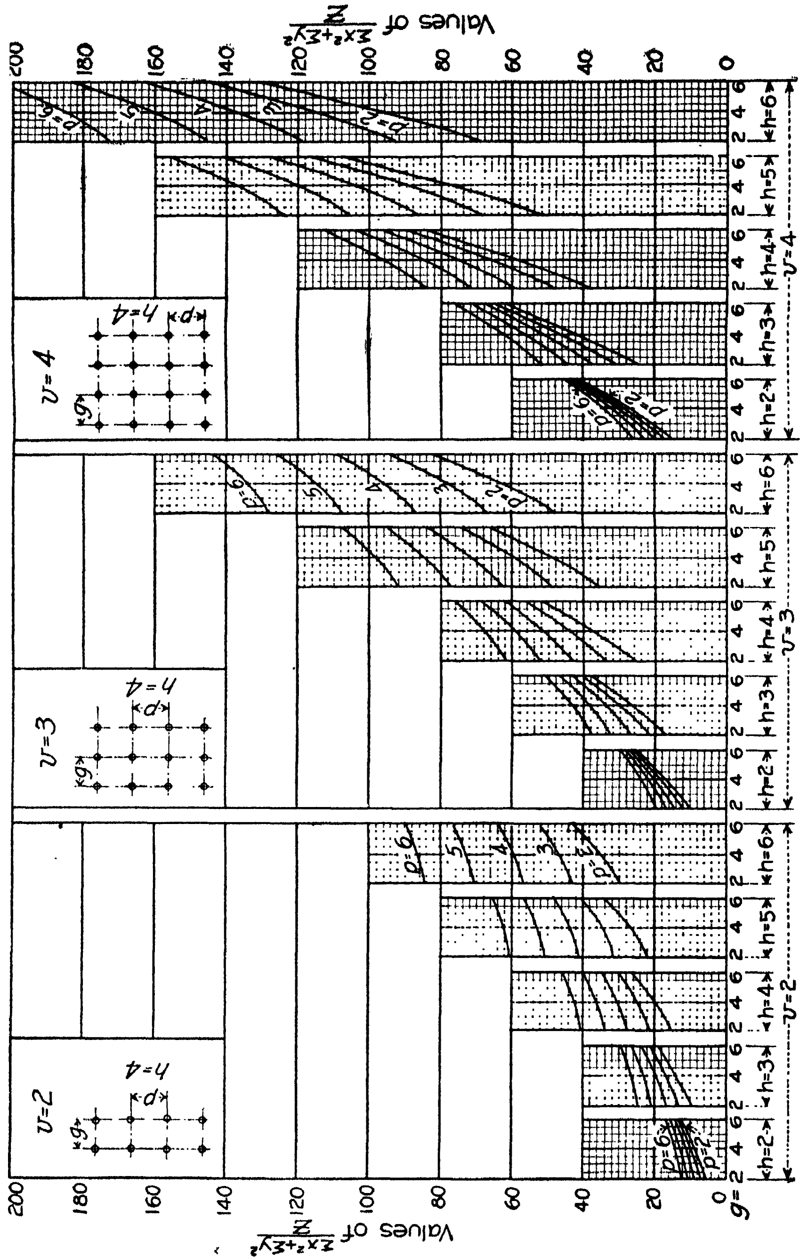

Values of $\frac{\Sigma}{\Sigma x^2+\Sigma y^2}$
v=2
v=3
v=4
h=4
h=2
h=3
h=4
h=5
h=6
g=

DIAGRAM 1.

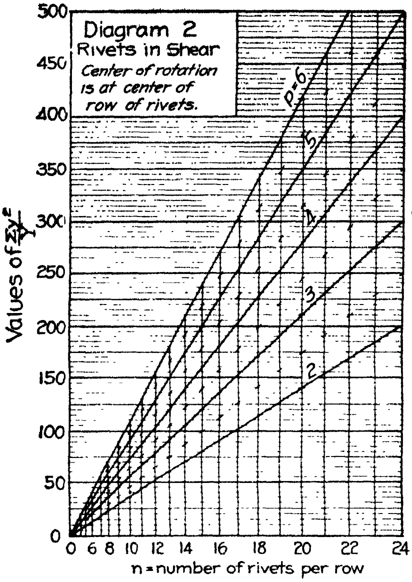


Diagram 2
Rivets in Shear
Center of rotation is at center of row of rivets.
Values of $\frac{\Sigma y^2}{Y}$
n = number of rivets per row
p=6
5
4
3
2
0
50
100
150
200
250
300
350
400
450
500
0 6 8 10 12 14 16 18 20 22 24

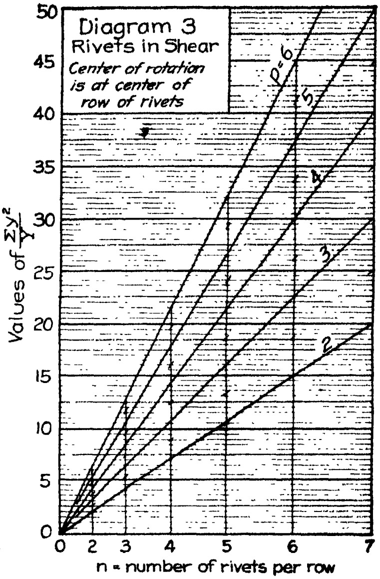


Diagram 3
Rivets in Shear
Center of rotation is at center of row of rivets
Values of $\frac{\Sigma y^2}{Y}$
n = number of rivets per row
p=6
5
4
3
2
0
5
10
15
20
25
30
35
40
45
50
0 2 3 4 5 6 7

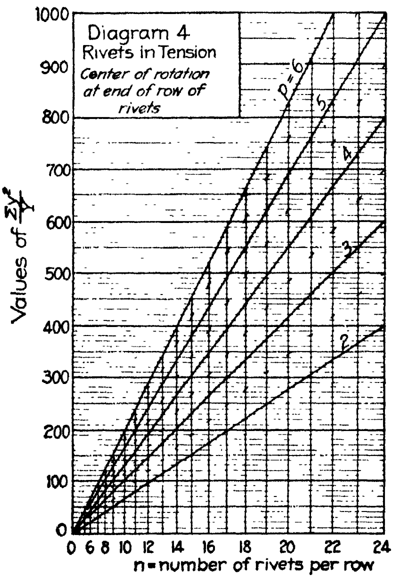


Diagram 4
Rivets in Tension
Center of rotation at end of row of rivets
Values of $\frac{\Sigma y^2}{Y}$
n = number of rivets per row
p=6
5
4
3
2
0
100
200
300
400
500
600
700
800
900
1000
0 6 8 10 12 14 16 18 20 22 24

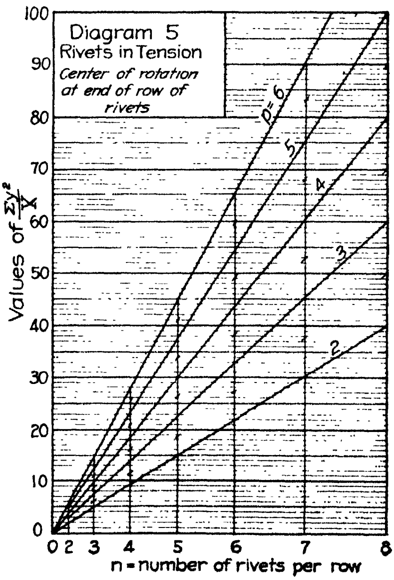


Diagram 5
Rivets in Tension
Center of rotation at end of row of rivets
Values of $\frac{\Sigma y^2}{Y}$
n = number of rivets per row
p=6
5
4
3
2
0
10
20
30
40
50
60
70
80
90
100
0 2 3 4 5 6 7 8

only necessary to design the connection for either the shear or the bending moment, whichever requires the larger number of rivets.

Due to the fact that the moment acts in a plane perpendicular to that of the connection, the factor $\frac{\Sigma x^2 + \Sigma y^2}{Z}$ of Art. **12** becomes $\frac{\Sigma y^2}{Y}$ for each row of rivets in a line parallel to the plane of the moment.

If V is the shear, M is the moment, f_v the stress in shear or bearing on the most highly stressed rivet and f_t the tensile stress on this rivet, then $f_v = \frac{V}{n}$ and $f_t = \frac{M}{\frac{\Sigma y^2}{Y}}$, or, for purposes of design

$$n = \frac{V}{f_v} \tag{9}$$

and

$$\frac{\Sigma y^2}{Y} = \frac{M}{f_t} \tag{10}$$

In this case values of the allowable stress per rivet in shear or bearing and in tension should be substituted for f_v and f_t respectively.

Illustrative Problem.—Design a connection for a shear of 50,000 lb. and a bending moment of 400,000 in.-lb. acting in a plane perpendicular to that of the connection. Assume the stress per rivet in shear is 4,420 lb. given by ¾-in. rivets at a unit stress of 10,000 lb. per sq. in. and assume the stress per rivet in tension is 3,100 lb. given by ¾-in. rivets at a unit stress of 7,000 lb. per sq. in. Only one row of rivets is to be used.

Substituting in eqs. (9) and (10)

$$n = \frac{50{,}000}{4{,}420} = 11.3 \text{ rivets.} \quad \text{At least 12 must be used.}$$

$$\frac{\Sigma y^2}{Y} = \frac{400{,}000}{3{,}100} = 129 \text{ in.}$$

When $n = 12$ rivets, and $\frac{\Sigma y^2}{Y} = 129$ in. we find from Diagram 4 that $p = 3$ in.

Illustrative Problem.—Design a connection for the same conditions as given in the preceding problem except that two rows of rivets are to be used.

The least number of rivets is 12 as before. With two row of rivets, $2\frac{\Sigma y^2}{Y} = 129$ in., or $\frac{\Sigma y_1}{Y} = 64.5$ in. for 6 rivets. From Diagram 5 we note that $p = 6$ in.

Assume that another condition is added that the connection shall be made as compact as possible.

In the first solution the height of the connection is $(n - 1)p = (12 - 1)3 = 33$ in. In the second solution the height is $\left(\frac{n}{2} - 1\right)p = \left(\frac{12}{2} - 1\right)6 = 30$ in. If we assume that the minimum spacing of 2.5 in. for ¾-in. rivets is used with $\frac{\Sigma y^2}{Y} = 64.5$ in. for two rows of rivets, we find from Diagram 4 that there must be 9 rivets in each row or 18 rivets altogether. The height of this connection is $\left(\frac{n}{2} - 1\right)p = \left(\frac{18}{2} - 1\right)2.5 = 20$ in.

14. Design of Connections Subjected to Bending, Direct Stress, and Shear.—The connection of a steel mill building column to its base is quite often subjected to bending, direct stress, and shear due to the vertical dead load and the horizontal wind load. In the ordinary case of bending, direct stress, and shear, all three act

in the plane of the connection and the direct stress and shear are perpendicular to each other. Then this case is the same as that of Art. **12** except that we have the shearing force added.

If we assume that the shear is the force acting parallel to the Y-axis and that the direct stress is the force acting parallel to the X-axis then the forces acting on the rivet which receives the maximum stress are as indicated in Fig. 13.

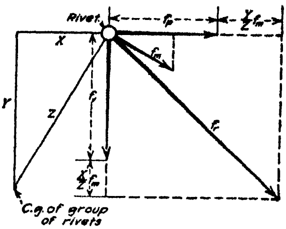


Fig. 13.

For the purpose of computing the resultant stress f_r, we may resolve f_m into two components parallel to f_p and f_v. Then the horizontal component of f_r is $f_p + \frac{Y}{Z} f_m$, and the vertical component of f_r is $f_v + \frac{X}{Z} f_m$. Therefore

$$f_r = \sqrt{\left(f_p + \frac{Y}{Z} f_m\right)^2 + \left(f_v + \frac{X}{Z} f_m\right)^2}$$

Since $X^2 + Y^2 = Z^2$, we find upon reducing that

$$f_r = \sqrt{(f_p)^2 + (f_v)^2 + (f_m)^2 + \frac{2}{Z} f_m\,(Y f_p + X f_v)} \tag{11}$$

If P = direct stress, V = shear, and f_r = allowable stress per rivet in shear or bearing, then in selecting a trial design let $n = \dfrac{P + \frac{V}{2}}{f_r}$ when P is greater than V, and let $n = \dfrac{V + \frac{P}{2}}{f_r}$ when V is greater than P. Then from Art. **12**, eq. (1)

$$f_m = \frac{Pe}{\dfrac{\Sigma x^2 + \Sigma y^2}{Z}}$$

From eq. (2), $f_p = \dfrac{P}{n}$; from Art. **13**, eq. (9), $f_v = \dfrac{V}{n}$, and values of X, Y and Z are obtained from eqs. (6), (7), and (8) of Art. **12.** When these quantities are determined they may be substituted in eq. (11), from which the true value of f_r is found. If this value is not close to the allowable stress, the result will indicate which way the design should be changed.

Illustrative Problem.—Design a connection for a direct stress of 55,000 lb., a bending moment of 30,000 in.-lb., and a shear of 20,000 lb. Assume that each rivet can take a stress of 5,300 lb. given by ¾-in. rivets in single shear at a unit stress of 12,000 lb. per sq. in.

Substituting in the equation

$$n = \frac{P + \frac{V}{2}}{f_r},$$

we have

$$n = \frac{50,000 + \frac{20,000}{2}}{5,300} = 11.3 \text{ rivets}$$

Let us assume that $n = 12$ rivets, $v = 3$ rows, $h = 4$ rows, $g = 4$ in., and $p = 4$ in. From Diagram 1, $\dfrac{\Sigma x^2 + \Sigma y^2}{Z} = 51$ in. and from eqs. (6), (7), and (8) $X = \dfrac{(3-1)4}{2} = 4$ in., $Y = \dfrac{(4-1)4}{2} = 6$ in., and $Z = \sqrt{(4)^2 + (6)^2} = 7.21$ in.

If $\dfrac{\Sigma x^2 + \Sigma y^2}{Z} = 51$ in., from eq. (1)

$$f_m = \frac{30,000}{51} = 590 \text{ lb.}$$

From eq. (2)

$$f_p = \frac{55,000}{12} = 4,580 \text{ lb.}$$

and from eq. (9)

$$f_v = \frac{15,000}{12} = 1,250 \text{ lb.}$$

Substituting these values in eq. (11) we have

$$f_r = \sqrt{(4,580)^2 + (1,250)^2 + (590)^2 + \left(\frac{2}{7.21}\right)(590)[6(4,580) + 4(1,250)]}$$
$$= 5,310 \text{ lb.}$$

This design is satisfactory.

15. Design of Connections Subjected to Torsion.—Suppose that I-beams carrying a floor load are riveted to one side of the channel girder shown in Fig. 14. Then torsion is induced in the channel due to the end moments of the floor beams. Therefore the connection shown in Fig. 14 is subjected to torsion. In the ordinary case of a connection subjected to torsion the connection must be designed for bending moments in two mutually perpendicular planes in addition to a shear or a direct stress. One of these moments which we shall call the torsional moment acts in a plane perpendicular to that of the connection and hence produces tensile stresses in most of the rivets of the connection. The other moment, which we shall call the bending moment, acts in the plane of the connection and hence produces shearing or bearing stresses in the rivets of the connection. The shear or direct stress which is present also produces shearing or bearing stresses in the rivets of the connection.

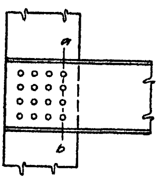


FIG. 14.

The connection may be designed to resist bending and direct stress by the method of Art. **12** and then its resistance to torsional moment may be investigated by the method of Art. **13.** The designer should use a lower working stress for the torsional moment which produces tension in the rivets than for the bending and direct stress which produce shearing or bearing stresses in the rivets.

Due to the lack of adequate experimental data the exact nature of the stress distribution in a riveted joint subjected to torsional moment is not known. However, if a connection such as the one shown in Fig. 14 is acted upon by a torsional moment, it is quite evident that the row of rivets nearest the force producing the torque, i.e., the row of rivets on line ab must carry more of this torque than any other single row. Therefore if v represents the number of rows of rivets resisting the torsional moment, it is recommended that the row of rivets nearest the force producing the torque T (row ab of Fig. 14) be designed for a torsional moment of $\dfrac{1.5T}{v}$ instead of a torsional moment of $\dfrac{T}{v}$ which would result on the assumption of a uniform distribution. This means that with two rows of

rivets, one row must carry 75 per cent of the total torsional moment; with three rows, one row must carry 50 per cent; and with six rows, one row must carry 25 per cent.

Illustrative Problem.—Design a connection for a direct stress of 90,000 lb., a bending moment of 60,000 in.-lb., and a torque of 20,000 in.-lb. Assume that the working stresses are 10,000 lb. per sq. in. and 7,000 lb. per sq. in. in shear and tension respectively, and that ⅞-in. rivets are used. Therefore $f_r = 6{,}010$ lb. and $f_t = 4{,}210$ lb.

A connection was designed in Art. **12** for the bending and direct stress specified. There it was decided to make $n = 16$ rivets, $v = 4$ rows, $h = 4$ rows, $p = 6$ in., and $g = 6$ in. From Diagram 5 we note that $\frac{\Sigma y^2}{Y} = 28$ in. when $v = 4$ in. and $p = 6$ in. The torsional moment which one row of rivets must carry is

$$M = \frac{1.5T}{v}$$

$$= \frac{1.5(20{,}000)}{4} = 7{,}500 \text{ in.-lb.}$$

From eq. (10) of Art. **13** we have

$$f_t = \frac{M}{\frac{\Sigma y^2}{Y}} = \frac{7{,}500}{28} = 268 \text{ lb.}$$

This design is satisfactory.

16. Plate Girder Web Splices.—Plate girder web splices are necessary when it is impossible to get the size of web plate desired or when it is inadvisable to ship the web plate from the shop to the place of erection in one piece. Both web and flange splices are almost invariably made in the form shown in Fig. 3c. The thickness of each cover plate is made half the thickness of the part which is spliced. A plate girder web splice should be made equivalent to the web net section as nearly as possible. In order that this may be accomplished the splice must be designed for shearing and bending stresses. It is reasonable to assume that all rivets are stressed equally by the vertical shear and that the stress on each rivet due to the bending moment is in direct proportion to its distance from the neutral axis. Therefore the best splice for the portion of the web plate between the flange angles is one in which the rivets are uniformly spaced as in Fig. 15. There are several ways of transmitting the stress in that portion of the web plate underneath the flange angles. One way is to use splice plates along the flanges as in Fig. 15. Sometimes it is possible to locate the splice at a section where the flange area is enough in excess of that actually required so that the excess area can be assumed to carry this stress. Another way is to create excessive flange area intentionally by extending a cover plate beyond its theoretical point of cutoff far enough to lap over the web splice.

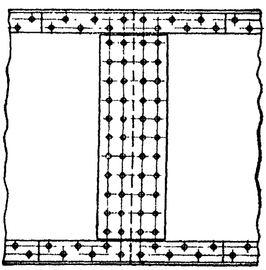


FIG. 15.

Since the resistance of a rivet to bending moment is proportional to its distance from the neutral axis, some designers have reasoned that a more effective web splice would result if the rivets of Fig. 15 were rearranged as shown in Fig. 16. Undoubtedly the moment of resistance of the rivets in the latter case is greater

than in the former case. However, the variable rivet spacing of the splice of Fig. 16, disturbs the natural linear distribution of moment stress in the web plate since it produces higher stresses near the flange angles and lower stresses near the neutral axis than would normally exist in the web plate. Therefore such a design is not in accordance with the principle that the web splice should be made equivalent to the web net section as nearly as possible.

The web splice of Fig. 17 is also a result of the idea that a splice is improved by placing the rivets as far as possible from the neutral axis. In this form of splice the total shear is carried by the rivets in group *efgh* while all of the rivets carry moment stress. The stress in the portion of the web underneath the flange angles may be carried by splice plates on the flanges as in Fig. 15, by excessive flange area or the main web splice may be designed for the total bending moment. The first and second methods are better than the third.

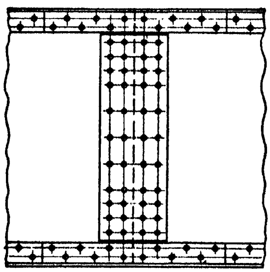


Fig. 16.

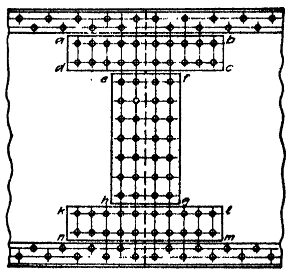


Fig. 17.

While neither of the web splices of Figs. 16 and 17 are in accordance with the principle that the splice should be equivalent to the web net section, yet both forms have been used and there seems to be no record of any serious effects resulting from their use.

In the design of a web splice such as the one shown in Fig. 15 it is generally possible to locate the splice at a section where there will be an excessive flange area. Then the main web splice is designed for the total shear and for a moment of resistance equal to that of the portion of the web between the flanges.

If the allowable extreme fiber stress on the net section of the girder is 16,000 lb. per sq. in. then the extreme fiber stress on the gross section may be assumed at 14,400 lb. per sq. in.

Let h = total height of girder
h_1 = clear depth of main portion of web between flanges
t = thickness of web plate
n = number of rivets on one side of splice
V = total shear on the section
M = resisting moment of portion of web between flanges

or

$$M = f\left(\frac{I}{c}\right) = 14{,}400\left(\frac{h_1}{h}\right)\left(\frac{th_1^2}{6}\right) = \frac{2{,}400th_1^3}{h} \tag{12}$$

Then for the horizontal component of the stress in the most highly stressed rivet we have

$$f_m = \frac{M}{\frac{\Sigma y^2}{Y}} = \frac{2{,}400th_1^3}{h\left(\frac{\Sigma y^2}{Y}\right)}$$

and for the vertical component we have

$$f_v = \frac{V}{n}$$

Since the resultant stress,

$$f_r = \sqrt{(f_m)^2 + (f_v)^2},$$

we have

$$f_r = \sqrt{\left[\frac{2{,}400th_1^3}{h\left(\frac{\Sigma y^2}{Y}\right)}\right]^2 + \left(\frac{V}{n}\right)^2} \tag{13}$$

or

$$\frac{\Sigma y^2}{Y} = \frac{2{,}400th_1^3}{h\sqrt{(f_r)^2 - \left(\frac{V}{n}\right)^2}} \tag{14}$$

Since a web splice is usually placed at a section where the moment is much more important than the shear, we may neglect the shear in making a trial design. Equation (14) then becomes

$$\frac{\Sigma y^2}{Y} = \frac{2{,}400th_1^3}{hf_r} \tag{15}$$

in which f_r is the allowable value in bearing or shear.

Values of the quantity $\frac{\Sigma y^2}{Y}$ for a single row of rivets are given in Diagrams 2 and 3. Values of this quantity for several rows of rivets may be obtained by multiplying the value found in the diagrams by the number of rows. This may be done because here the bending moment produces flexure and not rotation about a point as in the case of an eccentric connection. Hence the moment stress on any rivet varies in proportion to its distance from the neutral axis and not in proportion to its distance from the center of gravity of the group of two or more rows of rivets.

After a trial design has been made by means of eq. (15) and either Diagram 2 or 3, the exact value of the resultant stress f_r can be found by substituting in eq. (13). If this value is not sufficiently close to the allowable value the design may be revised quite readily.

Illustrative Problem.—Design a splice similar to Fig. 15 for a plate girder web for the following conditions: $V = 150{,}000$ lb., $h = 80$ in., $h_1 = 68$ in., $t = \frac{7}{16}$ in., and $f_r = 9{,}190$ lb. given by $\frac{7}{8}$-in. rivets in bearing on a $\frac{7}{16}$-in. plate at 24,000 lb. per sq. in.

Substituting in eq. (15) we have

$$\frac{\Sigma y^2}{Y} = \frac{2{,}400(\frac{7}{16})(68)^3}{80(9{,}190)} = 450 \text{ in.}$$

From Diagram 2 we note that for one row of 16 rivets, with $p = 4$, $\frac{\Sigma y^2}{Y} = 180$. For three rows of rivets $\frac{\Sigma y^2}{Y} = (3)(180) = 540$ in. and $n = (3)(16) = 48$

Substituting these values in eq. (13) we get

$$f_r = \sqrt{\left[\frac{2{,}400(7/16)(68)^3}{80(540)}\right]^2 + \left(\frac{150{,}000}{48}\right)^2} = 8{,}270 \text{ lb.}$$

This design is satisfactory.

Since plate girders are usually designed on the assumption that one-eighth of the web area is effective in resisting bending moment and since this in turn is based upon the assumption that the net area of the web is equal to three-fourths of the gross area, it is necessary that the rivet spacing in the web splice should conform to this practice. This means that the rivet pitch should equal four times the diameter of the rivet hole.

The general principles which have just been developed and illustrated for the design of a splice of the type shown in Fig. 15 also apply to the design of a splice of the type shown in Fig. 17. However, in this case, Diagrams 2 and 3 cannot be used in the determination of the quantity $\frac{\Sigma y^2}{Y}$. In finding the value of this quantity for an irregular group of rivets such as *abmn*, Fig. 17, which is composed of several elementary rectangular groups of rivets such as *abcd*, *efgh* and *klmn* it has been found simplest to find the value of Σy^2 for each group of rivets with respect to its own horizontal axis of symmetry first. Then the value of Σy^2 for the group of rivets *abcd* with respect to any other axis such as the horizontal axis of symmetry for the whole group may be found by an application of the same principle used in finding the moment of inertia of an area with respect to another axis which does not pass through the center of gravity of the area. In Art. **16**, Sec. 1, it has been stated that[1]

$$I_x = I_o + Ad^2$$

where I_o = moment of inertia of the figure about an axis through its center of gravity.

I_x = moment of inertia about a parallel axis.

A = area of the figure.

d = distance between axes.

If we regard the area of each rivet as equal to unity, as we have been doing, then Σy^2 is analogous to I and A is analogous to the number of rivets. Hence if we desire the value of Σy^2 for an elementary rectangular group when y is measured from an axis other than the axis of symmetry of the group we may use the following relation:

$$\Sigma y_1^2 = \Sigma y^2 + nd^2 \tag{16}$$

where y is measured from an axis of symmetry.

y_1 is measured from an axis parallel to an axis of symmetry.

n = number of rivets.

d = distance between axes.

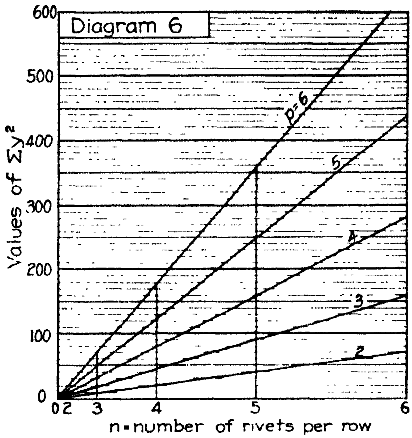
Values of Σy^2 for one row of rivets containing from 2 to 12 rivets and for rivet pitches varying from 2 to 6 in. are given in Diagrams 6 and 7.

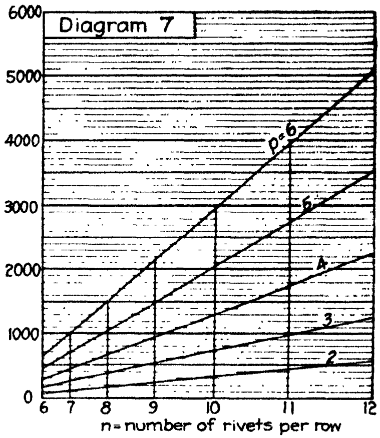
In the design of a splice of the type shown in Fig. 17 it is generally assumed that the rivets in the group *efgh* take all of the shear and a portion of the moment which is governed by the fact that the resultant rivet stress should not exceed the

[1] For proof of this statement, Appendix B, p. 578.

allowable stress and by the fact that the net area should be maintained at about three-fourths of the gross area. The remaining and generally the major portion of the moment is carried by the groups of rivets *abcd* and *klmn*. It is also generally assumed that the resisting moment of the full height of the web plate is to be developed and not just that portion of the web plate between flanges. Therefore h_1 of eq. (12) becomes h and we have for the total moment carried by the splice

$$M = 2{,}400th^2 \tag{17}$$





To get the portion of the total moment carried by the middle group of rivets, let

V = total shear.

m = portion of moment carried by middle group of rivets *efgh*.

n and $\dfrac{\Sigma y^2}{Y}$ refer to this group alone.

Then

$$f_m = \frac{m}{\dfrac{\Sigma y^2}{Y}}$$

$$f_v = \frac{V}{n}$$

$$f_r^2 = f_m^2 + f_v^2 = \left(\frac{m}{\dfrac{\Sigma y^2}{Y}}\right)^2 + \left(\frac{V}{n}\right)^2$$

from which

$$m = \frac{\Sigma y^2}{Y}\sqrt{f_r^2 - \left(\frac{V}{n}\right)^2} \tag{18}$$

Since the outer groups of rivets carry none of the shear

$$f_r = f_m = \frac{M - m}{\dfrac{\Sigma y_1{}^2}{Y}} = \frac{M - m}{\dfrac{\Sigma y^2 + nd^2}{Y}}$$

or

$$\Sigma y^2 + nd^2 = \frac{(M - m)\,Y}{f_r} \tag{19}$$

Illustrative Problem.—Design a splice similar to Fig. 17 for a plate girder web for the conditions given in the preceding problem; namely, that $V = 150{,}000$ lb., $h = 80$ in., $h_1 = 68$ in., $t = \frac{7}{16}$ in., and $f_r = 9{,}190$ lb. given by $\frac{7}{8}$-in. rivets in bearing on a $\frac{7}{16}$-in. plate at 24,000 lb. per sq. in.

In order that the center group of rivets may carry the shear, the number of rivets in this group must equal at least

$$\frac{V}{f_r} = \frac{150{,}000}{9{,}190} = 16.3, \text{ or } 17 \text{ rivets}$$

We shall use two rows of rivets with 9 rivets in each row and a rivet pitch of 4 in. For each row of rivets we find from Diagram 2 that $\frac{\Sigma y^2}{Y} = 58$ in. Hence for two rows of rivets $\frac{\Sigma y^2}{Y} = (2)(58) = 116$ in. and also $n = (2)(9) = 18$ rivets. Substituting these values in eq. (18) we find for the moment carried by the center group of rivets that

$$m = 116\sqrt{(9{,}190)^2 - \left(\frac{150{,}000}{18}\right)^2} = 448{,}000 \text{ in.-lb.}$$

From eq. (17)

$$M = 2{,}400(\tfrac{7}{16})(80)^2 = 6{,}720{,}000 \text{ in.-lb.}$$

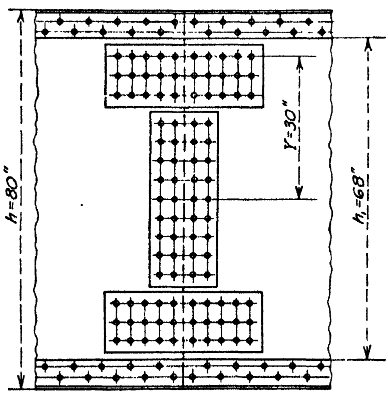


FIG. 18.

In this case we can assume that

$$Y = \frac{h_1}{2} - 4 \text{ in.} = \frac{68}{2} - 4 = 30 \text{ in.}$$

Substituting in eq. (19) we get

$$\Sigma y^2 + nd^2 = \frac{(6{,}720{,}000 - 448{,}000)30}{9{,}190} = 20{,}500 \text{ in.}^2$$

This value is for both outer groups of rivets (*abcd* and *klmn* of Fig. 17), so the value for one group is

$$\frac{20{,}500}{2} = 10{,}250 \text{ in.}^2$$

Since we know the size of the center group of rivets, the value of h_1 and Y we find that each outer group of rivets may be composed of three horizontal rows of rivets 4 in. apart and hence that $d = Y - 4$ in. $= 30 - 4 = 26$ in. The value of Σy^2 for one vertical row of three rivets with a 4-in. pitch is found from Diagram 6 to be 31 in.^{2} With $\Sigma y^2 = 31$ in.^{2}, $n = 3$ rivets, and $d = 26$ in., we find for one vertical row of rivets of this outer group that

$$\Sigma y^2 + nd^2 = 31 + 3(26)^2 = 2{,}159 \text{ in.}^2$$

Therefore the number of vertical rows of rivets in the outer group on each side of the break in the web plate must be

$$\frac{10{,}250}{2{,}159} = 5$$

The completed design is shown in Fig. 18.

17. Plate Girder Flange Splices.—The following quotation from the 1920 Specifications of the American Railway Engineering Association is representative of good practice regarding the splicing of plate girder flanges:

> Splices in flange members shall not be used except by special permission of the engineer. Two members shall not be spliced at the same cross-section and, if practicable, splices shall be located at points where there is an excess of section. The net section of the splice shall exceed by 10 per cent the net section of the member spliced. Flange angle splices shall consist of two angles, one on each side.

There must be enough rivets on each side of the splice so that their strength in single shear shall be equal to the strength of the splice angles or splice plates. The rivets which are used in the flange for other purposes are also available for this purpose and therefore by making the splice angles or splice plates long enough no excess rivets will be required. However, it is considered better practice to use a closer spacing of rivets in a flange splice so as to reduce its length to a minimum.

18. Pin Connections.—As stated in Art. **1**, where it is necessary to have several members meet at a joint in such a way that the members will be free to turn, a pin connection must be used. Pin connections are particularly well suited to designs in which most of the tension members are composed of eye-bars, since such members may be arranged on the pin with a good deal of facility. Pin connections are not so well suited to designs in which built-up members are to be connected, since it is not convenient to have the different widths of members which would be necessary where several such members meet at a joint. However, it is sometimes necessary to use pin connections in such cases.

19. Stresses Induced in Pins and Pin Plates.—In addition to the shearing and bearing stresses to which both rivets and bolts are subjected, pins are subjected to heavy bending stresses which are not usually considered in riveted connections. Therefore it is necessary to analyze a pin as a short cylindrical beam subjected to shearing, bearing, and bending stresses. Quite frequently the bearing stress on the web of a member, such as a channel, becomes excessive and hence it is necessary to reinforce the web by riveting a plate to it at the pin. Such plates are called *pin plates*.

In calculating the shearing stresses on a pin, it is convenient to get the horizontal and vertical components of these stresses and then sum up these components separately and get the resultant from the formula $V = \sqrt{(V_h)^2 + (V_v)^2}$, where V = resultant shear, and V_h and V_v are the summations of the horizontal and vertical shearing stresses respectively. However, it is only in special cases that the shearing stresses will govern the size of the pin.

It is generally assumed that the loads are concentrated at the centers of bearing of the members when the bending moments are being calculated. Strictly speaking the loads are distributed over the bearing areas but the difference in results obtained by the two methods is small ordinarily and therefore the simpler method is adopted. The horizontal and vertical components of the forces acting at a joint are calculated as in the determination of the shear, and then the resultant moment is found by the formula $M = \sqrt{(M_h)^2 + M_v)^2}$, where M = resultant moment, and M_h and M_v are the moments produced by the horizontal and vertical components respectively.

In general the maximum stresses in all the members meeting at a joint will not occur simultaneously. Therefore in calculating moments it is usual to consider two conditions of loading, one in which the stress in a chord member is a maximum, and the other in which the stress in a web member is a maximum. In each case the simultaneous stress occurring in one other member is calculated and then the simultaneous stresses in the other members intersecting at the pin may be obtained quite readily by means of a small force diagram.

Horizontal compression chord members are built continuous at the joints so that only the difference in stresses between the two segments is in action on the pin.

In the Specifications of the American Railway Engineering Association an allowable bending stress on pins of 24,000 lb. per sq. in. is recommended. This comparatively high value is allowed because of the fact that the assumption of concentrated loads gives bending moments in excess of the true values and also because of the fact that any slight bending of the pin produces some change in the magnitude of the forces which effect a reduction in the bending moments. Another reason for the use of a higher working stress for pins than for members is due to the fact that the large secondary stresses which often occur in members do not exist in the pins.

As soon as the maximum bending moment is found, the diameter of the pin may be obtained from the formula $f = \frac{Mc}{I}$. For pins c = radius, r, and $I = \frac{\pi r^4}{4}$. Substituting $\frac{d}{2}$ for r we get

$$f = \frac{M\left(\frac{d}{2}\right)4}{\pi\left(\frac{d}{2}\right)^4} = \frac{Md(4)(16)}{2\pi d^4} = \frac{32M}{\pi d^3} \text{ or } d = \sqrt[3]{\frac{10.187M}{f}}$$

If

$$f = 24{,}000 \text{ lb. per sq. in.}, d = \sqrt[3]{\frac{M}{2{,}356}}$$

Values of the allowable bending moments on pins of different sizes with several unit stresses are given in Table 9.

The bearing stresses are found as in riveted connections. The bearing area for riveted members is provided by adding pin plates until the resulting bearing stresses come within the allowable. With the allowable unit tension in the eye-bars known and the allowable unit bearing stress known, the limiting value of the ratio of pin diameter to width of widest bar which will result in a safe bearing stress may be found. For instance, if the unit tensile stress in the eye-bar is 16,000 lb. per sq. in. and the unit bearing stress is 24,000 lb. per sq. in. then this allowable bearing stress will not be exceeded if the diameter of the pin is equal to or greater than $\frac{16{,}000}{24{,}000} = \frac{2}{3}$ of the width of the widest member. Values of the allowable bearing stresses on pins of different sizes with several unit stresses are given in Table 10.

Quite a little attention must be given to the arrangement or packing of the members on the pin in order that the resulting bending moments will be reduced to a minimum. In general, members producing stresses in opposite directions

TABLE 9.[1]—PINS—BEARING VALUES IN POUNDS ON METAL ONE INCH THICK

Bearing Value = Diameter of Pin × Bearing Stress per Square Inch

Pin		Bearing stresses in pounds per square inch				
Diameter, inches	Area, sq. in.	12,000	15,000	20,000	22,000	24,000
1	0.785	12,000	15,000	20,000	22,000	24,000
1¼	1.227	15,000	18,800	25,000	27,500	30,000
1½	1.767	18,000	22,500	30,000	33,000	36,000
1¾	2.405	21,000	26,300	35,000	38,500	42,000
2	3.142	24,000	30,000	40,000	44,000	48,000
2¼	3.976	27,000	33,800	45,000	49,500	54,000
2½	4.909	30,000	37,500	50,000	55,000	60,000
2¾	5.940	33,000	41,300	55,000	60,500	66,000
3	7.069	36,000	45,000	60,000	66,000	72,000
3¼	8.296	39,000	48,800	65,000	71,500	78,000
3½	9.621	42,000	52,500	70,000	77,000	84,000
3¾	11.045	45,000	56,300	75,000	82,500	90,000
4	12.566	48,000	60,000	80,000	88,000	96,000
4¼	14.186	51,000	63,800	85,000	93,500	102,000
4½	15.904	54,000	67,500	90,000	99,000	108,000
4¾	17.721	57,000	71,300	95,000	104,500	114,000
5	19.635	60,000	75,000	100,000	110,000	120,000
5¼	21.648	63,000	78,800	105,000	115,500	126,000
5½	23.758	66,000	82,500	110,000	121,000	132,000
5¾	25.967	69,000	86,300	115,000	126,500	138,000
6	28.274	72,000	90,000	120,000	132,000	144,000
6¼	30.680	75,000	93,800	125,000	137,500	150,000
6½	33.183	78,000	97,500	130,000	143,000	156,000
6¾	35.785	81,000	101,300	135,000	148,500	162,000
7	38.485	84,000	105,000	140,000	154,000	168,000
7¼	41.282	87,000	108,800	145,000	159,500	174,000
7½	44.179	90,000	112,500	150,000	165,000	180,000
7¾	47.173	93,000	116,300	155,000	170,500	186,000
8	50.265	96,000	120,000	160,000	176,000	192,000
8¼	53.456	99,000	123,800	165,000	181,500	198,000
8½	56.745	102,000	127,500	170,000	187,000	204,000
8¾	60.132	105,000	131,300	175,000	192,500	210,000
9	63.617	108,000	135,000	180,000	198,000	216,000
9¼	67.201	111,000	138,800	185,000	203,500	222,000
9½	70.882	114,000	142,500	190,000	209,000	228,000
9¾	74.662	117,000	146,300	195,000	214,500	234,000
10	78.540	120,000	150,000	200,000	220,000	240,000
10¼	82.516	123,000	153,800	205,000	225,500	246,000
10½	86.590	126,000	157,500	210,000	231,000	252,000
10¾	90.763	129,000	161,300	215,000	236,500	258,000
11	95.033	132,000	165,000	220,000	242,000	264,000
11¼	99.402	135,000	168,800	225,000	247,500	270,000
11½	103.869	138,000	172,500	230,000	253,000	276,000
11¾	108.434	141,000	176,300	235,000	258,500	282,000
12	113.097	144,000	180,000	240,000	264,000	288,000

[1] From Pocket Companion, 20th edition, Carnegie Steel Co., Pittsburgh, Pa.

TABLE 10.[1]—PINS—BENDING MOMENTS IN INCH POUNDS

Bending Moment = (Diameter of Pin)^3 $\times$ 0.098175 $\times$ Stress per Square Inch

Pin		Fiber stress in pounds per square inch						
Diameter, inches	Area, sq. in.	15,000	18,000	20,000	22,000	22,500	24,000	25,000
1	0.785	1,500	1,800	2,000	2,200	2,200	2,400	2,500
1¼	1.227	2,900	3,500	3,800	4,200	4,300	4,600	4,800
1½	1.767	5,000	6,000	6,600	7,300	7,500	8,000	8,300
1¾	2.405	7,900	9,500	10,500	11,600	11,800	12,600	13,200
2	3.142	11,800	14,100	15,700	17,300	17,700	18,800	19,600
2¼	3.976	16,800	20,100	22,400	24,600	25,200	26,800	28,000
2½	4.909	23,000	27,600	30,700	33,700	34,500	36,800	38,300
2¾	5.940	30,600	36,800	40,800	44,900	45,900	49,000	51,000
3	7.069	39,800	47,700	53,000	58,300	59,600	63,600	66,300
3¼	8.296	50,600	60,700	67,400	74,100	75,800	80,900	84,300
3½	9.621	63,100	75,800	84,200	92,600	94,700	101,000	105,200
3¾	11.045	77,700	93,200	103,500	113,900	116,500	124,300	129,400
4	12.566	94,200	113,100	125,700	138,200	141,400	150,800	157,100
4¼	14.186	113,000	135,700	150,700	165,800	169,600	180,900	188,400
4½	15.904	134,200	161,000	178,900	196,800	201,300	214,700	223,700
4¾	17.721	157,800	189,400	210,400	231,500	236,700	252,500	263,000
5	19.635	184,100	220,900	245,400	270,000	276,100	294,500	306,800
5¼	21.648	213,100	255,700	284,100	312,500	319,600	340,900	355,200
5½	23.758	245,000	294,000	326,700	359,300	367,500	392,000	408,300
5¾	25.967	280,000	336,000	373,300	410,600	419,900	447,900	466,600
6	28.274	318,100	381,700	424,100	466,500	477,100	508,900	530,100
6¼	30.680	359,500	431,400	479,400	527,300	539,300	575,200	599,200
6½	33.183	404,400	485,300	539,200	593,100	606,600	647,100	674,000
6¾	35.785	452,900	543,500	603,900	664,300	679,400	724,600	754,800
7	38.485	505,100	606,100	673,500	740,800	757,700	808,200	841,800
7¼	41.282	561,200	673,400	748,200	823,100	841,800	897,900	935,300
7½	44.179	621,300	745,500	828,400	911,200	931,900	994,000	1,035,400
7¾	47.173	685,500	822,600	914,000	1,005,400	1,028,200	1,096,800	1,142,500
8	50.265	754,000	904,800	1,005,300	1,105,800	1,131,000	1,206,400	1,256,600
8¼	53.456	826,900	992,300	1,102,500	1,212,800	1,240,400	1,323,000	1,378,200
8½	56.745	904,400	1,085,300	1,205,800	1,326,400	1,356,600	1,447,000	1,507,300
8¾	60.132	986,500	1,183,900	1,315,400	1,446,900	1,479,800	1,578,500	1,644,200
9	63.617	1,073,500	1,288,300	1,431,400	1,574,500	1,610,300	1,717,700	1,789,200
9¼	67.201	1,165,500	1,398,600	1,554 000	1,709,400	1,748,300	1,864,800	1,942,500
9½	70.882	1,262,600	1,515,100	1,683,500	1,851,800	1,893,900	2,020,100	2,104,300
9¾	74.662	1,364,900	1,637,900	1,819,900	2,001,900	2,047,400	2,183,900	2,274,900
10	78.540	1,472,600	1,767,100	1,963,500	2,159,800	2,208,900	2,356,200	2,454,400
10¼	82.516	1,585,900	1,903,000	2,114,500	2,325,900	2,378,800	2,537,400	2,643,100
10½	86.590	1,704,700	2,045,700	2,273,000	2,500,300	2,557,100	2,727,600	2,841,200
10¾	90.763	1,829,400	2,195,300	2,439,200	2,683,200	2,744,100	2,927,100	3,049,100
11	95.033	1,960,100	2,352,100	2,613,400	2,874,800	2,940,100	3,136,100	3,266,800
11¼	99.402	2,096,800	2,516,100	2,795,700	3,075,200	3,145,100	3,354,800	3,494,600
11½	103.869	2,239,700	2,687,600	2,986,200	3,284,900	3,359,500	3,583,500	3,732,800
11¾	108.434	2,388,900	2,866,700	3,185,300	3,503,800	3,583,400	3,822,300	3,981,600
12	113.097	2,544,700	3,053,600	3,392,900	3,732,200	3,817,000	4,071,500	4,241,200

[1] From Pocket Companion, 20th edition, Carnegie Steel Co., Pittsburgh, Pa.

should be placed close together. Each pair of members should be so located that the couple which is produced will be of opposite sign to that of the pair adjacent on each side in order that the moment will not accumulate toward the center of the pin.

Illustrative Problem.—Compute the maximum moment on the pin in the joint shown in Fig. 19.

The stress in each diagonal bar is $\frac{16{,}960}{2} = 8{,}480$ lb. The horizontal and vertical components of 8,480 lb. are

$$8{,}480\ (\sin 45°) = 8{,}480(\cos 45°) = 6{,}000 \text{ lb.}$$

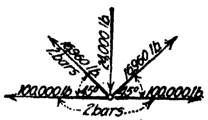


FIG. 19.

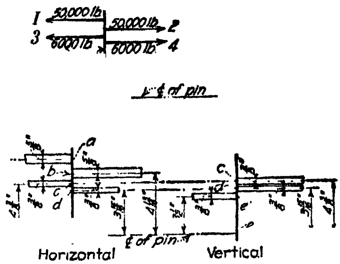


FIG. 20.

Figure 20 shows the stresses in their assumed positions with the distance of each from the center line of the pin.

Hor. mom. about b = $(50{,}000)(\frac{11}{16})$ = 34,380 in.-lb.
Hor. mom. about c = $(50{,}000)(1\frac{1}{8}) - (50{,}000)(\frac{7}{16})$ = 34,380 in.-lb.
Hor. mom. about d = $(50{,}000)(1\frac{9}{16}) - (50{,}000)(\frac{7}{8}) + (6{,}000)(\frac{7}{16})$ = 37,000 in.-lb.
Hor. mom. about e = $(50{,}000)(5\frac{1}{2}) - (50{,}000)(4\frac{13}{16}) + (6{,}000)(4\frac{3}{8}) - (6{,}000)(3\frac{15}{16})$ = 37,000 in.-lb.

Vert. mom. about c = 0
Vert. mom. about d = $(6{,}000)(\frac{7}{16})$ = 2,630 in.-lb.
Vert. mom. about e' = $(6{,}000)(\frac{7}{8}) + (6{,}000)(\frac{7}{16})$ = 7,880 in.-lb.
Vert. mom. about e = $(6{,}000)(4\frac{3}{8}) + (6{,}000)(3\frac{15}{16}) - (12{,}000)(3\frac{1}{2})$ = 7,880 in.-lb.
Max. mom. then, is at e and is.

$$\sqrt{(37{,}000)^2 + (7{,}880)^2} = 37{,}800 \text{ in.-lb.}$$

SECTION 4

DESIGN OF WOODEN MEMBERS

By Henry D. Dewell

WOODEN BEAMS

Under this heading will be considered only timber beams and girders of solid and uniform section.

1. Factors to be Considered in Design.—The factors determining the selection of the size of a wooden beam are:

(*a*) The maximum unit fiber stress in bending must not be excessive.

(*b*) The maximum unit stress due to horizontal shear must not be excessive.

(*c*) The deflection of the beam under maximum loading must be within the allowable limit.

(*d*) The depth in building construction must be within any limits of space between floor and ceiling, or in accordance with any restrictions as to clear story height.

(*e*) The cross-sectional dimensions should be of a size easy to obtain.

(*f*) The cross-sectional dimensions should be considered as to requirements of details of connection.

(*g*) One or both of the cross-sectional dimensions may be limited.

The fundamental bending formula used in the design of beams, is treated in the chapter on "Simple and Cantilever Beams" in Sec. 1. Shear and deflection are also treated in the same chapter.

2. Allowable Unit Stresses.—Unit stresses for design of wooden beams are usually prescribed by building ordinances for the various kinds of timber. These allowable stresses vary widely in different cities, the older ordinances in general prescribing lower limits than the more recent ones. The tendency in revising ordinances is to increase the allowable unit stresses in timber, at least for timber in bending. This feature is due largely to the efforts of the lumber manufacturers' organizations in competition with the constantly widening use of reinforced concrete. At the same time these manufacturers, in conjunction with engineering organizations, are giving more attention to the grading rules and to furnishing timber of uniform high quality. In comparing the allowable unit stresses found in various building ordinances the prescribed live loading must also be taken into consideration. For example, a limit of 1,500 lb. per sq. in. in bending with a 60-lb. live load will give the same size beam as a 40-lb. live load with a limiting fiber stress in bending of 1,000 lb. per sq. in.

It is obvious that the allowable unit stresses are dependent on the quality of timber used. In this respect most of the newer building ordinances allow higher

stresses for a select grade of lumber, whereas older ordinances make no distinction in grade, or, more accurately speaking, they prescribe for the grade of timber most likely to be used.

Probably the most comprehensive study ever made of the strength values of structural timber was that made by the American Railway Engineering Association, through their Committee on Wooden Bridges and Trestles. The ultimate and working stresses recommended by that committee are given in the table of Appendix D, p. 597. The table gives no working unit stresses for pure tension. The working unit resistance to tension may be taken the same as for bending.

3. Kinds of Timber.—The timbers most commonly used for wooden beams are long-leaf yellow pine and Douglas fir, the first being employed almost exclusively throughout the Eastern states, and the latter having its widest use in the Pacific Coast states. Less extensively employed, may be mentioned short-leaf yellow pine, white pine, Norway pine, spruce, hemlock and redwood.

4. Quality of Timber.—The desired quality of timber is determined by specifications or by referring to grading rules established by the lumber manufacturers. Thus, the timber for joists or girders may be specified by the designer to be Select Structural, Dense Grade, Sound Grade, No. 1 Common, or Select No. 1 Common. In the Pacific Coast states, the two latter terms are generally used, very little structural timber entering into a building being above No. 1 Common. Both the Southern Pine Association and the West Coast Lumbermen's Association have established a Structural Grade for long-leaf yellow pine and Douglas fir, and in the larger cities lumber of this quality can probably be obtained. In many cases, however, the lumber may be purchased from the smaller yards, and, even if specified as No. 1 Common, may contain a considerable percentage of No. 2 Common, since it is a practice of some lumber yards to purchase No. 2 Common material and select the better pieces to be sold as No. 1 Common.

5. Horizontal Shear.—In deep short beams the safe unit stress in horizontal shear may be the determining feature. This will seldom be the case in the design of joists, but may be a factor in the selection of the proper size for girders. In this connection the effect of possible checks at the ends of the beam, in or near the horizontal plane, should be considered. Such checks obviously decrease the section of beam for resisting shearing stresses.

6. Bearing at Ends of Beams.—Sufficient bearing must be provided at the ends of all beams, so that, with the maximum reaction at the support, the timber may not crush in side bearing. Most structural timbers are comparatively weak in cross bearing. The details at the ends of timber beams are often poor, insufficient bearing area being provided, so that the beams could never develop their safe loads as determined by bending strength. In general no beam should have a smaller bearing area than given by the product of the width of the beam by 4 in. Details of end connections of beams and girders are discussed in Arts. **15** and **16**, Sec. 5.

7. Deflection.—If a beam has insufficient depth for its span, it will deflect excessively. The result may be a cracked ceiling, if the latter is plastered, or, in an unplastered building, merely a floor that shakes when walked upon. The limit of deflection of a timber joist is generally placed at $\frac{1}{360}$ of the span.

Timber is different from the other building materials, such as steel or concrete, in that, if loaded excessively with a constant load, its deflection will continue to

increase with no increase of load, even though the maximum unit stress in bending be within the elastic limit of the particular timber. For this reason, many specifications require that the modulus of elasticity for "dead," or constant, loads be taken as one-half the modulus of elasticity used for "live," or occasional, loading, the latter quantity being the value determined from a short-time loading test. For example, the Am. Ry. Eng. Assoc. through the committee on "Wooden Bridges and Trestles," recommends "To compute the deflection of a beam under long-continued loading instead of that when the load is first applied, only 50 per cent of the corresponding modulus of elasticity for short time loading is to be employed." Tests by Tieneman[1] indicate that a beam may be loaded to within 20 per cent of its elastic limit without danger of increase of deflection.

The recommendation is here made that for constant or "dead" loads the modulus of elasticity be taken at three-fourths that for short time loading, while for occasional or "live" loading the full value be used.

8. Lateral Support for Beams.—A timber beam needs to be supported laterally in the same manner as a beam of steel or concrete. Floor joists are braced by the flooring and also by the bridging, while the girders are held by the attachment of joists.

In the case of a beam unsupported laterally, the maximum unit fiber stress in flexure should not exceed the value

$$f = f_1\left(1 - \frac{1}{90} \cdot \frac{l}{b}\right)$$

where f_1 = basic unit flexural fiber stress, l = span of beam in inches, and b = breadth of beam in inches.[2]

9. Sized and Surfaced Timbers.—The fact must always be borne in mind by the designer of timber beams that a variation from the nominal size of timbers is allowed by all grading rules; also, that if timber beams are sized, the actual depth is less than the nominal depth. Further, if timber is bought from a local lumber yard, joists may come surfaced one side. In general, all-rail shipments of timbers are surfaced one side one edge (S1S1E) while all-water shipments are not surfaced. *The actual dimensions of the finished stick must be used in all calculations.* Tables 1, 2, 3, 6, 7, 8, and 9 show the relation between actual sizes and nominal sizes.

10. Joists.—Joists usually carry only a uniform load composed of the weight of the joists themselves plus the flooring plus superimposed loads of people, furniture, etc. The latter loads are commonly termed "live" loads in contrast with the constant loads due to the weight of the floor construction itself, called "dead" loads. The joists carry the flooring directly on their upper surfaces, and are in turn supported at their ends by girders, bearing partitions or bearing walls. Joists are always single sticks of timber. Joists may, and often do, carry concentrated loads in addition to the uniform loads mentioned above. Such concentrations may be caused by heavy pieces of furniture, safes, etc., by cross partitions resting on the floor, or by special floor framing as required by openings in the floor.

[1] See *Eng. News*, vol. 62, pp. 216–217.

[2] Properly the factor 1/90 holds only for the case of simple beams loaded uniformly and at the third points, and for cantilever beams with uniform loading. For a simple beam with single concentrated load at any point of span the factor is 1/120, while for quarter point loading the factor is 1/80.

Many designs of joists or girders are faulty in that the designer has not considered such concentrated floor loads in addition to the uniform loading. In design, with the use of tables giving safe loads for timber, the beams selected thereby may not be sufficient for all cases of framing where loading has been assumed to be uniform. For such cases, the concentrations are sometimes reduced to equivalent uniform loads before entering the tables. A correct and satisfactory method, except for the simpler cases, is to compute the separate bending moments due to each load and combine these partial moments to get the amount and position of the maximum moment. The combination of the partial moments may be quickly accomplished by graphical methods, as illustrated in Art. **15**. Having this, the required section is easily found (see chapter on "Simple and Cantilever Beams," Sec. 1).

Table 6 gives the resisting moments of rectangular beams, computed on the basis of the actual finished sections, for maximum unit fiber stresses varying from 1,000 to 2,000 lb. per sq. in.; also the factors by which the moments in the tables are to be multiplied to get the resisting moments of the rough sections.

11. Girders.—Girders may be single sticks or composite sections. Girders usually support joists, and in turn are supported by columns or bearing walls. When girders are carried otherwise than by columns, the fact must always be borne in mind that such girders deliver a concentrated load of some magnitude to the wall, or bearing partition, and care must be taken to see that such wall or partition is strong enough in column action to carry the load imposed upon it by the girders.

For ordinary building construction, where timber not better than No. 1 Common is likely to be used, it is recommended that the maximum unit fiber stress in bending for long-leaf yellow pine or Douglas fir be limited to 1,500 lb. per sq. in., and the maximum unit longitudinal shearing stress be limited to 150 lb. per sq. in. For timber of the grade of Select Structural, or Select No. 1 Common, the unit flexural stress, computed always on the basis of actual finished sections, may be increased to 1,800 lb. per sq. in., and the unit longitudinal shearing stress to 175 lb. per sq. in.

Tables 1, 2, and 3 give the safe loads, deflection, and maximum unit shearing stresses for 2-, 3- and 4-in. joists, respectively. The maximum unit fiber stress in bending is 1,500 lb. per sq. in., computed on the finished size of joist. The deflection is based on a modulus of clasticity of 1,643,000. The maximum intensity of horizontal shearing stress is given for the shortest span. To use this table for other unit flexural fiber stresses, the values in the tables must be multiplied by the factors of Tables 4 and 5.

Illustrative Problem.—Required to find proper size of joist to support a load of 5,500 lb. on a 14-ft. span, with a fiber stress of 1,200 lb. per sq. in.

From Table 5 we find factor of multiplication to be 1.250. The new load to use in entering Tables 1, 2, and 3 is therefore 5,500 lb. × 1.250 = 6,870 lb. From Table 2 it is seen that a 3- × 16-in. joist on a 14-ft. span has a safe carrying capacity of 7,150 lb. (at 1,500 lb. per sq. in.).

Illustrative Problem.—Given a 2- × 14-in. joist on a 16-ft. span. Required the safe load as limited by a maximum unit fiber stress of 1,200 lb. per sq. in. in bending. From Table 1, the safe load for 1,500 lb. per sq. in. in bending is seen to be 3,085 lb. From Table 4, the factor of multiplication is seen to be 0.80, giving the safe load as 2,468 lb.

Diagram on p. 361 gives a simple method for solving the strength of any timber beam as determined by maximum unit strength in bending, also for determining the proper size of any timber beam to support a given load in bending.

Illustrative Problem.—Given a total floor load, dead and live, of 174 lb. per sq. ft., span 13 ft. 1 in. What size joists, spaced 16 in. on centers, will support this load with a maximum unit fiber stress of 1,800 lb. per sq. in.

Lay a flexible straight edge, such as a card, on the diagram, p. 361, joining point *A* (174 lb. per sq. ft.) with *B* (16-in. spacing), and mark intersection *C* on Working Line. Pivoting card about *C*, connect *C* with *D* (13 ft. 1 in.) and read 5,000 ft.-lb. at *E*. Connect *E* with *F* (1,800 lb. per sq. in.), crossing Working Line at *G*. Pivoting card about *G*, set card on 1¾ in. (width of beam) at *H* and read 11½ in. (depth of beam) at *K*.

12. Explanation of Tables.—In Tables 1, 2, and 3, the first line of figures in each group represents the safe load for the particular beam, including the weight of the beam itself. The second line of figures gives the deflection in inches for the beam at the maximum safe load, computed for a modulus of elasticity of 1,643,000 lb. per sq. in. The third figure, where such figure occurs, indicates the maximum unit horizontal shearing stress. The shearing stress is given, only in those cases in which such shear is in excess of 150 lb. per sq. in. All quantities in these tables are based upon the surfaced sizes of sticks. To obtain the safe loads for the rough or full nominal sizes of timber, the loads of Tables 1, 2, and 3 must be multiplied by the "multiplying factors" of Table 6. These tables have been adapted from similar tables in the "Structural Timber Handbook on Pacific Coast Woods" published by the West Coast Lumbermen's Association.

Tables 7, 8 and 9 give for timber joists: (1) The safe loads corresponding to a maximum flexural stress of 1,800 lb. per sq. in., indicated in the tables by the letter "*B*"; (2) the safe load, uniformly distributed, limited by a maximum intensity of horizontal shear of 175 lb. per sq. in., indicated in the tables by the letters "*HS*"; (3) the uniformly distributed load that produces a deflection of $\frac{1}{30}$ in. per foot of span, indicated in the tables by the letter "*D*"; and (4); the deflection in inches for a load of 1,000 lb., uniformly distributed, indicated in the tables by "*D*1." All deflections are computed for a modulus of elasticity of 1,620,000 lb. per sq. in. All loads and deflections are computed on the finished or surfaced sizes of joists. For convenience, the section moduli of the various sizes of joists are given, based on finished sizes. These tables are taken from the "Southern Pine Manual" published by the Southern Pine Association.

Attention is called to the variation of sizes of finished joists in Tables 1, 2, 3 and Tables 7, 8 and 9, representing the difference between the standards of the West Coast Lumbermen's Association and the Southern Pine Association, the finished sections of the Southern Pine Association utilizing a greater percentage of the rough timber than the standards of the West Coast Lumbermen's Association. All sizes of joists in Tables 1, 2, and 3 (West Coast Lumbermen's Association) are for joists surfaced one side and one edge, or surfaced four sides (S4S). All sizes in Tables 7, 8, and 9 (Southern Pine Association) are for joists surfaced one side and one edge (S1S1E).

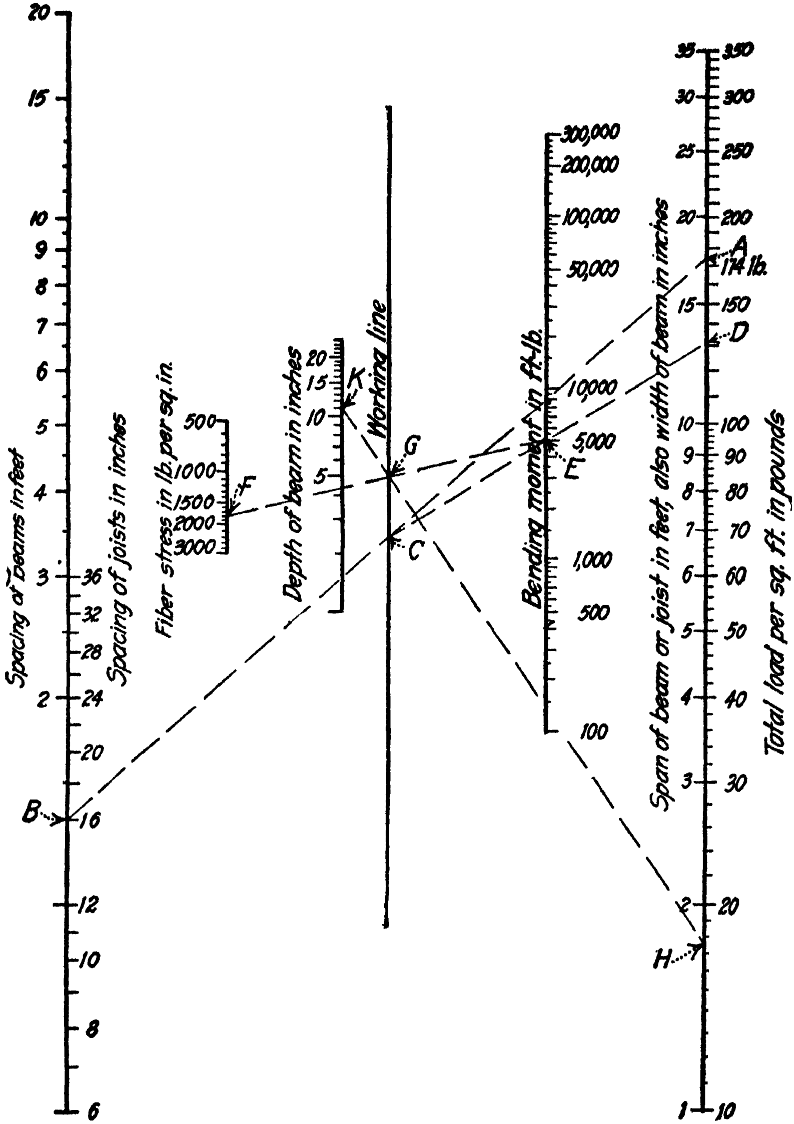


Diagram for capacities of timber beams as determined by bending strength.

TABLE 1.—TABLE OF SAFE LOADS AND DEFLECTIONS FOR TIMBER JOISTS WITH NOMINAL WIDTH OF 2 IN., UNIFORMLY LOADED, BASED ON MAXIMUM FLEXURAL FIBER STRESS OF 1,500 LB. PER SQ. IN.

Sizes	Rough size	2×4	2×6	2×8	2×10	2×12	2×14	2×16	2×18
	Surfaced size S1S1E or S4S[1]	1⅝×3⅝	1⅝×5⅝	1⅝×7½	1⅝×9½	1⅝×11½	1⅝×13½	1⅝×15½	1⅝×17½
	Section modulus	3.56	8.57	15.23	24.44	35.82	49.36	65.07	82.94
Spans in feet	3	1,187 0.0681 151							
	4	890 0.121	2,142 0.0780 176						
	5	712 0.189	1,714 0.122						
	6	593 0.272	1,428 0.176	2,538 0.131 156					
	7	509 0.370	1,224 0.239	2,176 0.179	3,491 0.141 170				
	8		1,071 0.312	1,904 0.234	3,055 0.185	4,478 0.153 180			
	9		953 0.395	1,692 0.296	2,716 0.243	3,980 0.193			
	10		857 0.487	1,523 0.365	2,444 0.289	3,582 0.238	4,936 0.203 169		
	11		779 0.589	1,385 0.442	2,222 0.349	3,256 0.288	4,487 0.245	5,915 0.214 167	
	12			1,269 0.526	2,037 0.415	2,985 0.343	4,113 0.292	5,423 0.254	6,912 0.225 182
	13			1,172 0.617	1,880 0.487	2,755 0.403	3,797 0.343	5,005 0.299	6,380 0.265
	14			1,088 0.716	1,746 0.565	2,559 0.467	3,526 0.397	4,648 0.347	5,924 0.307
	15			1,015 0.822	1,629 0.649	2,388 0.536	3,291 0.456	4,338 0.398	5,529 0.352
	16				1,528 0.738	2,239 0.610	3,085 0.519	4,067 0.453	5,184 0.401
	17				1,438 0.834	2,107 0.688	2,904 0.586	3,828 0.511	4,879 0.452
	18				1,358 0.935	1,990 0.773	2,742 0.657	3,615 0.572	4,608 0.504
	19					1,885 0.860	2,598 0.732	3,426 0.637	4,365 0.565
	20					1,791 0.953	2,468 0.811	3,254 0.706	4,147 0.626
	21						2,350 0.895	3,099 0.779	3,949 0.690
	22						2,244 0.981	2,958 0.855	3,770 0.758
	23							2,829 0.935	3,606 0.829
	24								3,456 0.901
	25								3,318 0.979

[1] S1S1E = surfaced one side and one edge.
S4S = surfaced four sides.

TABLE 2.—TABLE OF SAFE LOADS AND DEFLECTIONS FOR TIMBER JOISTS WITH NOMINAL WIDTH OF 3 IN., UNIFORMLY LOADED, BASED ON MAXIMUM FLEXURAL FIBER STRESS OF 1,500 LB. PER SQ. IN.

Sizes	Rough size	3×6	3×8	3×10	3×12	3×14	3×16	3×18
	Surfaced size S1S1E or S4S[1]	2½×5½	2½×7½	2½×9½	2½×11½	2½×13½	2½×15½	2½×17½
	Section modulus	12.60	23.42	37.61	55.10	75.94	100.10	127.60
Spans in feet	4	3,150 0.0797 172						
	5	2,520 0.125						
	6	2,100 0.179	3,903 0.131 156					
	7	1,800 0.244	3,346 0.179	5,373 0.141 170				
	8	1,575 0.319	2,928 0.234	4,701 0.185	6,888 0.153 180			
	9	1,400 0.404	2,602 0.296	4,179 0.234	6,122 0.193			
	10	1,260 0.498	2,342 0.365	3,761 0.289	5,510 0.238	7,594 0.203 169		
	11	1,145 0.603	2,129 0.442	3,419 0.349	5,009 0.288	6,904 0.245	9,100 0.214 176	
	12		1,952 0.526	3,134 0.415	4,592 0.343	6,328 0.292	8,342 0.254	10,633 0.225 182
	13		1,802 0.617	2,893 0.487	4,239 0.403	5,842 0.343	7,700 0.299	9,815 0.265
	14		1,673 0.716	2,686 0.565	3,936 0.467	5,424 0.397	7,150 0.347	9,114 0.307
	15		1,561 0.822	2,507 0.649	3,673 0.536	5,063 0.456	6,673 0.398	8,507 0.352
	16			2,351 0.738	3,444 0.610	4,746 0.519	6,256 0.453	7,975 0.401
	17			2,212 0.834	3,241 0.688	4,467 0.586	5,889 0.511	7,506 0.452
	18			2,089 0.935	3,061 0.773	4,219 0.657	5,561 0.572	7,089 0.507
	19				2,900 0.917	3,997 0.732	5,268 0.637	6,716 0.565
	20					3,797 0.811	5,005 0.706	6,380 0.626
	21					3,616 0.895	4,767 0.779	6,076 0.690
	22					3,452 0.981	4,550 0.855	5,800 0.758
	23						4,352 0.935	5,548 0.829
	24							5,317 0.901
	25							5,104 0.979

[1] S1S1E = surfaced one side and one edge.
S4S = surfaced four sides.

TABLE 3.—TABLE OF SAFE LOADS AND DEFLECTIONS FOR TIMBER JOISTS WITH NOMINAL WIDTH OF 4 IN., UNIFORMLY LOADED, BASED ON MAXIMUM FLEXURAL FIBER STRESS OF 1,500 LB. PER SQ. IN.

Sizes									
	Rough size	4×4	4×6	4×8	4×10	4×12	4×14	4×16	4×18
	Surfaced size S1S1E or S4S[1]	3½×3½	3½×5½	3½×7½	3½×9½	3½×11½	3½×13½	3½×15½	3½×17½
	Section modulus	7.15	17.64	32.81	52.65	77.15	106.31	140.15	178.65
Spans in feet	3	2,383 0.0705 146							
	4	1,788 0.125	4,410 0.0797 172						
	5	1,430 0.196	3,528 0.125						
	6	1,192 0.282	2,940 0.179	5,468 0.131 156					
	7	1,021 0.384	2,520 0.244	4,687 0.179	7,521 0.141 170				
	8		2,205 0.319	4,101 0.234	6,581 0.185	9,644 0.153 180			
	9		1,960 0.404	3,646 0.296	5,850 0.234	8,572 0.193			
	10		1,764 0.498	3,281 0.365	5,265 0.289	7,715 0.238	10,631 0.203 169		
	11		1,604 0.603	2,983 0.442	4,786 0.349	7,015 0.288	9,665 0.245	12,741 0.214 176	
	12			2,734 0.526	4,388 0.415	6,429 0.343	8,861 0.292	11,679 0.254	14,888 0.225 182
	13			2,524 0.617	4,050 0.487	5,935 0.403	8,178 0.343	10,781 0.299	13,742 0.265
	14			2,344 0.715	3,761 0.565	5,511 0.467	7,594 0.397	10,011 0.347	12,761 0.307
	15			2,187 0.822	3,510 0.649	5,143 0.536	7,087 0.456	9,343 0.398	11,910 0.352
	16				3,291 0.738	4,822 0.610	6,644 0.519	8,759 0.453	11,166 0.401
	17				3,097 0.834	4,538 0.688	6,254 0.586	8,244 0.511	10,508 0.452
	18				2,924 0.935	4,286 0.773	5,906 0.657	7,786 0.572	9,925 0.507
	19					4,061 0.860	5,595 0.732	7,376 0.637	9,403 0.565
	20					3,858 0.953	5,316 0.811	7,008 0.706	8,933 0.626
	21						5,063 0.895	6,674 0.779	8,507 0.690
	22						4,832 0.981	6,370 0.855	8,120 0.758
	23							6,093 0.935	7,767 0.829
	24								7,444 0.901
	25								7,146 0.979

[1] S1S1E = surfaced one side and one edge.
S4S = surfaced four sides.

TABLE 4.—FACTORS BY WHICH SAFE LOADS IN TABLES 1, 2 AND 3 MUST BE MULTIPLIED TO FIND SAFE LOADS THAT GIVEN SIZE OF JOIST WILL SUPPORT AT A UNIT FLEXURAL STRESS OTHER THAN 1,500 LB. PER SQ. IN.

TABLE 5.—FACTORS BY WHICH GIVEN LOAD MUST BE MULTIPLIED TO FIND EQUIVALENT LOAD TO BE USED IN ENTERING TABLES 1, 2, AND 3 TO FIND PROPER SIZE OF JOIST

TABLE 4

Desired unit fiber stress	Factor of multiplication
1,000	0.667
1,100	0.734
1,200	0.800
1,300	0.867
1,400	0.933
1,500	1.000
1,600	1.067
1,700	1.133
1,800	1.200
2,000	1.333

TABLE 5

Desired unit fiber stress	Factor of multiplication
1,000	1.500
1,100	1.363
1,200	1.250
1,300	1.153
1,400	1.071
1,500	1.000
1,600	0.939
1,700	0.883
1,800	0.833
2,000	0.750

TABLE 6.—MAXIMUM BENDING OR RESISTING MOMENTS IN FOOT-POUNDS FOR RECTANGULAR BEAMS

Values in this table are based on surfaced sizes.[1] To get values for rough sizes, multiply Resisting Moment for any given size by "multiplying factor" in dark type in same horizontal line

Size		"Multiplying factor"	Section modulus (in.³)	Moment of inertia (in.⁴)	Resisting moments in foot-pounds for safe fiber stresses in pounds per square inch, as indicated								
Nominal (inches)	Actual[1] (inches)				1,000	1,100	1,200	1,300	1,400	1,500	1,600	1,800	2,000
2 × 4	1⅝ × 3⅝	1.50	3.56	6.45	297	327	356	386	416	446	475	535	59-
2 × 6	1⅝ × 5⅝	1.40	8.57	24.10	714	785	857	928	1,000	1,071	1,142	1,285	1,428
2½ × 6	2¼ × 5½	1.32	11.34	31.18	945	1,040	1,134	1,228	1,323	1,417	1,512	1,701	1,890
2 × 8	1⅝ × 7½	1.40	15.23	57.13	1,269	1,396	1,523	1,650	1,777	1,904	2,030	2,284	2,538
2½ × 8	2¼ × 7½	1.26	21.10	79.10	1,758	1,934	2,110	2,285	2,462	2,637	2,813	3,165	3,518
2 × 10	1⅝ × 9½	1.36	24.44	116.10	2,037	2,241	2,444	2,648	2,852	3,056	3,259	3,667	4,074
2½ × 10	2¼ × 9½	1.23	33.84	160.76	2,820	3,102	3,384	3,666	3,948	4,230	4,512	5,076	5,640
2 × 12	1⅝ × 11½	1.34	35.82	205.95	2,985	3,284	3,582	3,881	4,179	4,478	4,776	5,373	5,970
2½ × 12	2¼ × 11½	1.21	49.59	285.16	4,133	4,546	4,958	5,371	5,786	6,197	6,612	7,436	8,265
2 × 14	1⅝ × 13½	1.32	49.36	333.18	4,113	4,524	4,936	5,347	5,758	6,170	6,581	7,403	8,226
2 × 14	1¾ × 13½	1.23	53.16	358.80	4,430	4,873	5,316	5,759	6,202	6,645	7,088	7,974	8,860
2½ × 14	2¼ × 13½	1.20	68.34	461.32	5,695	6,264	6,834	7,403	7,973	8,542	9,112	10,251	11,390
2 × 16	1⅝ × 15½	1.31	65.07	504.28	5,423	5,965	6,507	7,050	7,592	8,135	8,677	9,761	10,846
2 × 16	1¾ × 15½	1.22	70.10	543.06	5,842	6,426	7,010	7,594	8,178	8,762	9,347	10,515	11,683
2½ × 16	2¼ × 15½	1.18	90.10	698.23	7,508	8,260	9,010	9,760	10,512	11,262	12,013	13,515	15,018
2 × 18	1⅝ × 17½	1.30	82.94	725.75	6,912	7,603	8,294	8,986	9,677	10,368	11,059	12,442	13,824
2 × 18	1¾ × 17½	1.21	89.32	781.57	7,446	8,188	8,932	9,676	10,421	11,165	11,901	13,398	14,887
2½ × 18	2¼ × 17½	1.17	114.84	1,004.88	9,570	10,527	11,484	12,441	13,398	14,355	15,312	17,226	19,140
3 × 6	2½ × 5½	1.43	12.60	34.66	1,050	1,155	1,260	1,365	1,470	1,575	1,680	1,890	2,100
3 × 6	2¾ × 5½	1.30	13.86	38.13	1,155	1,271	1,386	1,501	1,617	1,732	1,848	2,079	2,310
3 × 8	2½ × 7½	1.37	23.42	87.89	1,952	2,147	2,342	2,538	2,733	2,928	3,123	3,514	3,904

3 × 8	2¾ × 7½	1.24	25.78	96.68	2,148	2,363	2,578	2,793	3,008	3,222	3,437	3,867	4,297
3 × 10	2½ × 9½	1.33	37.61	178.62	3,134	3,447	3,761	4,074	4,388	4,701	5,014	5,641	6,268
3 × 10	2¾ × 9½	1.21	41.36	196.48	3,447	3,791	4,136	4,480	4,825	5,170	5,515	6,204	6,893
3 × 12	2½ × 11½	1.31	55.10	316.85	4,592	5,051	5,510	5,970	6,429	6,888	7,347	8,266	9,184
3 × 12	2¾ × 11½	1.19	60.61	348.53	5,051	5,556	6,060	6,565	7,071	7,575	8,081	9,090	10,102
3 × 14	2½ × 13½	1.29	75.94	512.58	6,328	6,961	7,594	8,226	8,859	9,492	10,125	11,390	12,656
3 × 14	2¾ × 13½	1.17	83.53	563.84	6,961	7,657	8,353	9,049	9,745	10,441	11,137	12,529	13,922
3 × 16	2½ × 15½	1.28	100.10	775.81	8,342	9,176	10,010	10,845	11,679	12,513	13,347	15,016	16,684
3 × 16	2¾ × 15½	1.16	110.11	853.39	9,176	10,093	11,011	11,928	12,846	13,763	14,681	16,516	18,352
3 × 18	2½ × 17½	1.27	127.60	1,116.54	10,633	11,696	12,760	13,823	14,886	15,950	17,013	19,139	21,266
3 × 18	2¾ × 17½	1.15	140.36	1,228.19	11,696	12,866	14,036	15,205	16,375	17,545	18,715	21,054	23,393
4 × 4	3½ × 3½	1.49	7.15	12.51	596	656	715	775	834	894	954	1,073	1,192
4 × 4	3⅝ × 3⅝	1.34	7.94	14.39	662	728	794	860	926	992	1,059	1,190	1,323
4 × 6	3½ × 5½	1.36	17.64	48.53	1,470	1,617	1,764	1,911	2,058	2,205	2,352	2,646	2,940
4 × 6	3⅝ × 5⅝	1.26	19.12	53.76	1,593	1,753	1,912	2,071	2,231	2,390	2,549	2,868	3,187
4 × 8	3½ × 7½	1.30	32.81	123.05	2,734	3,007	3,281	3,554	3,828	4,101	4,374	4,921	5,468
4 × 8	3¾ × 7½	1.21	35.16	131.83	2,930	3,223	3,516	3,809	4,102	4,395	4,688	5,274	5,860
4 × 10	3½ × 9½	1.27	52.65	250.07	4,388	4,827	5,265	5,704	6,143	6,582	7,021	7,898	8,776
4 × 10	3¾ × 9½	1.18	56.41	267.93	4,701	5,171	5,641	6,110	6,581	7,050	7,521	8,461	9,402
4 × 12	3½ × 11½	1.25	77.15	443.59	6,429	7,072	7,715	8,358	9,001	9,644	10,286	11,572	12,858
4 × 12	3¾ × 11½	1.16	82.66	475.27	6,888	7,577	8,266	8,955	9,644	10,332	11,021	12,399	13,777
4 × 14	3½ × 13½	1.23	106.31	717.61	8,859	9,745	10,631	11,517	12,403	13,289	14,174	15,946	17,718
4 × 14	3¾ × 13½	1.15	113.91	768.87	9,493	10,442	11,391	12,340	13,289	14,238	15,188	17,086	18,985
4 × 16	3½ × 15½	1.22	140.15	1,086.13	11,679	12,847	14,015	15,183	16,351	17,519	18,686	21,022	23,358
4 × 16	3¾ × 15½	1.14	150.16	1,163.71	12,513	13,765	15,016	16,267	17,519	18,770	20,021	22,524	25,027
4 × 18	3½ × 17½	1.21	178.65	1,563.15	14,888	16,377	17,865	19,354	20,843	22,332	23,821	26,798	29,776
4 × 18	3¾ × 17½	1.13	191.41	1,674.80	15,951	17,546	19,141	20,736	22,331	23,926	25,521	28,711	31,902
6 × 6	5½ × 5½	1.30	27.73	76.26	2,311	2,542	2,773	3,004	3,235	3,467	3,698	4,160	4,622
6 × 8	5½ × 7½	1.24	51.56	193.36	4,297	4,727	5,156	5,586	6,016	6,446	6,875	7,735	8,594
6 × 10	5½ × 9½	1.21	82.73	392.96	6,894	7,583	8,273	8,962	9,652	10,341	11,030	12,409	13,788
6 × 12	5½ × 11½	1.19	121.23	697.07	10,103	11,113	12,123	13,134	14,144	15,155	16,165	18,185	20,206
6 × 14	5½ × 13½	1.17	167.06	1,127.67	13,922	15,314	16,706	18,099	19,491	20,883	22,275	25,060	27,844
6 × 16	5½ × 15½	1.16	220.23	1,706.78	18,353	20,188	22,023	23,859	25,694	27,530	29,365	33,035	36,706
6 × 18	5½ × 17½	1.16	280.73	2,456.38	23,394	25,733	28,073	30,412	32,752	35,091	37,430	42,109	46,788
6 × 20	5½ × 19½	1.15	348.56	3,398.49	29,047	31,952	34,856	37,761	40,666	43,571	46,475	52,285	58,094

TABLE 6.—MAXIMUM BENDING OR RESISTING MOMENTS IN FOOT-POUNDS FOR RECTANGULAR BEAMS—(*Continued*)

Size: Nominal (inches)	Size: Actual[1] (inches)	"Multiplying factor"	Section modulus (in.³)	Moment of inertia (in.⁴)	Resisting moments in foot-pounds for safe fiber stresses in pounds per square inch, as indicated: 1,000	1,100	1,200	1,300	1,400	1,500	1,600	1,800	2,000
8 × 8	7½ × 7½	1.21	70.31	263.67	5,859	6,445	7,031	7,617	8,203	8,789	9,374	10,546	11,718
8 × 10	7½ × 9½	1.18	112.81	535.86	9,401	10,341	11,281	12,221	13,161	14,102	15,042	16,922	18.802
8 × 12	7½ × 11½	1.16	165.31	950.55	13,776	15,154	16,531	17,909	19,286	20,664	22,042	24,797	27,552
8 × 14	7½ × 13½	1.15	227.81	1,537.74	18,984	20,882	22,781	24,679	26,578	28,476	30,374	34,171	37,968
8 × 16	7½ × 15½	1.14	300.31	2,327.43	25,026	27,529	30,031	32,534	35,036	37,539	40,042	45,047	50,052
8 × 18	7½ × 17½	1.13	382.81	3,349.61	31,901	35,091	38,281	41,471	44,661	74,852	51,042	57,422	63,802
8 × 20	7½ × 19½	1.12	475.31	4,634.30	39,609	43,570	47,531	51,492	55,453	59,414	63,374	71,296	79,218
10 × 10	9½ × 9½	1.17	142.89	678.76	11,908	13,099	14,289	15,480	16,671	17,862	19,053	21,434	23,816
10 × 12	9½ × 11½	1.15	209.40	1,204.03	17,450	19,195	20,940	22,685	24,430	26,175	27,920	31,410	34,900
10 × 14	9½ × 13½	1.13	288.56	1,947.80	24,047	26,452	28,856	31,261	33,666	36,071	38,475	43,285	48,094
10 × 16	9½ × 15½	1.12	380.40	2,948.07	31,700	34,870	38,040	41,210	44,380	47,550	50,720	57,060	63,400
10 × 18	9½ × 17½	1.11	484.90	4,242.84	40,408	44,449	48,490	52,530	56,571	60,612	64,653	72,734	80,816
10 × 20	9½ × 19½	1.11	602.06	5,870.11	50,172	55,189	60,206	65,224	70,241	75,258	80,275	90,310	100,344
12 × 12	11½ × 11½	1.14	253.48	1,457.51	21,123	23,235	25,348	27,460	29,572	31,685	33,797	38,021	42,246
12 × 14	11½ × 13½	1.12	349.31	2,357.86	29,109	32,020	34,931	37,842	40,753	43,664	46,574	52,396	58,218
12 × 16	11½ × 15½	1.11	460.48	3,568.72	38,373	42,210	46,018	49,885	53,722	57,560	61,397	59,071	76,746
12 × 18	11½ × 17½	1.10	586.98	5,136.07	48.915	53,807	58.698	63,590	68,481	73,373	78,264	88,047	97,830
12 × 20	11½ × 19½	1.10	728.81	7,105.93	60,734	66,807	72,881	78,954	85,028	91,101	97,174	109,321	121,468
14 × 14	13½ × 13½	1.12	410.06	2,767.93	34,172	37,589	41,006	44,424	47,481	51,258	54,675	61,510	68,344
14 × 16	13½ × 15½	1.11	540.56	4,189.37	45,047	49,552	54,056	58,561	63,066	67,571	72,075	81,085	90,094
14 × 18	13½ × 17½	1.10	689.06	6,029.30	57,422	63,164	68,906	74,649	80,391	86,133	91,875	103,360	114,844
14 × 20	13½ × 19½	1.09	855.56	8,341.74	71,297	78,427	85,556	92,686	99,816	106,946	114,075	128,335	142,594

16 × 16	15½ × 15½	1.10	620.64	4,810.01	51,720	56,892	62,064	67,236	72,408	77,580	82,752	93,096	103,440
16 × 18	15½ × 17½	1.09	791.15	6,922.53	65,929	72,522	79,115	85,708	92,301	98,894	105,486	118,672	131,858
16 × 20	15½ × 19½	1.09	982.31	9,577.55	81,859	90,045	98,231	106,417	114,603	122,789	130,974	147,346	163,718
16 × 22	15½ × 21½	1.08	1,194.15	12,837.07	99,513	109,464	119,415	129,367	139,318	149,270	159,221	179,123	199,026
16 × 24	15½ × 23½	1.08	1,426.65	16,763.10	118,888	130,777	142,665	154,554	166,443	178,332	190,221	213,998	237,776
18 × 18	17½ × 17½	1.09	893.23	7,815.76	74,436	81,880	89,323	96,767	104,210	111,654	119,098	133,985	148,872
18 × 20	17½ × 19½	1.08	1,109.06	10,813.37	92,422	101,664	110,906	120,149	129,391	138,633	147,875	166,360	184,844
18 × 22	17½ × 21½	1.08	1,348.23	14,493.47	112,353	123,588	134,823	146,059	157,294	168,530	179,765	202,235	224,706
18 × 24	17½ × 23½	1.07	1,610.73	18,926.08	134,228	147,651	161,073	174,496	187,919	201,342	214,765	241,610	268,456
18 × 26	17½ × 25½	1.07	1,896.56	24,181.18	158,047	173,852	189,656	205,461	221,266	237,071	252,875	284,485	316,094
20 × 20	19½ × 19½	1.08	1,235.81	12,049.18	102,984	113,282	123,581	133,879	144,178	154,476	164,774	185,371	205,968
20 × 22	19½ × 21½	1.07	1,502.31	16,149.87	125,193	137,712	150,231	162,751	175,270	187,790	200,309	225,347	250,386
20 × 24	19½ × 23½	1.07	1,794.81	21,089.06	149,568	164,525	179,481	194,438	209,395	224,352	239,309	269,222	299,136
20 × 26	19½ × 25½	1.07	2,113.31	26,944.74	176,109	193,720	211,331	228,942	246,553	264,164	281,774	316,996	352,218
20 × 28	19½ × 27½	1.06	2,457.81	33,794.90	204,818	225,300	245,781	266,263	286,745	307,227	327,709	368,672	409,636
20 × 30	19½ × 29½	1.06	2,828.31	41,717.62	235,693	259,262	282,831	306,401	329,970	353,540	377,109	424,247	471,386

[1] This table is based on tables from the "Southern Pine Manual" of the Southern Pine Association, and the "Structural Timber Handbook on Pacific Coast Woods" of the West Coast Lumbermen's Association.

The standards of the latter association are as follows:

"*Dimensions, Plank and Small Timbers.*—Sizes—S1S1E or S4S: 2 × 3 to 1⅝ × 2⅝; 2 × 4 to 1⅝ × 3⅝; 2 × 6 to 1⅝ × 5⅝; 2 × 8 to 1⅝ × 7½; 2 × 10 to 1⅝ × 9½; 2 × 12 to 1⅝ × 11½; 2 × 14 to 1⅝ × 13½; 2 × 16 to 1⅝ × 15½; etc.; 3 × 4 to 2½ × 3⅝; 3 × 6 to 2½ × 5½; 3 × 8 to 2½ × 7½; 3 × 10 to 2½ × 9½; 3 × 12 to 2½ × 11½; 3 × 14 to 2½ × 13½; 3 × 16 to 2½ × 0 15½; etc.; 4 × 4 to 3½ × 3½; 4 × 6 to 3½ × 5½, etc.; 5 × 5 to 4½ × 4½; etc.; 6 × 6 and 8, ½ in. off each way.

"*Timbers.*—Sizes—S1S, S1E, S1S1E, or S4S: 8 × 8 and larger ½ in. off each way. Standard lengths are multiples of 2 ft."

The standards of the Southern Pine Association for timber surfaced four sides are the same as those of the West Coast Lumbermen's Association, *i.e.*, ½ in. off the nominal width and depth. For material surfaced one side one edge (S1S1E) their standards are ¼ in. off the nominal width and ½ in. off the nominal depth.

Table 7.—Table of Safe Loads and Deflections for Timber Joists with Nominal Width of 2 In., Uniformly Loaded, Based on Maximum Flexural Stress of 1,800 Lb. per Sq. In.

		2×4	2×6	2×8	2×10	2×12	2×14	2×16	2×18
Sizes	Rough size	2×4	2×6	2×8	2×10	2×12	2×14	2×16	2×18
	Surfaced size S1S1E[1]	1⅝×3⅝	1⅝×5⅝	1⅝×7½	1⅝×9½	1⅝×11½	1¾×13½	1¾×15½	1¾×17½
	Section modulus	3.56	8.57	15.23	24.44	35.82	53.16	70.10	89.32
Spans in feet: 3	HS	1,372							
	D1	0.0581							
4	B	1,068							
	D	967							
	D1	0.1379	0.0369						
	HS	......	2,125						
5	B	854	2,056						
	D	619	......						
	D1	0.2693	0.0720						
6	B	712	1,714						
	D	430	1,607						
	D1	0.4651	0.1244	0.0525					
	HS	......	......	2,843					
7	B	610	1,469	2,611					
	D	316	1,180	......					
	D1	0.7384	0.1977	0.0834					
8	B	534	1,285	2,284					
	D	242	904	2,142					
	D1	1.1020	0.2950	0.1245	0.0612				
	HS	......	......	......	3,601				
9	B		1,143	2,031	3,258				
	D		714	1,693	......				
	D1		0.4202	0.1772	0.0872	0.0492			
	HS		......	......	......	4,361			
10	B		1,028	1,828	2,933	4,298			
	D		578	1,371	2,786	......			
	D1		0.5767	0.2431	0.1196	0.0674			
11	B		935	1,661	2,666	3,908			
	D		478	1,133	2,303	......			
	D1		0.7671	0.3236	0.1592	0.0897	0.0515		
	HS		......	......	......	......	5,512		
12	B		857	1526	2444	3,582	5,316		
	D		402	952	1,935	3,432	......		
	D1		0.9950	0.4202	0.2067	0.1165	0.0669		
13	B			1,406	2,256	3,306	4.907		
	D			811	1,648	2,924	......		
	D1			0.5343	0.2630	0.1482	0.0850	0.0562	
	HS			......	......	......	......	6,328	
14	B			1,306	2,095	3,070	4,556	6,008	
	D			700	1,422	2,521	4,393	......	
	D1			0.6667	0.3282	0.1851	0.1062	0.0702	
15	B			1,218	1,955	2,865	4,253	5,608	
	D			609	1,238	2,196	3,827	......	
	D1			0.8210	0.4040	0.2277	0.1306	0.0863	0.0599
	HS			......	......	......	......	......	7,147
16	B			1,142	1,833	2,686	3,987	5,257	6,699
	D			536	1,089	1,931	3,364	5,091	......
	D1			0.9950	0.4898	0.2762	0.1585	0.1047	0.0728

[1] S1S1E = surfaced one side and one edge.

TABLE 7.—TABLE OF SAFE LOADS AND DEFLECTIONS FOR TIMBER JOISTS WITH NOMINAL WIDTH OF 2 IN., UNIFORMLY LOADED, BASED ON MAXIMUM FLEXURAL STRESS OF 1,800 LB. PER SQ. IN.—(*Continued*)

Sizes	Rough size	2×4	2×6	2×8	2×10	2×12	2×14	2×16	2×18
	Surfaced size S1S1E[1]	1⅝×3⅝	1⅝×5⅝	1⅝×7½	1⅝×9½	1⅝×11½	1¾×13½	1¾×15½	1¾×17½
	Section modulus	3.56	8.57	15.23	24.44	35.82	53.16	70.10	89.32
Spans in feet 17	B				1,725	2,528	3,753	4,948	6,305
	D				964	1,710	2,980	4,510	......
	D1				0.5878	0.3314	0.1902	0.1256	0.0873
18	B				1,629	2,388	3,544	4,673	5,954
	D				860	1,525	2,658	4,023	5,790
	D1				0.6977	0.3934	0.2254	0.1492	0.1036
19	B				1,544	2,262	3,358	4,427	5,641
	D				772	1,369	2,385	3,610	5,196
	D1				0.8204	0.4626	0.2655	0.1754	0.1219
20	B				1,466	2,149	3,190	4,206	5,359
	D				697	1,236	2,153	3,258	4,690
	D1				0.9565	0.5395	0.3097	0.2046	0.1422
21	B					2,047	3,038	4,006	5,104
	D					1,121	1,953	2,956	4,254
	D1					0.6244	0.3585	0.2368	0.1646
22	B					1,954	2,900	3,824	4,872
	D					1,021	1,779	2,693	3,876
	D1					0.7183	0.4122	0.2723	0.1892
23	B					1,869	2,774	3,657	4,660
	D					934	1,628	2,464	3,546
	D1					0.8208	0.4710	0.3112	0.2162
24	B					1,791	2,658	3,505	4,466
	D					858	1,495	2,263	3,257
	D1					0.9324	0.5351	0.3535	0.2456
25	B						2,552	3,365	4,287
	D						1,378	2,085	3,001
	D1						0.6048	0.3996	0.2777
26	B						2,454	3,235	4,122
	D						1,274	1,928	2,775
	D1						0.6804	0.4495	0.3123
27	B						2,363	3,116	3,970
	D						1,181	1,788	2,573
	D1						0.7619	0.5034	0.3498
28	B						2,278	3,004	3,828
	D						1,098	1,663	2,392
	D1						0.8498	0.5614	0.3902
29	B							2,901	3,696
	D							1,550	2,230
	D1							0.6238	0.4334
30	B							2,804	3,573
	D							1,448	2,085
	D1							0.6905	0.4798
31	B								3,457
	D								1,952
	D1								0.5294
32	B								3,349
	D								1,832
	D1								0.5823

[1] S1S1E = surfaced one side and one edge.

TABLE 8.—TABLE OF SAFE LOADS AND DEFLECTIONS FOR TIMBER JOISTS WITH NOMINAL WIDTH OF 3 IN. UNIFORMLY LOADED, BASED ON MAXIMUM FLEXURAL STRESS OF 1,800 LB. PER SQ. IN.

		3×6	3×8	3×10	3×12	3×14	3×16	3×18
Sizes	Rough size	3×6	3×8	3×10	3×12	3×14	3×16	3×18
Sizes	Surfaced size S1S1E[1]	2¾×5½	2¾×7½	2¾×9½	2¾×11½	2¾×13½	2¾×15½	2¾×17½
Sizes	Section modulus	13.86	25.78	41.36	60.61	83.53	110.11	140.36
Spans in feet: 4	HS	3,528						
4	D1	0.0233						
5	B	3,326						
5	D1	0.0455						
6	B	2,773						
6	D	2,542						
6	D1	0.0787	0.0310					
6	HS	......	4,812					
7	B	2,376	4,419					
7	D	1,867	......					
7	D1	0.1250	0.0493					
8	B	2,079	3,867					
8	D	1,429	3,625					
8	D1	0.1866	0.0735	0.0362				
8	HS	......	......	6,097				
9	B	1,848	3,437	5,515				
9	D	1,129	2,865	......				
9	D1	0.2657	0.1047	0.0515	0.0291			
9	HS	......	......	......	7,378			
10	B	1,663	3,093	4,963	7,273			
10	D	915	2,320	4,715	......			
10	D1	0.3643	0.1437	0.0707	0.0398			
11	B	1,512	2,812	4,512	6,612			
11	D	756	1,918	3,897	......			
11	D1	0.4850	0.1912	0.0941	0.0530	0.0328		
11	HS	......	......	......	......	8,662		
12	B	1,386	2,578	4,136	6,061	8,353		
12	D	635	1,612	3,275	5,808	......		
12	D1	0.6299	0.2481	0.1221	0.0689	0.0426		
13	B	......	2,380	3,818	5,595	7,710		
13	D	......	1,373	2,790	4,949	......		
13	D1	......	0.3156	0.1553	0.0875	0.0541	0.0357	
13	HS	......	......	......	......	......	9,947	
14	B	......	2,210	3,545	5,195	7,159	9,438	
14	D	......	1,184	2,406	4,267	6,904	......	
14	D1	......	0.3941	0.1939	0.1094	0.0676	0.0446	
15	B	......	2,065	3,309	4,849	6,682	8,809	
15	D	......	1,031	2,096	3,717	6,014	......	
15	D1	......	0.4850	0.2386	0.1345	0.0831	0.0549	0.0382
15	HS	......	......	......	......	......	......	11,228
16	B	......	1,934	3,102	4,546	6,264	8,258	10,527
16	D	......	906	1,842	3,267	5,286	8,001	.....
16	D1	......	0.5887	0.2895	0.1632	0.1009	0.0666	0.0463
17	B	......	......	2,920	4,278	5,896	7,772	9,908
17	D	......	......	1,632	2,894	4,682	7,087	......
17	D1	......	......	0.3472	0.1958	0.1210	0.0799	0.0555

[1] S1S1E = surfaced one side and one edge.

TABLE 8.—TABLE OF SAFE LOADS AND DEFLECTIONS FOR TIMBER JOISTS WITH NOMINAL WIDTH OF 3 IN. UNIFORMLY LOADED, BASED ON MAXIMUM FLEXURAL STRESS OF 1,800 LB. PER SQ. IN.—(*Continued*)

Sizes	Rough size	3×6	3×8	3×10	3×12	3×14	3×16	3×18
	Surfaced size S1S1E[1]	2¾×5½	2¾×7½	2¾×9½	2¾×11½	2¾×13½	2¾×15½	2¾×17½
	Section modulus	13.86	25.78	41.36	60.61	83.53	110.11	140.36
Spans in feet								
18	B			2,758	4,041	5,568	7,341	9,356
	D	......	......	1,455	2,582	4,177	6,322	9,098
	D1	......	......	0.4124	0.2324	0.1437	0.0949	0.0659
19	B	......	......	2,612	3,828	5,275	6,954	8,865
	D	......	......	1,306	2,317	3,748	5,673	8,165
	D1	......	......	0.4849	0.2733	0.1689	0.1116	0.0775
20	B	......	......	2,481	3,636	5,012	6,606	8,421
	D	......	......	1,179	2,091	3,383	5,120	7,369
	D1	......	......	0.5655	0.3188	0.1971	0.1302	0.0904
21	B	......	......	......	3,463	4,773	6,292	8,020
	D	......	......	......	1,897	3,069	4,644	6,684
	D1	......	......	......	0.3690	0.2281	0.1507	0.1047
22	B	......	......	......	3,306	4,556	6,006	7,656
	D	......	......	......	1,728	2,796	4,232	6,090
	D1	......	......	......	0.4244	0.2623	0.1733	0.1204
23	B	......	......	......	3,162	4,358	5,745	7,323
	D	......	......	......	1,581	2,558	3,872	5,572
	D1	......	......	......	0.4849	0.2997	0.1980	0.1376
24	B	......	......	......	3,031	4,176	5,505	7,018
	D	......	......	......	1,452	2,350	3,556	5,118
	D1	......	......	......	0.5510	0.3405	0.2250	0.1563
25	B	......	......	......	......	4,009	5,285	6,737
	D	......	......	......	......	2,165	3,277	4,716
	D1	......	......	......	......	0.3849	0.2543	0.1767
26	B	......	......	......	......	3,855	5,082	6,478
	D	......	......	......	......	2,002	3,030	4,361
	D1	......	......	......	......	0.4329	0.2860	0.1987
27	B	......	......	......	......	3,712	4,894	6,238
	D	......	......	......	......	1,856	2,810	4,043
	D1	......	......	......	......	0.4848	0.3203	0.2226
28	B	......	......	......	......	3,579	4,719	6,015
	D	......	......	......	......	1,726	2,612	3,760
	D1	......	......	......	......	0.5407	0.3573	0.2482
29	B	......	......	......	......	......	4,556	5,808
	D	......	......	......	......	......	2,436	3,505
	D1	......	......	......	......	......	0.3969	0.2758
30	B	......	......	......	......	......	4,404	5,614
	D	......	......	......	......	......	2,276	3,275
	D1	......	......	......	......	......	0.4394	0.3053
31	B	......	......	......	......	......	......	5,433
	D	......	......	......	......	......	......	3,067
	D1	......	......	......	......	......	......	0.3369
32	B	......	......	......	......	......	......	5,263
	D	......	......	......	......	......	......	2,879
	D1	......	......	......	......	......	......	0.3708

[1] S1S1E = surfaced one side and one edge.

Table 9.—Table of Safe Loads and Deflections for Timber Joists with Nominal Width of 4 In. Uniformly Loaded, Based on Maximum Flexural Stress of 1,800 Lb. per Sq. In.

Sizes		4×4	4×6	4×8	4×10	4×12	4×14	4×16	4×18
Rough size		4×4	4×6	4×8	4×10	4×12	4×14	4×16	4×18
Surfaced size S1S1E[1]		3⅝×3⅝	3⅝×5⅝	3¾×7½	3¾×9½	3¾×11½	3¾×13½	3¾×15½	3¾×17½
Section modulus		7.94	19.12	35.16	56.41	82.66	113.91	150.16	191.41
Spans in feet									
3	HS	3,066							
	D1	0.0261							
4	B	2,382							
	D	2,152							
	D1	0.0618	0.0165						
	HS	......	4,760						
5	B	1,905	4,588						
	D	1,382	......						
	D1	0.1206	0.0323						
6	B	1,588	3,824						
	D	960	3,584						
	D1	0.2083	0.0558	0.0227					
	HS	......	......	6,562					
7	B	1,361	3,277	6,027					
	D	705	2,633	......					
	D1	0.3307	0.0886	0.0361					
8	B	1,191	2,868	5,274					
	D	540	2,016	4,944					
	D1	0.4938	0.1323	0.0539	0.0265				
	HS	......	......	......	8,312				
9	B	......	2,549	4,688	7,521				
	D	......	1,593	3,906	......				
	D1	......	0.1883	0.0768	0.0378	0.0213			
	HS	......	......	......	......	10,062			
10	B	......	2,294	4,219	6,769	9,919			
	D	......	1,290	3,164	6,430	......			
	D1	......	0.2584	0.1053	0.0518	0.0292			
11	B	......	2,086	3,835	6,154	9,017			
	D	......	1,066	2,615	5,315	......			
	D1	......	0.3440	0.1402	0.0690	0.0389	0.0240		
	HS	......	......	......	......	......	11,812		
12	B	......	1,912	3,516	5,641	8,266	11,391		
	D	......	896	2,197	4,466	7,921	......		
	D1	.....	0.4464	0.1821	0.0896	0.0505	0.0312		
13	B	......	......	3,246	5,207	7,630	10,515		
	D	......	......	1,873	3,805	6,750	......		
	D1	......	......	0.2313	0.1139	0.0642	0.0397	0.0262	
	HS	......	......	......	......	......	......	13,562	
14	B	......	......	3,014	4,835	7,085	9,764	12,870	
	D	......	......	1,615	3,281	5,820	9,415	......	
	D1	......	......	0.2890	0.1422	0.0802	0.0496	0.0327	
15	B	......	......	2,813	4,513	6,613	9,113	12,013	
	D	......	......	1,406	2,858	5,070	8,201	......	
	D1	......	......	0.3556	0.1750	0.0986	0.0610	0.0403	0.0279
	HS	......	......	......	......	......	......	......	15,312
16	B	......	......	2,637	4,230	6,199	8,543	11,262	14,356
	D	......	......	1,236	2,511	4,456	7,208	10,909	......
	D1	......	......	0.4316	0.2124	0.1197	0.0740	0.0489	0.0339

[1] S1S1E = surfaced one side and one edge.

TABLE 9.—TABLE OF SAFE LOADS AND DEFLECTION FOR TIMBER JOISTS WITH NORMAL WIDTH OF 4 IN. UNIFORMLY LOADED, BASED ON MAXIMUM FLEXURAL STRESS OF 1,800 LB. PER SQ. IN.—(*Continued*)

Sizes	Rough size	4×4	4×6	4×8	4×10	4×12	4×14	4×16	4×18
	Surface size S1S1E[1]	3⅝×3⅝	3⅝×5⅝	3¾×7½	3¾×9½	3¾×11½	3¾×13½	3¾×15½	3¾×17½
	Section modulus	7.94	19.12	35.16	56.41	82.66	113.91	150.16	191.41
Spans in feet 17	B	.....	.....	.....	3,982	5,835	8,041	10,599	13,511
	D	.....	.....	.....	2,225	3,947	6,385	9,664	.....
	D1	.....	.....	.....	0.2547	0.1436	0.0887	0.0586	0.0407
18	B	.....	.....	.....	3,760	5,510	7,594	10,010	12,760
	D	.....	.....	.....	1,985	3,521	5,695	8,620	12,406
	D1	.....	.....	.....	0.3023	0.1704	0.1053	0.0696	0.0483
19	B	.....	.....	.....	3,563	5,221	7,194	9,484	12,089
	D	.....	.....	.....	1,782	3,160	5,112	7,737	11,134
	D1	.....	.....	.....	0.3554	0.2004	0.1239	0.0818	0.0569
20	B	.....	.....	.....	3,384	4,959	6,834	9,009	11,484
	D	.....	.....	.....	1,608	2,852	4,613	6,982	10,049
	D1	.....	.....	.....	0.4146	0.2337	0.1445	0.0955	0.0663
21	B	.....	.....	.....	.....	4,724	6,509	8,580	10,938
	D	.....	.....	.....	.....	2,587	4,184	6,330	9,115
	D1	.....	.....	.....	.....	0.2706	0.1673	0.1105	0.0768
22	B	.....	.....	.....	.....	4,509	6,213	8,190	10,440
	D	.....	.....	.....	.....	2,357	3,813	5,770	8,305
	D1	.....	.....	.....	.....	0.3111	0.1923	0.1271	0.0883
23	B	.....	.....	.....	.....	4,313	5,943	7,834	9,986
	D	.....	.....	.....	.....	2,156	3,488	5,280	7,598
	D1	.....	.....	.....	.....	0.3556	0.2198	0.1452	0.1009
24	B	.....	.....	.....	.....	4,133	5,695	7,508	9,570
	D	.....	.....	.....	.....	1,980	3,204	4,849	6,978
	D1	.....	.....	.....	.....	0.4040	0.2497	0.1650	0.1146
25	B	.....	.....	.....	.....	.....	5,468	7,208	9,188
	D	.....	.....	.....	.....	.....	2,952	4,469	6,431
	D1	.....	.....	.....	.....	.....	0.2822	0.1865	0.1296
26	B	.....	.....	.....	.....	.....	5,257	6,930	8,834
	D	.....	.....	.....	.....	.....	2,730	4,132	5,946
	D1	.....	.....	.....	.....	.....	0.3175	0.2099	0.1457
27	B	.....	.....	.....	.....	.....	5,063	6,674	8,507
	D	.....	.....	.....	.....	.....	2,531	3,831	5,514
	D1	.....	.....	.....	.....	.....	0.3555	0.2349	0.1632
28	B	.....	.....	.....	.....	.....	4,882	6,435	8,203
	D	.....	.....	.....	.....	.....	2,354	3,562	5,127
	D1	.....	.....	.....	.....	.....	0.3965	0.2620	0.1820
29	B	.....	.....	.....	.....	.....	.....	6,214	7,920
	D	.....	.....	.....	.....	.....	.....	3,321	4,780
	D1	.....	.....	.....	.....	.....	.....	0.2911	0.2022
30	B	.....	.....	.....	.....	.....	.....	6,006	7,656
	D	.....	.....	.....	.....	.....	.....	3,103	4,466
	D1	.....	.....	.....	.....	.....	.....	0.3222	0.2239
31	B	.....	.....	.....	.....	.....	.....	.....	7,409
	D	.....	.....	.....	.....	.....	.....	.....	4,183
	D1	.....	.....	.....	.....	.....	.....	.....	0.2470
32	B	.....	.....	.....	.....	.....	.....	.....	7,178
	D	.....	.....	.....	.....	.....	.....	.....	3,925
	D1	.....	.....	.....	.....	.....	.....	.....	0.2717

[1] S1S1E = surfaced one side and one edge.

WOODEN GIRDERS

The loads coming upon the girders of a floor system consist of the loads delivered by the floor joists, plus the weight of the girders themselves, plus any loads coming directly upon the girder, as distinguished from loads transmitted by the joists. Girders often carry partition loads directly.

In office buildings, dwelling houses, and certain areas of other buildings, exclusive of warehouses and storage buildings, where crowds of people cannot congregate, the live load coming upon the girders is reduced in intensity. The reduction factor is specified in building ordinances, and is usually taken as 20 per cent.

Horizontal shear at the ends of girders often governs the girder section, as in the case of short spans with heavy loading, and this stress should always be checked.

The end connections of girders are of much more importance than the end connections of joists, as the girders of a building, together with the posts, usually form the stiffening frame of the building against lateral forces. Particular attention also needs to be paid to the design of the support of wooden girders, as failure of a girder would mean the probable collapse of at least a whole floor bay.

Wooden girders, even if continuous over two spans, are generally computed as simple beams.

The detail of end connection of building girders will depend on the type of building. If such building is of mill construction with heavy masonry walls, the wall ends of girders should be encased in wall boxes, the inner end connections designed to allow the girders to fall, in case of fire, without pulling the columns with them. In other types of buildings, as the mill type, stiff rigid connections of girders to posts may be desirable.

13. Girders of Solid Section.—The section of wooden girders composed of solid sticks of timber are to be designed exactly as treated under "Wooden Beams."

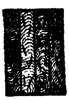


Fig. 1.—Built-up girder—type (1).

14. Built-up Wooden Girders.—Built-up wooden girders may be divided into the following types:

1. Girders constructed of planking, set side by side, the width of plank vertical, as in Fig. 1.

2. Girders constructed of two or more timbers set on top of one another, but not fastened together, as in Fig. 2.

3. Girders constructed of two or more timbers set on top of one another, and diagonally sheathed with boards or planking, as in Fig. 3.

4. Girders constructed of two or more timbers set on top of one another, and effectively fastened together by means of hard wood or metal keys or pins, combined with bolting, as in Fig. 4.

Fig. 2.—Built-up girder—type (2).

Type (1).—A girder, or beam, of this type, if all planking extends the full length of girder, is of full nominal thickness, and is well spiked and bolted together. It is generally given credit for being somewhat stronger than a girder or beam of solid section of the same dimensions, since the planking is assumed to

be better seasoned and freer from defects, particularly checks, than the larger solid timber. A construction of this type is often observed in small buildings where planks are more easily obtained than heavy timbers, and where the solid section construction might incur purchase of additional material by the contractor. Insufficient spiking, lack of proper bolting, probability of planking under-running in thickness, thus giving an actual size of finished beam less than the solid section, possibility of some planks being spliced, and the probability of upper surface of girder being uneven—i.e., one plank projecting higher than another, giving uneven bearings for the joists—are practical reasons for always

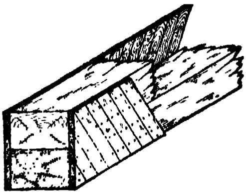


Fig. 3.—Built-up girder—type (3).

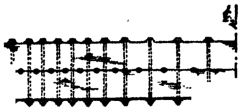


Fig. 4.—One-half typical built-up girder—type (4).

advocating the beam of solid section. Incidentally, no building ordinance gives the built-up girder any advantage in strength. Solid sections should be insisted upon for important beams. When it is necessary to use this type of built-up girder, provide two bolts at each end, and pairs of bolts at intervals of 2 ft. along the length of beam, the size of bolts to be not less than ⅝ in., and preferably ¾ in.

Type (2).—This type of girder should never be used. The strength of the combined section is practically no more than the sum of the strengths of the component sticks, each stick acting as a separate beam. Even if such a girder should be constructed of planking, well spiked together, the above statement of resulting strength would hold, as the nailing would be insufficient to prevent one plank from slipping on another.

Type (3).—In this type of built-up girder, as in the following type, the object of all connections between the component sticks (usually two) is to prevent relative motion along the plane of contact. If this condition of no-slip could be attained, the compound girder would have the strength of a single stick of timber of the same outside total dimensions. Type (3) is considerably less efficient than Type (4), both as regards ultimate strength and deflection under load. The diagonal sheathing is spiked to the timbers, and the sheathing should be at 45 deg. with the length of girder.

Tests made by Edgar Kidwell (see Trans. Am. Soc. Mining Engineers, 1897, vol. 27) showed an efficiency of approximately 70 per cent, based on the ultimate strength, as compared to a beam of solid section, while the efficiency factor based on deflection was about 50 per cent.

The sheathing for such girders should be not less than 1¼ in. and not over 2 in. in thickness. With such sheathing the nails should be 10- or 12-D for the smaller thickness, and 20- to 30-D for the 2-in. sheathing. For a girder supporting uniform load the diagonals near the ends require the most spikes. The spiking in each diagonal should be concentrated near the plane of junction of the timbers, and at the ends of the diagonals.

In designing a girder of this type, it must be remembered that the case is not similar to that of a truss. In a truss are two chords, in each of which, due to the small depth of chord as compared to the large depth of truss, the stress is practically uniform throughout the cross-section of each chord, and the diagonals take either tension or compression. The side planking in the built-up girder under discussion is subjected to bending moments, and, consequently, the nails take unequal loading. Any slip of the nails under stress allows a corresponding slip in the plane of contact of the two main timbers, with a consequent deflection of the girder. By referring to p. 404 it will be found that nails under lateral or shearing strain slip at a small load.

Type (4).—In the girders of this class, the tendency of one timber to slip over the other is resisted by wedges, keys, or pins driven into the contact faces of the timbers. These wedges, whether rectangular, square, or round, perform their main function through bearing against the ends of the fibers of the timbers. A second action is pressure across the fibers of the timbers. The action of these wedges tends to separate the two timbers, resulting in tension in the bolts. The amount of such tension depends primarily upon the shape of wedge. For example, a square key will produce a greater bolt tension than a rectangular key with long axis parallel to the length of girder, while a circular key or pin will give the greatest tension in the bolts.

The number and size of keys is to be determined directly from consideration of horizontal shear in the girder, in accordance with the principles of Sec. 1, Art. **51**, and illustrated in the typical example hereafter.

The bolts in such a girder are assumed to take only tension, although, due to their resistance to lateral forces, they add somewhat to the strength of the girder. However, it is always advisable, and on the safe side, to neglect such lateral resistance of the bolts.

Kidwell's series of tests on girders of this type showed a maximum efficiency of 75 to 80 per cent of an equivalent girder of solid section, the former figure representing girders with white oak keys and the latter figure with keys of iron.

Any shrinkage in the timbers will allow the component parts of the girder to separate, with a consequent loss of efficiency, and an increased deflection. As fully seasoned timber is not always available, this type of girder should be avoided for cases in which the major portion of the load is a constant load. For situations in which the girder carries live load for the greater part, in which access may be had to tighten the bolts as the wood seasons, and when it is reasonably certain that such maintenance will be given, this girder may be used with confidence. Obviously, the keyed girder is particularly unsuited for such locations as will prohibit access for tightening the bolts, as in a floor system ceiled underneath.

15. Examples of Design of Solid and Built-up Girders.—The following typical examples will illustrate the method of design for the most common cases that will be encountered:

Conditions of Design:
Span: 26 ft.
Loading: Uniform load of 1,500 lb. per lin. ft.
One concentrated load of 6,000 lb., 7 ft. from left support.
One concentrated load of 14,000 lb. at center of span.
One concentrated load of 2,000 lb., 9 ft. from right support.
Timber: Long-leaf yellow pine, dense structural grade.

The reactions are given in Fig. 5 and the bending moment curves in Fig. 6. The parabola of moments for uniform load is plotted about the base line, and the polygon of moments for concentrated loads below this line.

The following unit stresses will be used:

Bending stress on outer fibers	1,800 lb. per sq. in.
Longitudinal shear	175 lb. per sq. in.
Bearing across grain	400 lb. per sq. in.
Bearing against grain	1,800 lb. per sq. in.

Solid Girder.—Maximum bending moment = 248,100 ft.-lb. From Table 6, p. 369 an 18 × 24-in. girder, surfaced to $17\frac{1}{2} \times 23\frac{1}{2}$ in., has a resisting moment of 241,610 ft.-lb., which will be near enough to be used, or a double girder may be used. For example, 2 – 14 × 20-in. sticks would have a safe resisting moment of 256,670 ft.-lb. The required cross-section for longitudinal shear is

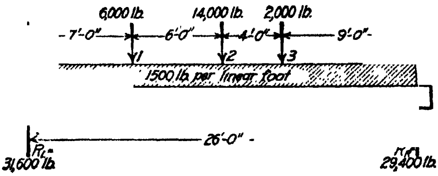


Fig. 5.– Loads and reactions for girder of Art. 56.

$$\frac{(\frac{3}{2})(31,600)}{175} = 271 \text{ sq. in.}$$

Either of the above girders has an excess of timber for shear.

Built-up Girders.— Type (1) could not be considered, as no standard planking 20 or 24 in. is made.

Type (2) would require 2 – 14 × 20-in. sticks, one on top of the other—an impractical consideration.

Type (3).—Maximum bending moment = 248,100 ft.-lb. Using an efficiency factor of 70 per cent the moment to be designed for is 355,000 ft.-lb. Assume a width of 14 in. The required section modulus

$$S = \frac{(355,000)(12)}{1,800} = 2,370$$

$$d = \sqrt{\frac{(2,370)(6)}{13.5}} = 32.4 \text{ in.}$$

Use 2 – 14 × 18-in. sticks, finished section $13\frac{1}{2} \times 35$ in.

Use 2 × 12-in. sheathing both sides, spiked with 40-D nails—detail similar to that of Fig. 3.

Type (4).—Assume efficiency factor of 80 per cent

Designing moment

$$= \frac{248,100}{0.80} = 310,000 \text{ ft.-lb.}$$

$$S = \frac{(310,000)(12)}{1,800} = 2,070$$

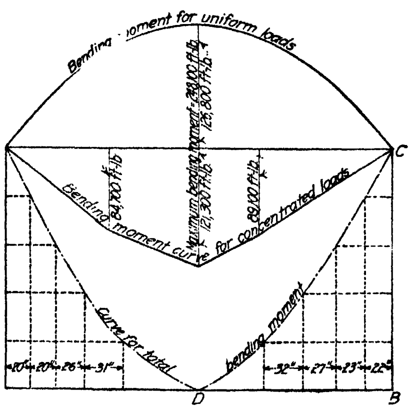


Fig. 6.—Diagram for bending moments and spacing of shear keys for girder of Art. 46.

Assuming a width of $13\frac{1}{2}$ in., the required depth is found to be 30.2 in. Use 2 – 14 × 16-in. sticks, S4S,[1] actual combined section $13\frac{1}{2} \times 31$ in., section modulus 2,160.

A shear diagram is next constructed, as shown in Fig. 7*a*. Each ordinate of this diagram represents the total vertical shear at the point where the ordinate is taken, and this total vertical shear is proportional to the maximum intensity of the horizontal shear at the same point. Considering Point (1), directly under the concentrated load of 6,000 lb., the total vertical shear just to the left of this point is 31,600 – (7)(1,500) = 21,100 lb. The ordinate one foot to the left will have a value of 31,600 – (6)(1,500) = 22,600 lb.

[1] Surfaced four sides.

The area of the trapezoid between these two ordinates is therefore $\frac{22,600 + 21,100}{2} =$ 21,850 ft.-lb. The maximum intensity of horizontal shear at a point immediately to the right of point (1), is

$$v = \frac{3}{2} \cdot \frac{V}{bd} = \frac{3}{2} \frac{21,100}{(13\frac{1}{2})(31)} = 76 \text{ lb. per sq. in.}$$

The next step is to find a means for determining the proper spacing of keys. Two methods will be explained.

Method 1.—For this purpose, the total vertical shear between the point of zero shear and each point of division of beam is computed by adding together the differential shears between these two points. The corresponding ordinates are drawn, giving the line *ABC* in Fig. 7*b*. The summation of the vertical shears to the left of the point of zero shear is found to be 248,050 ft.-lb.; agreeing with the value of the bending moment, which furnishes a check on the work. Similarly, the summation of the vertical shears to the right of the point of zero shear will give the same value.

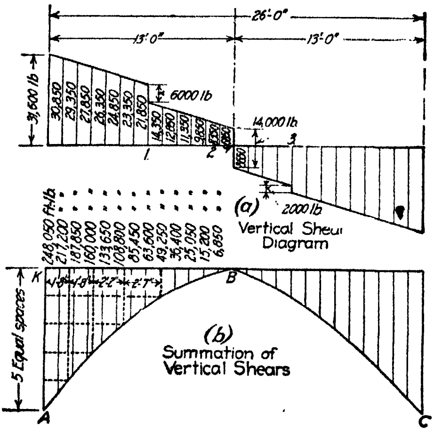


FIG. 7.

Since, for practical reasons, all keys will be of uniform size, and must therefore be stressed uniformly, the spacing of same must vary. The number of keys for the left half of girder will be taken at 5.

Method 2.—A much simpler method for constructing the total shear diagram will now be shown. In Fig. 6 the dot-dash line represents the curve of the total bending moment, the ordinates of this curve being the sums of the corresponding ordinates of the moment curves for the uniform and concentrated loadings.

If the horizontal line *AB* be drawn through the apex of this total moment curve, the latter curve referred to the line *AB* becomes the curve for the total vertical shears—in other words, the figure *ABCDE* becomes the total shear diagram.

To find the proper spacing of the keys for the left half of beam, the vertical ordinate (248,100 ft.-lb.) of the total shear diagram is divided into 5 equal spaces, horizontals drawn from these division points to the curve of total shear, and vertical ordinates drawn

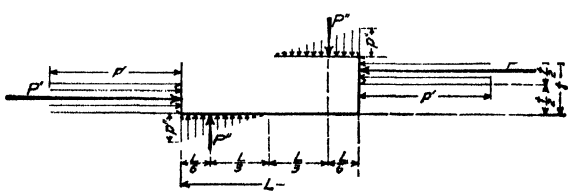


FIG. 8.—Diagram of distribution of pressures on rectangular key.

from these intersection points to the base line. These ordinates divide the area *ABK*, (Fig. 7*b*) or *ADE* (Fig. 6) between the curve and base line, into 5 equal divisions. The points on the girder thus found determine the position of keys. Referring to either Fig. 7*b* or Fig. 6, the proper spacing of keys for the left half of the girder is found to be two spaces at 20 in., one at 26 in., and one at 31 in. The spacing of keys for the right of the center of girder may be found in the same manner.

Girder with Rectangular Keys.—In the above example the girder will first be designed for rectangular cast-iron keys. Assume 5 keys between the left support and the point of zero shear. Each key will therefore resist one-fifth of the total horizontal shear.

The required dimensions of each key will be determined from the following consideration. The forces acting upon the key are shown in Fig. 8.

Let p' = maximum allowable intensity of pressure against ends of fibers.
p'' = maximum allowable intensity of pressure across fibers of timber.
t = thickness of key.
L = length of key.
P' = resultant pressure against fibers of timber for section of key 1 in. in width.
P'' = resultant pressure across fibers of timber for section of key 1 in. in width.

Then

$$P'\left(\frac{t}{2}\right) = P''\left(\frac{2}{3}L\right)$$

$$P' = p'\left(\frac{t}{2}\right)$$

$$P'' = \left(\frac{p''}{2}\right)\left(\frac{L}{2}\right)$$

Whence

$$p'\left(\frac{t}{2}\right)\left(\frac{t}{2}\right) = \frac{p''}{2}\left(\frac{L}{2}\right)\left(\frac{2}{3}L\right)$$

$$\frac{p't^2}{4} = \frac{p''L^2}{6}$$

$$L^2 = \frac{6p't^2}{4p''} = \frac{3}{2}\cdot\frac{p't^2}{p''}$$

$$L = 1.225t\sqrt{\frac{p'}{p''}}$$

For

$$p' = 1800, \text{ and } p'' = 400$$

$$\sqrt{\frac{p'}{p''}} = \sqrt{4.5} = 2.12$$

Whence

$$L = (1.225)(2.12)t = 2.6$$

The total horizontal shear is

$$\left(\frac{3}{2}\right)\left(\frac{248,100}{31}\right)(12) = 144,000 \text{ lb.}$$

Each key must therefore resist 28,800 lb. At 1,800 lb. per sq. in. in bearing against the grain, and with a width of key of 13½ in., one-half the depth of key must be $\frac{28,800}{(13.5)(1,800)}$ =

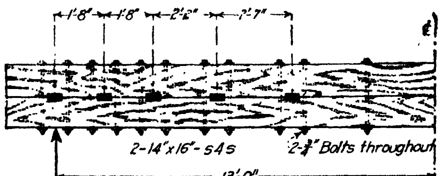


Fig. 9.—Detail of built-up girder with cast-iron keys.

1.19 in., or the total thickness of key must be 2.38 in. The minimum length of key must therefore be 2.38 ×2 .6 = 6.19 in.

The minimum distance between keys, considering shear, must not exceed $\frac{28,800}{(175)(1.53)}$ = 12.2; adding the width of key, the minimum spacing of keys, center to center, must not be less than 18¼ in., which is less than the smallest spacing found.

The bolts for each key should be spaced on each side of each key and equidistant from the center line. Assume four bolts for each key. The stress in each bolt will then be

$\frac{1}{2} \cdot \frac{(P')(t/2)}{2/3L} = \frac{1}{2} \cdot \frac{(28,800)(2.375)}{(2/3)(6)(2)} = 4{,}276$ lb., or four ¾-in. bolts are required. Washers 4 × 4 × 7/16 in. will be used.

The detail of the left half of the girder is shown in Fig. 9.

Girder with Circular Shear Pins.—For this design circular pins, 2 in. in diameter, of solid iron, extra heavy steel pipe, Australian Ironbark or Hawaiian Ohia will be used. Each pin will be considered capable of resisting a shear of 800 lb. per lin. in. of pin. With a 13½-in. length of pin, therefore, one pin will have a resisting value of 13½ × 800 = 10,800 lb. Since the total horizontal shear is 144,000, the total number of pins required is $\frac{144{,}100}{10{,}800} = 14.4$. Dividing the end ordinate into 15 divisions and proceeding as before, it will be found that the minimum spacing of the pins near the end of the girder is 6 in. The spacings of all pins for the left half of girder, commencing at the center line of support of girder, are as follows: 2 spaces at 6 in.; 6 spaces at 7 in.; 1 space at 8 in.; 1 space at 9 in.; 1 space at 10 in.; 1 space at 13 in.; 1 space at 16 in.; and one space at 19 in. For each pin there will be required bolts sufficient in tension for 10,800 lb. Two ¾-in. bolts will be used, with 4- × 4- × 7/16-in. washers. The detail of one-half of girder is shown in Fig. 10.

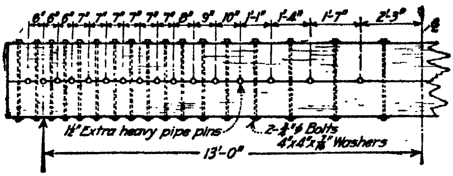


FIG. 10.—Details of built-up girder with circular shear pins.

16. Flitch-plate Girders.—A flitch-plate girder is a combination girder of timber and steel, composed of two sticks of timber with a steel plate between them or three sticks of timber with two steel plates, bolted together, the contact planes between timber and steel plate being parallel to the plane of bending (see Fig. 11). This combination girder is seldom used at the present time, the usual availability of steel structural shapes making the flitch-plate girder practically obsolete. Situations may sometimes exist, however, when the use of this type of girder may be warranted.

Consider any plane cross-section of such a combination girder: the deflection and also the deformation of all points in such section on a line normal to the plane of bending must be the same. Since the modulus of elasticity is the ratio of stress to deformation, it follows that the extreme fiber stresses of timber and steel will be in proportion to their moduli of elasticity, or

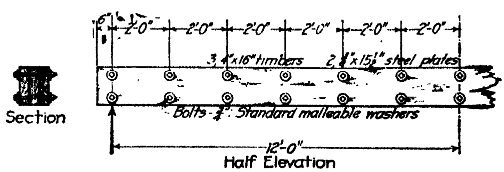


FIG. 11.—Detail of flitched-plate girder.

$$\frac{f_t}{f_s} = \frac{E_t}{E_s}$$

where the subscripts "*t*" and "*s*" represent timber and steel, respectively. This relation of extreme fiber stresses means practically that with the steel plate working efficiently (extreme unit fiber stress of 16,000 lb. per sq. in.) the limiting extreme unit fiber stress in the timbers is approximately 1/18 to 1/20 of the allowable working stress for steel. In the case of a flitch-plate girder of long-leaf yellow pine and steel, the timber would be stressed to approximately 900 lb.

per sq. in. The timber is therefore working at an efficiency of about 50 per cent, while that steel plate in the rectangular section is only approximately 55 per cent efficient as compared to an I-beam of equal depth and weight.

As an illustration of the computation for the strength of a flitch-plate girder, assume a girder composed of 3-4- × 16-in. timbers of No. 1 Common Douglas fir (finished section $3\frac{1}{2} \times 15\frac{1}{2}$ in.), with two $\frac{3}{8} \times 15\frac{1}{2}$-in. steel plates between the timbers. With a span of 24 ft., it is desired to find the safe load, uniformly distributed, that the girder will support.

Maximum allowable unit fiber stress in timber = 1,500 lb. per sq. in.

Maximum unit fiber stress for steel plate = 16,000 lb. per sq. in.

E for Douglas fir = 1,600,000.

E for steel = 29,000,000.

Therefore, for flitch-plate girder, the maximum unit fiber stress in bending can be only

$$\frac{1,600,000}{29,000,000}(16,000) = 880 \text{ lb. per sq. in.}$$

The resisting moment of the three timbers in foot-pounds (see illustrative problem following Art. **50**, Sec. 1) is

$$M = \tfrac{1}{6}fbd^2(\tfrac{1}{12}) = \frac{(880)(10.5)(240)}{(6)(12)} = 30,800 \text{ ft.-lb.}$$

The resisting moment of the two steel plates is

$$M = \tfrac{1}{6}fbd^2(\tfrac{1}{12}) = \frac{(16,000)(0.75)(240)}{(6)(12)} = 40,000 \text{ ft.-lb.}$$

The combined resisting moment is therefore

$$30,800 + 40,000 = 70,800 \text{ ft.-lb.}$$

$$M = \tfrac{1}{8}WL = 70,800 \text{ ft.-lb.}$$

$$W = \frac{(70,800)(8)}{24} = 23,600 \text{ lb.}$$

The detail of this girder is shown in Fig. 11. The timbers and steel of the flitch-plate girder should be well bolted together; such bolting should consist of not less than two $\frac{3}{4}$-in. bolts, 2-ft. centers.

In designing a flitch-plate girder for a definite span and loading, the thickness of timber should be from 16 to 18 times the thickness of steel.

17. Trussed Girders.—For situations in which the span or loading, or both, are too great for a girder of single timber section, the trussed girder type is effective, if space limitations will allow its use. The trussed girder is preferable to either the built-up or deepened girder, or to the flitch-plate girder, principally on account of its efficiency and reliability of action. In the trussed girder no fear need be entertained as to decrease of initial efficiency or increase of deflection from initial conditions, due to shrinkage of timber, with consequent slip of fastenings.

Trussed girders may be divided into four types, as follows

1. King Post trussed girder.
2. Queen Post trussed girder.
3. Reversed King Post trussed girder.
4. Reversed Queen Post trussed girder.

These types are illustrated in Figs. 12, 13, 14 and 15.

Trussed girders are adapted particularly for either uniform loading or concentrated loads situated symmetrically with respect to the center line of girder. Both the Queen Post girder and the Reversed Queen Post girder are unsuited for unsymmetrical loading. Since each contains a rectangular panel, loading unsymmetrical in distribution with respect to the center line of girder will cause bending stresses in the joints of the girder, which cannot take such stresses.

The determination of the stresses in a trussed girder is a problem in least work. For practical purposes the following approximate formulas are sufficient:

Uniformly Distributed Loading:

Figs. 12 and 14. (King Post and Reversed King Post types)

Tension in DB (Fig. 12) or compression in BD (Fig. 14) $= \frac{5}{8}W$

Tension in AB and BC (Fig. 12) or compression in AB and BC (Fig. 14) $= \frac{5}{32}\frac{Wl}{h}$

Compression in AD and DC (Fig. 12) or tension in AD and DC (Fig. 14) $= \frac{5}{16}\frac{Wa}{h}$

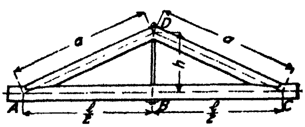


FIG. 12.—King post girder.

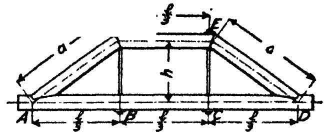


FIG. 13.—Queen post girder

To the stresses thus found in members AB and BC, must be added the flexural stresses resulting from these members acting as beams carrying the uniform loading between A and B, and B and C.

The bending moment in inch pounds in AB and BC is $M = (\frac{1}{8})(W/2)(l/2)(12) = \frac{3}{8}Wl$; also $M = fS = f(\frac{1}{6}bd^2)$. The maximum unit flexural stress is, therefore,

$$f = \frac{2.25Wl}{bd^2}$$

Figs. 13 and 15. (Queen Post and Reversed Queen Post types)

Tension in FB and EC (Fig. 13) or compression in BF and CE (Fig. 15) $= \frac{11}{30}W$

Tension in AB, BC and CD (Fig. 13) or compression in AB, BC and CD (Fig. 15) $= \frac{11}{90}\frac{Wl}{h}$

Compression in FE (Fig. 13) or tension in FE (Fig. 15) = 11/[illegible] $\frac{Wl}{h}$

Compression in AF and ED (Fig. 13) or tension in AF and DE (Fig. 15) $= \frac{11}{30}\frac{Wa}{h}$

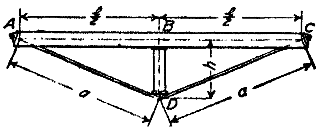


FIG. 14.—Reversed King post girder.

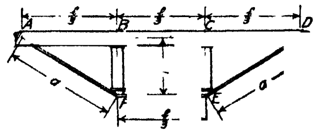


FIG. 15.—Reversed Queen post girder.

As in the king post truss, to the unit stress in the members AD from the formula above must be added the flexural stress due to the timber acting as a beam. The extreme fiber stress due to this bending may be taken as

$$f = \frac{Wl}{bd^2}$$

Concentrated Loading:

Figs. 12 and 14. (King Post and Reversed King Post types)

Concentrated load P at center of span.

Tension in DB (Fig. 12) or compression in BD (Fig. 14) $= P$

Tension in AB and BC (Fig. 12) or compression in AB and BC (Fig. 14) $= \frac{Pl}{4h}$

Compression in AD and DC (Fig. 12) or tension in AD and DC (Fig. 14) $= \frac{Pa}{2h}$

Obviously, there are no flexural stresses in this case to be added to the primary stresses found above.

Figs. 13 and 15. (Queen Post and Reversed Queen Post types)
Concentrated load P at B and C

Tension in FB and EC (Fig. 13) or compression in BF and CE (Fig. 15) $= P$

Tension in AB, BC and CD (Fig. 13) or compression in AB, BC and CD (Fig. 15) $= \frac{\frac{1}{3}Pl}{h}$

Compression in FE (Fig. 13) or tension in FE (Fig. 15) $= \frac{\frac{1}{3}Pl}{h}$

Compression in AF and ED (Fig. 13) or tension in AF or ED (Fig. 15) $= \frac{Pa}{h}$

The stresses resulting from these formulas are all that need to be considered.

17a. Details of Trussed Girders.—In the girders of Figs. 12 and 13, the vertical members only are of iron or steel, in the form of rods. Since such rods are short, plain rods—i.e., without upset ends—should be used. Attention must be given to the washers, to the end that sufficient area be provided to avoid crushing the fibers of the timber. As great a depth as possible should be given to these girders, not alone to reduce the stresses and the deflection but in order that the stresses of the end connections may be kept within limits. With a small depth of girder, the inclination of the members AD and DC of Fig. 12, and AF and ED of Fig. 13 will be so small that it may be found impossible to design connections at A and C of Fig. 12 and A and D of Fig. 13 that will hold. As a matter of fact, trussed girders of these types are seldom used.

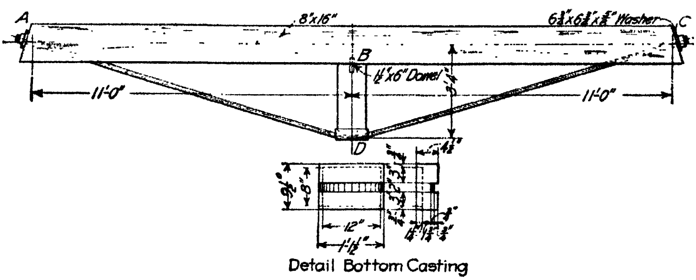


FIG. 16.—Detail of trussed girder.

The horizontal timbers of the girders of Figs. 14 and 15 may be single sticks or double or triple sticks of timber, spaced with a distance between sufficient to allow the diagonal rods to pass. One or two rods may be employed. The ends of the timbers are usually beveled off at the upper corners to provide a seat for the washers of the rods. The vertical struts may be of timber or of cast iron, and must be sufficient in section to take their stress acting as columns. The unit bearing stress between the upper end of the strut and the chord timber must be within the allowed limit for cross bearing. To accomplish this, the strut may be given the area required for bearing, or a smaller strut sufficient for column action may be employed, and a steel plate washer used. The strut should be designed with as wide a base as possible, as there is a tendency to pull the struts out of line, when the rods are tightened. Similarly, at the lower end of the struts, the bearing between rods and the strut must be examined. Cast-iron washers with grooves for the rods, are often used. To do away with the necessity for cast iron shoes, square bars are sometimes used instead of round rods, and a flat steel

washer placed at the bottom of the strut, the bend in the bars being made just outside the strut.

Illustrative Problem.—Required to design a trussed girder, as shown in Fig. 16, for a building to be used for light storage; span 22 ft., depth on center lines 3 ft. 4 in., loading uniform 2,000 lb. per lin. ft., material dense Southern yellow pine and steel.

The modulus of elasticity of the timber will be taken at 1,200,000,[1] the corresponding quantity for steel at 29,000,000. Assume dead weight of girder at 50 lb. per lin. ft. Then total load per lin. ft. = 2,050 lb.

$$\text{Total load} = (22)(2{,}050) = 45{,}000 \text{ lb.}$$

$$\text{Direct stress in beam } AB = BC = \frac{(5)(45{,}000)(22)}{(32)(3.33)} = 46{,}500 \text{ lb.}$$

$$\text{Stress in strut } BD = (5/8)(45{,}000) = 28{,}100 \text{ lb.}$$

$$\text{Stress in rod } AD = DC = \frac{(5)(45{,}000)(11.5)}{(16)(3.33)} = 48{,}600 \text{ lb.}$$

$$\text{Length } a = \sqrt{(11)^2 + (3.33)^2} = 11.5 \text{ ft.}$$

Size of rod:

At 16,000 lb. per sq. in., the required area of rod is

$$\frac{48{,}600}{16{,}000} = 3.00 \text{ sq. in.}$$

A $1\frac{3}{4}$-in. square bar is required, upset at the ends to $2\frac{1}{2}$ in.

Size of strut:

For bearing between the strut and beam the area required at 300 lb. per sq. in. is

$$\frac{28{,}100}{300} = 94 \text{ sq. in.}$$

For the column, the area required is

$$\frac{28{,}100}{1{,}000} = 28 \text{ sq. in.}$$

Size of beam:

$$M = \frac{(1/8)(45{,}000)(11)}{2} = 31{,}000 \text{ ft.-lb.}$$

Assume an 8- × 16-in. timber, S4S. The section modulus, from Table 6, p. 368, is 300.31. The maximum unit fiber stress is $\frac{(31{,}000)(12)}{300.3} = 1{,}240$ lb. per sq. in.

Since the area of section is 116.25, the direct stress is

$$\frac{46{,}500}{116.25} = 400 \text{ lb. per sq. in.}$$

The maximum unit stress on the extreme fibers is therefore

$$1{,}240 + 400 = 1{,}640 \text{ lb. per sq. in.}$$

End washer:

Angle between the plane of the washer and direction of the fibers of wood is

$$\cot^{-1}\frac{11.00}{3.33} = 3.30 = 73 \text{ deg.}$$

Allowable unit pressure by Diagram 3, p. 416 = 1,200 lb. per sq. in.

Area required is

$$\frac{48{,}600}{1{,}200} = 40 \text{ sq. in.}$$

Add area hole, or 40 + 5.4 = 45.4 sq. in. = total gross area required.

Side of square washer = $\sqrt{45.4}$ = 6.75 in.

The short diameter of a square nut for a $2\frac{1}{2}$-in. rod is $3\frac{7}{8}$ in.

The maximum bending moment is along the edge of nut when sides of nut and washer are at 45 deg., and is in amount 9,100 in.-lb.

The full width of plate along line of edge of nut is 5.67 in. and, with this width and a flexural stress of 24,000 lb. per sq. in., the required thickness of plate is 0.64 in.

Washer will be made $6\frac{3}{4} \times 6\frac{3}{4} \times \frac{11}{16}$ in.

An 8- × 12-in. timber will be used for the strut, and top and bottom castings used as detailed in Fig. 16.

[1] This low value will be used in computing deflection, since its assumed load is largely constant or fixed.

17b. Deflection.—The exact method for finding the deflection of a trussed girder is a problem in least work. An approximate solution will be illustrated below. In the example of Fig. 16, assume the average depth between center line of the 8- × 16-in. beam and the center line of rod as ⅝ total depth, or 25 in. This dimension is the depth at the third point of the length of girder. Compute the equivalent moment of inertia of the girder at this point.

Area 8- × 16-in. timber $= (7\frac{1}{2})(15\frac{1}{2}) = 116$ sq. in.

Equivalent area in steel $= (116)\left(\frac{1{,}200{,}000}{29{,}000{,}000}\right) = 4.81$ sq. in.

Area 1¾-in. square bar = 3.06 sq. in.

These equivalent areas are 25 in. on centers. Then center of gravity of combined sections is

$$25 - \frac{(4.81)(25)}{7.87} = 9.7 \text{ in.}$$

below center line of the 8- × 16-in. beam.

Moment of inertia of combined section:

$$\begin{aligned}(4.81)(9.7)^2 &= 452.5\\ (3.06)(25-9.7)^2 &= \underline{716.0}\\ &\ 1{,}168.5\end{aligned}$$

$$\text{Deflection} = \frac{5Wl^3}{384EI} = \frac{(5)(45{,}000)(18{,}399{,}744)}{(384)(29{,}000{,}000)(1{,}168.5)} = 0.318 \text{ in., say } \tfrac{5}{16} \text{ in.}$$

It must be realized that this method is approximate only, the principal indeterminate factor being the assumed average depth. For the case of the reversed Queen Post type, the depth should be taken as the distance between the center line of beam and the center line of the horizontal rods.

WOODEN COLUMNS

Interior columns of buildings, supporting floors only, are normally square in cross-section, while columns supporting roof trusses are usually made rectangular in order to attain greater stiffness in the plane of the roof truss than in the plane of the building wall. Columns supporting roof trusses may take bending stresses, due to wind, far in excess of the unit stresses produced by the weight of the roof and wall constructions.

Interior columns, when exposed, are usually surfaced four sides, and the corners chamfered. Sometimes the columns are bored from end to end with a 1½-in. hole, and with ¾-in. holes at top and bottom extending from the face of column to the core hole. That is done in order to prevent dry rot, and to relieve the usual condition of rapid drying out of the exterior of the column, and slow seasoning of the interior timber.

Wooden columns with a ratio of $\frac{L}{d}$ greater than 20 will fail by lateral buckling. No wooden column should be designed with a greater $\frac{L}{d}$ than 60, and good practice will lower this limiting slenderness ratio to 40.

A general treatment pertaining to columns and column loads is given in the chapter on "Columns" in Sec. 1. For splicing wooden columns and for column connections, see Arts. **14** and **16**. Bending and direct stress in columns is treated in Sec. 1.

18. Formulas for Wooden Columns.—All modern formulas for wooden columns assume the case of square-ended columns, and this condition of ends is the only condition recognized in practice. Practically all of the tests on wooden columns have been made with flat ends.

A number of formulas have been proposed and are in use for determining the safe working strength of wooden columns. With few exceptions these formulas are of the experimental type—that is, they are based on the results of tests. The straight-line formula is the type most favored by engineers. The two formulas of this type most generally used are: (1) The formula of the American Railway Engineering Association

$$p = C\left(1 - \frac{1}{60}\frac{L}{d}\right)$$

and (2), the Winslow formula

$$p = C\left(1 - \frac{1}{80}\frac{L}{d}\right)$$

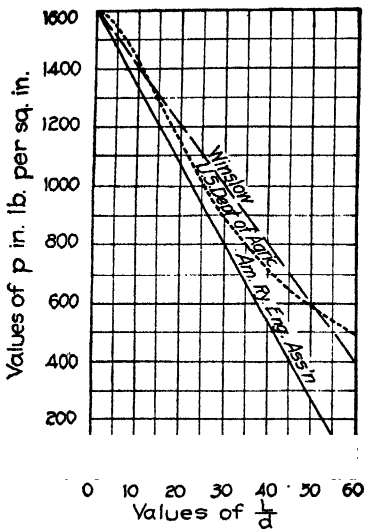


FIG. 17.—Curves of column formulas. (C = 1600.)

The second class of column formulas gives a curved graph. Of this type, the following formula of the U. S. Department of Agriculture is extensively employed

$$p = C\left(\frac{700 + 15c}{700 + 15c + c^2}\right)$$

In the above formulas, p = average unit compression (pounds per square inch).

C = compressive strength for short columns (pounds per square inch).

$c = \frac{L}{d}$.

L = length of column in inches.

d = least cross-sectional dimension of column in inches.

For the range of values of $\frac{L}{d}$ occurring in ordinary building construction, the three preceding formulas will give approximately the same results. Figure 17 shows the graphs of these formulas for working conditions, with $C = 1{,}600$. For columns with a slenderness ratio $\left(\frac{L}{d}\right)$ less than 15, the unit stress to be used is that for $\frac{L}{d} = 15$.

Table 10, p. 389, gives the unit stress for timber columns for various ratios of $\frac{L}{d}$ and values of C from 1,000 to 1,600 inclusive, corresponding to the formula of the U. S. Department of Agriculture. Table 11 gives similar quantities using the American Railway Engineering Association formula. Table 12 gives the

TABLE 10.—WORKING UNIT STRESSES IN POUNDS PER SQUARE INCH FOR TIMBER COLUMNS WITH SQUARE ENDS, SYMMETRICALLY LOADED

(Formula of U. S. Department of Agriculture)

L/d	Working unit stresses in pounds per square inch for values of "C" as indicated						
	1,000	1,100	1,200	1,300	1,400	1,500	1,600
15	804	884	965	1,046	1,127	1,206	1,284
16	785	864	943	1,022	1,100	1,179	1,255
17	767	844	921	998	1,075	1,150	1,226
18	749	823	899	974	1,050	1,124	1,199
19	730	805	878	950	1,025	1,097	1,170
20	712	786	857	928	1,000	1,071	1,143
21	695	768	837	905	975	1,046	1,117
22	679	750	817	883	951	1,020	1,090
23	663	731	796	861	929	996	1,063
24	647	714	778	841	906	971	1,039
25	631	697	759	821	884	949	1,013
26	617	681	741	802	864	927	989
27	601	664	724	784	844	905	965
28	587	648	707	766	824	883	942
29	573	632	690	748	805	862	920
30	559	617	674	730	787	841	899
31	547	601	659	713	768	821	878
32	534	587	643	696	750	801	856

TABLE 11.—WORKING UNIT STRESSES IN POUNDS PER SQUARE INCH FOR TIMBER COLUMNS WITH SQUARE ENDS, SYMMETRICALLY LOADED

(Formula of American Railway Engineering Association)

L/d	Working unit stresses in pounds per square inch for values of "C" as indicated						
	1,000	1,100	1,200	1,300	1,400	1,500	1,600
15	749	824	900	974	1,049	1,125	1,200
16	732	806	879	952	1,025	1,100	1,182
17	716	787	860	930	1,002	1,075	1,145
18	700	769	840	909	979	1,050	1,119
19	683	750	819	887	955	1,025	1,092
20	666	732	800	866	932	1,000	1,065
21	649	714	779	843	909	975	1,039
22	632	696	760	822	885	950	1,012
23	616	677	739	801	862	925	985
24	600	659	720	779	839	900	959
25	582	640	699	757	815	875	932
26	566	622	680	735	792	850	906
27	549	604	659	714	769	825	879
28	533	585	639	692	746	800	852
29	516	567	620	670	722	775	825
30	500	548	599	649	699	750	799
31	483	530	580	627	675	725	772
32	466	512	559	606	651	700	745

Table 12.—Table of Safe Bearing Loads in 1,000-pound Units for Timber Columns with Square Ends, Symmetrically Loaded

Values in this table are based on surfaced sizes. To get values for rough sizes, multiply bearing load by "multiplying factor" in dark type in same horizontal line. To get cross-section of rough size, multiply area given by factor in dark type directly below

(Based on Formula Adopted by American Railway Engineering Association)

Size: Rough	Size: Surfaced S1S1E or S4S1	Area cross-section	Length of column	L/d	"Multiplying factor"	Safe bearing loads in 1,000-pound uni s for values of C as indicated						
Inches	Inches	Square inches	Feet			1,000	1.100	1,200	1,300	1,400	1,500	1,600
6 × 6	5½ × 5½	30.25	6	13.1	**1.19**	30.25	33.28	36.30	39.33	42.35	45.38	48.40
		1.19	8	17.5	**1.23**	21.39	23.53	25.67	27.81	29.95	32.09	34.22
			10	21.8	**1.25**	19.21	21.13	23.05	24.97	26.89	28.82	30.74
			12	26.2	**1.27**	17.00	18.70	20.40	22.10	23.80	25.50	27.20
			14	30.5	**1.29**	14.85	16.34	17.82	19.31	20.79	22.28	23.76
			8	12.8	**1.14**	56.25	61.88	67.50	73.13	78.75	84.38	90.00
			10	16.0	**1.16**	41.18	45.30	49.42	53.53	57.65	61.77	65.89
8 × 8	7½ × 7½	56.25	12	19.2	**1.17**	38.19	42.01	45.83	49.65	53.47	57.29	61.10
		1.14	14	22.4	**1.18**	35.21	38.73	42.25	45.77	49.29	52.82	56.34
			16	25.6	**1.19**	32.18	35.40	38.62	41.83	45.05	48.27	51.49
			18	28.8	**1.20**	29.19	32.11	35.03	37.95	40.87	43.79	46.70
			8	10.1	**1.11**	90.25	99.28	108.30	117.33	126.35	135.38	144.40
			10	12.6	**1.11**	90.25	99.28	108.30	117.33	126.35	135.38	144.40
10 × 10	9½ × 9½	90.25	12	15.2	**1.13**	67.24	73.96	80.69	87.41	94.14	100.86	107.58
		1.11	14	17.7	**1.13**	63.54	69.89	76.25	82.60	88.96	95.31	101.66
			16	20.2	**1.14**	59.75	65.73	71.70	77.68	83.65	89.63	95.60
			18	22.7	**1.14**	56.05	61.66	67.26	72.87	78.47	84.08	89.68
			20	25.3	**1.15**	52.16	57.38	62.59	67.81	73.02	78.24	83.46

12 × 12	11½ × 11½	132.25	8 to	8.3	**1.09**	132.25	145.48	158.70	171.93	185.15	198.38	211.60
		1.09	14	14.6	**1.09**	132.25	145.48	158.70	171.93	185.15	198.38	211.60
			16	16.7	**1.11**	95.35	104.89	114.42	123.96	133.49	143.03	152.56
			18	18.8	**1.11**	90.72	99.79	108.86	117.94	127.01	136.08	145.15
			20	20.9	**1.11**	86.09	94.70	103.31	111.92	120.53	129.14	137.74
			22	23.0	**1.12**	81.47	89.62	97.76	105.91	114.06	122.21	130.35
			24	25.0	**1.12**	76.97	84.67	92.36	100.06	107.76	115.46	123.15
14 × 14	13½ × 13½	182.25	8 to	7.1	**1.08**	182.25	200.48	218.70	236.93	255.15	273.38	291.60
		1.08	16	14.2	**1.08**	182.25	200.48	218.70	236.93	255.15	273.38	291.60
			18	16.0	**1.09**	133.41	146.75	160.09	173.43	186.77	200.12	213.46
			20	17.8	**1.09**	128.12	140.93	153.74	166.56	179.37	192.18	204.99
			22	19.6	**1.09**	122.65	134.92	147.18	159.45	171.71	183.97	196.24
			24	21.3	**1.10**	117.37	129.11	140.84	152.58	164.32	176.06	187.79
16 × 16	15½ × 15½	240.25	10 to	7.7	**1.07**	240.25	264.28	288.30	312.33	336.35	360.38	384.40
		1.07	18	14.0	**1.07**	240.25	264.28	288.30	312.33	336.35	360.38	384.40
			20	15.5	**1.08**	177.79	195.58	213.35	231.13	248.91	266.69	284.46
			22	17.0	**1.08**	172.02	189.22	206.42	223.63	240.83	258.03	275.23
			24	18.6	**1.08**	165.53	182.08	198.64	215.19	231.74	248.30	264.85
18 × 18	17½ × 17½	306.25	10 to	6.9	**1.06**	306.25	336.88	367.50	398.13	428.75	459.38	490.00
		1.06	20	13.7	**1.06**	306.25	336.88	367.50	398.13	428.75	459.38	490.00
			22	15.1	**1.07**	229.08	251.99	274.90	297.80	320.71	343.62	366.53
			24	16.5	**1.07**	221.73	243.90	266.08	288.25	310.42	332.60	354.77
20 × 20	19½ × 19½	380.25	10 to	6.2	**1.05**	380.25	418.28	456.30	494.33	532.35	570.38	608.40
		1.05	24	14.8	**1.05**	380.25	418.28	456.30	494.33	532.35	570.38	608.40
22 × 22	21½ × 21½	462.25	10 to	5.6	**1.05**	462.25	508.48	554.70	600.93	647.15	693.38	739.60
		1.05	24	13.4	**1.05**	462.25	508.48	554.70	600.93	647.15	693.38	739.60
24 × 24	23½ × 23½	552.25	10 to	5.1	**1.04**	552.25	607.48	662.70	717.93	773.15	828.38	883.60
		1.04	24	12.3	**1.04**	552.25	607.48	662.70	717.93	773.15	828.38	883.60
26 × 26	25½ × 25½	650.25	10 to	4.7	**1.04**	650.25	715.28	780.30	845.33	910.35	975.38	1,040.40
		1.04	24	11.3	**1.04**	659.25	715.28	780.30	845.33	910.35	975.38	1,040.40

[1] S1S1E means surfaced one side and one end.
S4S means surfaced all four sides.

safe loads in thousands of pounds for surfaced square timber columns, by the American Railway Engineering Association formula.

19. Ultimate Loads for Columns.—It is sometimes necessary to investigate the ultimate strength of wooden columns. Unfortunately, the ultimate strength of a timber column, especially of a long column, or a column with an $\frac{L}{d}$ of from 40 to 60, is indeterminate. The tests which have been made on long columns of sections commensurate with those used in building construction are not sufficient in number to justify confidence in the values given by formulas.

From the results of tests made by the Watertown Arsenal, J. B. Johnson proposed for timber columns the following formulas:

Ultimate strength for partially seasoned yellow pine columns

$$p = 4{,}500 - 1.0\left(\frac{L}{d}\right)^2$$

Ultimate strength for partially seasoned white pine column

$$p = 2{,}500 - 0.5\left(\frac{L}{d}\right)^2$$

Ultimate strength for dry long-leaf pine column

$$p = 6{,}000 - 1.5\left(\frac{L}{d}\right)^2$$

Ultimate strength for dry white pine column

$$p = 3{,}600 - 0.72\left(\frac{L}{d}\right)^2$$

W. H. Burr, from a study of the same tests, recommends the formulas:
For yellow pine

$$p = 5{,}800 - 70\frac{L}{d}$$

For white pine

$$p = 3{,}800 - 47\frac{L}{d}$$

One other column formula needs to be mentioned, since it has been used quite extensively in the past. This is the formula of C. Shaler Smith who made some 1,200 tests on full-sized specimens of square and rectangular yellow pine columns for the Ordnance Department of the Confederate Government. For green, half-seasoned sticks of good merchantable lumber the formula of Smith is

$$p = \frac{5{,}400}{1 + \frac{1}{250}\frac{L^2}{d^2}}$$

This formula gives much lower strength values for wooden columns than any of the preceding formulas.

All of the above formulas for ultimate strengths are based on short-time loadings. J. B. Johnson, in some 75 tests made to investigate the effect of time on continued uniform loading of timber in end compression, found that but little more than one-half the short-time ultimate load will cause a column to fail, if left on permanently. In other words, the ultimate strength of a timber column under permanent loads is approximately one-half the ultimate strength of the same column, as determined from the results of an actual test in a testing machine.

20. Built-up Columns.—The preceding discussion applies only to columns consisting of single sticks of timber. Built-up columns may be divided into two types: (1) Those of solid section made up of thin planking and nailed, or nailed and bolted; and (2) columns of solid section bolted and keyed together, also latticed or trussed columns.

Type (1).—Columns of the first class are often used in cheap construction and, unfortunately, in situations where there is no excuse for not using a solid section Carpenters, in order to make use of material available or handy, will often build up posts spiked together instead of using a solid section, in the belief that they are furnishing a stronger column than the larger timber of one piece. Tests have conclusively shown that a column of two or three pieces of timber blocked apart and bolted together at the ends and middle has no greater strength than the sum of the strengths of the component sticks, each acting as a single column, entirely independent of the other sticks.

The strength of a built-up column of this class depends wholly upon the ability of the fastenings to resist initial deflection under loading. Such columns are usually designed with one of two typical sections: A column composed of a number of planks laid face to face and bolted or spiked together, as shown in Fig. 18*a*; or a column composed of planks face to face with their edges tied together by cover-plates, as in Fig. 18*b*. Of the two details, that of Fig. 18*b* is far superior to Fig. 18*a*. When a column of the type shown in Fig. 18*b* is thoroughly spiked, in addition to being bolted, the strength of column is undoubtedly greater than the sum of the strengths of the component planks acting as individual sticks. From tests made by the writer, it is recommended that the strength of a built-up column of the type of Fig. 18*a* be taken at 80 per cent of the mean of the strength computed, (1) as a solid stick, and (2) as a summation of the strength of the individual sticks considered as individual columns. For columns of the type of Fig. 18*b*, it is recommended that the strength be taken as 80 per cent of that of a solid stick of equal cross-section and length.

The preceding recommendations are for built-up columns taking no appreciable bending stresses; in other words, for columns whose loads are balanced about the gravity center of the column section. Obviously, the resistance to bending of a built-up column of this class is low, as has been pointed out in the case of built-up girders (see Art. 14).


(a) (b)

Fig. 18.—Sections of built-up columns.

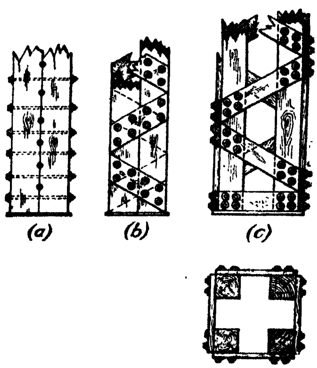

(a) (b) (c)

Fig. 19.—Heavy built-up columns.

Type (2).—In framing for large timber buildings, as for expositions, wooden columns are sometimes constructed of two posts bolted and keyed together (Fig. 19*a*), two posts laced with diagonal sheathing (Fig. 19*b*), or four posts

laced together (Fig. 19*c*). Such a construction may be necessary for the long story heights encountered in such buildings. The lacing shown in the detail of Fig. 19*c* may be spiked, bolted, or attached by means of lag screws, as determined usually by consideration of the stresses in the lacing due to wind shear. For dead loads, it is well to assume that the individual timbers act as separate columns, not held together by the fastenings. The lacing may be at 60 or at 45 deg. with the axis of the column, depending on the judgment of the designer. In general, the writer prefers the 60-deg. lacing.

21. Column Bases.—Except for temporary construction, building footings at the present time are constructed of concrete, reinforced concrete, or steel grillages incased in concrete. The statement may be made, therefore, that the first-story column of any building will rest on a concrete footing. A base plate between the bottom of post and top of footing is a necessity for two reasons: (1) To distribute the column pressure over the footing without exceeding the safe unit bearing pressure for concrete; and (2) to prevent moisture from entering the bottom of the column and causing rot. For this purpose a wooden plate, preferably of redwood or cedar, a standard metal column base, a cast-iron base, or a plain steel plate may be used. The latter is often found as satisfactory and more economical than the standard metal post base. If a single plate is used, the thickness must be sufficient to give strength to the plate, in flexure, to distribute the load uniformly over the footing, with a uniform distribution of pressure on the footing.

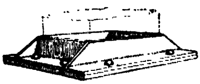


FIG. 20.

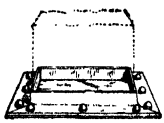


FIG. 21.—Duplex steel post base.

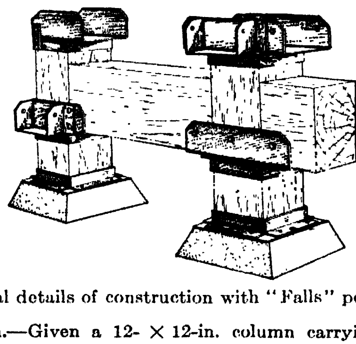


FIG. 22.—Typical details of construction with "Falls" post caps and bases.

Illustrative Problem.—Given a 12- × 12-in. column carrying a load of 130,000 lb. Using a working value of 400 lb. per sq. in. for bearing on the concrete, a base of 130,000/400 = 326 sq. in. is required, or 18-in. square. The plate will then project $3\frac{1}{4}$ in. from each face of column. The bending moment on the plate may be taken as $\left(\frac{310,000}{4}\right)(\frac{2}{3})(9) - \left(\frac{130,000}{4}\right)(\frac{3}{4})(5\frac{3}{4}) = (32,500)(2.17) = 70,500$ in.-lb. This moment is resisted by the full width of base. As the plate is in effect a short, thick beam, a maximum flexural fiber stress of 20,000 lb. per sq. in. for structural steel may be used, giving a required section modulus of 3.53. Therefore $S = (\frac{1}{6})(18)(d^2) = 3.53$, or $d = \sqrt{1.18} = 1.08$, or a $1\frac{1}{16}$-in. plate.

In detailing the base of column, it is well to set a dowel into the concrete and let the same project into the bottom of post. The size of dowel is a matter of judgment. For a 12- × 12-in. post, the dowel should be not less than $1\frac{1}{4} \times 6$ in.

If the use of a standard column base is contemplated, the particular base should be examined to make sure its composition is sufficiently strong to distribute its load equally over the foundation.

It remains to be stated that all metal bases should be well painted. The bottoms of columns should be given two coats of a good wood preservative. The top of the concrete footing should be set a few inches above the floor to prevent moisture standing around the bottom of the column.

Figures 20, 21 and 22 show standard post bases, taken from manufacturers' catalogs.

SECTION 5

SPLICES AND CONNECTIONS FOR WOODEN MEMBERS

BY HENRY D. DEWELL

1. Nails.—*Wire nails* are usually of steel, of circular cross-section without taper, and with a head and point. In size they are designated as 8-D (8 penny), 10-D (10 penny), etc., and, in class, as common, finishing, casing, barbed roofing, shingle, fine, cement coated, etc.

Cut nails are of steel or iron, with a rectangular cross-section, and taper from head to point, the latter being cut square, i.e., not pointed. The sizes are designated as for wire nails.

Spikes designate the large sizes of nails.

The sizes of nails and spikes are given in Tables 1 to 9 inclusive. For quantity of nails required in timber construction, see Table 10.

Boat spikes are employed in heavy timber construction. They are made from square bars of steel or wrought iron, have a forged head and a wedge-shaped point. The common sizes and weights are given in Table 11.

2. Screws.—Screws may be classified as (1) *common wood screws*, and (2) *lag*, or *coach screws*.

Wood screws have slotted heads; the shank is smooth for a portion of its length adjacent to the head, the remainder of the length being threaded, and tapering to a point. Wood screws are usually of steel, but are made also of bronze and brass The ordinary wood screw has a flat head, but screws are also made with round heads. Wood screws are designated by gage and length. Given the gage number, the diameter of the smooth shank may be found from the formula

$$d = 0.0578 + 0.01316G$$

where d = diameter in inches, and G = gage number of screw. Table 12 gives the length and gage numbers of wood screws, flat head, bright steel.

Lag screws are of heavier stock than the common wood screws and have a square head without slot. Table 13 gives the sizes, lengths, and weights of lag screws.

3. Bolts.—Bolts, in timber construction, may be divided into two classes, (1) *common, ordinary*, or *machine bolts*, and (2) *drift bolts*.

Machine bolts are of steel or wrought iron, of circular cross-section without taper, having a square head upset on one end, and the other end threaded to receive a nut. The length of a bolt is the length from underside or inside of head to end of thread. Nuts are usually square unless otherwise ordered, but hexagonal nuts may be obtained where desired. Table 14 gives the weights of 100 machine bolts with square heads and nuts. Table 15*a* gives the values in tension of bolts at various stresses, based on the areas of the bolts at the root of thread. Table 15*b* gives the strength of round rods with upset ends.

TABLE 1.—WIRE NAILS—COMMON

Size	Length, inches	Gage, number	Diameter, inches	Approximate number to pound
2d	1	15	0.072	876
3d	1¼	14	0.083	568
4d	1½	12½	0.102	316
5d	1¾	12½	0.102	271
6d	2	11½	0.115	181
7d	2¼	11½	0.115	161
8d	2½	10¼	0.124	106
10d	3	9	0.148	69
12d	3¼	9	0.148	63
16d	3½	8	0.165	49
20d	4	6	0.203	31
30d	4½	5	0.220	24
40d	5	4	0.238	18
50d	5½	3	0.259	14
60d	6	2	0.284	11

TABLE 2.—WIRE NAILS—FINISHING

Size	Length, inches	Gage, number	Approximate number to pound
2d	1	16½	1,351
3d	1¼	15½	807
4d	1½	15	584
5d	1¾	15	500
6d	2	13½	309
7d	2¼	13	238
8d	2½	12½	189
10d	3	11½	121
12d	3¼	11½	113
16d	3½	11	90
20d	4	10	62

TABLE 3.—WIRE NAILS—CASING

Size	Length, inches	Gage, number	Approximate number to pound
2d	1	15½	1,010
3d	1¼	14½	635
4d	1½	14	473
5d	1¾	14	407
6d	2	12½	236
7d	2¼	12½	210
8d	2½	11½	145
10d	3	10½	94
12d	3¼	10½	87
16d	3½	10	71
20d	4	9	52
30d	4½	9	46
40d	5	8	35

TABLE 4.—WIRE NAILS—FINE

Size	Length, inches	Gage, number	Approximate number to pound
2d	1	16½	1,351
3d	1⅛	15	778

TABLE 5.—WIRE NAILS—SHINGLE

Size	Length, inches	Gage, number	Approximate number to pound
3d	1¼	13	429
4d	1½	12	274

TABLE 6.—WIRE NAILS—BARBED ROOFING

Length, inches	Gage, number	Approximate number to pound
¾	12	548
⅞	12	469
⅞	13	613
⅞	14	811
1	12	411
1	13	536
1	14	710
2	9	103

TABLE 7.—WIRE NAILS—FELT ROOFING (GALVANIZED)

Length, inches	Gage, number	Diameter of head, inches	Approximate number to pound
⅞	12	⅝	215
1	12	⅝	198

TABLE 8.—WIRE SPIKES

Length, inches	Diameter	Approximate number to pound
6	1 gage	8
7	5/16 in.	7
8	⅜ in.	6
9	⅜ in.	5
10	⅜ in.	4
12	⅜ in.	3

Table 9.—Cut Nails

Size	Length, inches	Size	Length, inches
3d	$1\frac{1}{4}$	12d	$3\frac{1}{4}$
4d	$1\frac{1}{2}$	16d	$3\frac{1}{2}$
5d	$1\frac{3}{4}$	20d	4
6d	2	30d	$4\frac{1}{2}$
7d	$2\frac{1}{4}$	40d	5
8d	$2\frac{1}{2}$	50d	$5\frac{1}{2}$
10d	3	60d	6

Table 10.—Quantity of Nails Required for Timber Construction

		Size nail	Nails in pounds for various spacing of joists and studding					
			12 in.	16 in.	20 in.	36 in.	48 in.	60 in.
1,000 M.B.M.	Joists, frame building	20d	20	16	14			
	Joists, brick building	20d	12	10	8			
1,000 pcs.	Bridging, 1 × 4	8d	..	....	35			
	Bridging, 2 × 4	10d	..	....	50			
1,000 M.B.M.	Studding	20d	15	12				
	Studding	10d	5	4				
	Sheathing, 1 × 8	8d	26	20	17			
	Flooring, 1 × 4	8d	26	22				
	Flooring, 1 × 4	10d	40	32				
	Flooring, 1 × 6	8d	17	13	11			
	Flooring, 1 × 6	10d	26	20	17			
	Planking, 3 × 6, 2 nailings	60d	..	....	..	51	40	34
	Planking, 3 × 8, 2 nailings	60d	..	....	..	39	30	26
	Planking, 3 × 10, 2 nailings	60d	..	....	..	31	24	20
	Planking, 3 × 12, 3 nailings	60d	..	....	..	39	30	26
	Planking, 2 × 6, 2 nailings	20d	..	51	42	27	21	18
	Planking, 2 × 10, 2 nailings	20d	..	30	25	16	13	11
	Finishing	8d	..	20				
100 lin. ft.	Base	8 × 6d	..	1				
1	Door	8 × 6d	..	$\frac{1}{2}$				
1	Window	8 × 6d	..	$\frac{3}{8}$				

TABLE 11.—BOAT SPIKES—(WROUGHT IRON)

Size	Average number 100 lb.	Size	Average number in 100 lb.
¼ × 3	1,500	⅜ × 7	325
3½	1,350	8	300
4	1,187	9	263
4½	1,110	10	238
5	1,025	7⁄16 × 6	300
5½	975	7	295
6	913	8	255
5⁄16 × 4	680	9	200
4½	650	10	180
5	615	½ × 6	225
5½	605	7	188
6	588	8	168
⅜ × 5	470	9	150
6	400	10	138
		12	120

TABLE 12.—WOOD SCREWS
(Flat Head. Bright Steel)

Length, inches	Gage numbers																	
¼	0	1	2	3	4													
⅜	*0	1	2	3	4	5	6	7	8	9								
½	*1	2	3	4	5	6	7	8	9	10	*11	*12						
⅝	*1	2	3	4	5	6	7	8	9	10	11	12						
¾	*2	3	4	5	6	7	8	9	10	11	12	*13	14	*15	*16			
⅞	*2	*3	4	5	6	7	8	9	10	11	12	*13	14	*15	*16			
1	*3	4	5	6	7	8	9	10	11	12	13	14	15	16	*17	*18		
1¼	*4	5	6	7	8	9	10	11	12	13	14	15	16	*17	*18	*20		
1½	*5	6	7	8	9	10	11	12	13	14	15	16	17	18	20	*22		
1¾	6	7	8	9	10	11	12	13	14	15	16	17	18	20	*22	*24		
2	6	7	8	9	10	11	12	13	14	15	16	*17	18	20	*22	*24		
2¼	6	7	8	9	10	11	12	13	14	*15	16	*17	18	*20	*22	*24		
2½	8	9	10	11	12	13	14	*15	16	*17	18	20	22	24				
*2¾	*10	*11	*12	*13	*14	*15	*16	*17	*18	*20	*22	*24						
3	*10	11	12	*13	14	15	16	*17	18	20	22	*24	*26					
3½	*10	*11	*12	*13	14	*15	16	*17	18	20	*22	24	*26					
4	12	14	16	18	20	22	24	*26										
4½	*16	*18	20	22	24	26												
5	*18	20	*22	*24	*26	*28												
*6	*20	*22	*24	*26	*28	*30												

* Sizes not usually carried in stock.

TABLE 13.—LAG SCREWS
(Gimlet Point. Square Head)

Length, inches	Diameter, inches						
	1/4	5/16	3/8	1/2	5/8	3/4	7/8
	Weight in pounds of 100 screws						
1 1/4	2.6	3.9	5.1	10.4			
1 1/2	2 7	4.0	6.0	11.0			
1 3/4	2.8	4.4	5.8	11.7			
2	3.1	4.8	6 7	13 0	24.0		
2 1/2	3 7	5.6	8.4	15.6	27.2	39.0	
3	4 2	6.5	9.1	18.2	30.5	45.0	66.0
3 1/2	4.8	7.3	10.6	20.6	33.7	51.0	72.0
4	5.4	8.2	12.0	22.9	37.0	57.0	78.0
4 1/2	6.0	9.0	13.0	25.8	40.2	62.0	85.0
5	6.6	9.9	14.0	27.5	43.5	67.0	92.0
5 1/2	...	10 8	15.0	30.3	47.0	72.0	100.0
6	...	11.7	16.0	32.0	50.6	77.0	107.0
7	...	....	...	36.5	57.8	87.0	122.0
8	...	....	...	41.0	64.7	97.0	137.0
9	....	....	....	45.5	72.0	107.0	152.0
10	...	....	...	50.0	79.2	117.0	167.0
11	...	...	..	54.5	86 5	127.0	180.0
12	...	..	...	59.0	94.0	137.0	191.0

TABLE 14.—MACHINE BOLTS[1]

Length, inches	Diameter, inches								
	1/4	5/16	3/8	7/16	1/2	5/8	3/4	7/8	1
	Threads per inch								
	20	18	16	14	13	11	10	9	8
	Weight in pounds of 100 bolts with square heads and nuts								
3/4	2.4	4.4	6.9	10.4					
1	2.8	4.9	7.6	11.5	16.3				
1 1/4	3.1	5.5	8.4	12.5	17.7	31.7	52.2		
1 1/2	3.4	6.0	9.2	13.6	19.1	33.8	55.3	83.4	
2	4.1	7.1	10.8	15.7	21.8	38.1	61.5	91.8	129.0
2 1/2	4.8	8.2	12.3	17.8	24.6	42.4	67.7	99.7	140.1
3	5.5	9.2	13.8	19.9	27.4	46.7	73.9	108.1	151.1
3 1/2	6.2	10.3	15.3	21.8	29.8	51.0	80.1	116.6	162.2
4	6.9	11.4	16.9	24.0	32.6	55.4	86.3	125.0	173.2
4 1/2	7.5	12.4	18.4	26.1	35.4	59.3	92.1	132.9	182.7
5	8.2	13.5	19.9	28.2	38.1	63.6	98.3	141.3	193.7
5 1/2	8.9	14.6	21.5	30.3	40.9	67.9	104.5	149.8	204.8
6	9.6	15.6	23.0	32.4	43.7	72.3	110.7	158.2	215.8
6 1/2	10.3	16.7	24.6	34.5	46.4	76.6	116.9	166.7	226.9
7	11.0	17.8	26.1	36.6	49.2	80.9	123.1	175.1	237.9
7 1/2	11.7	18.9	27.7	38.8	51.9	85.2	129.4	183.6	248.9
8	12.4	20.0	29.2	40.9	54.7	89.5	135.6	192.0	260.0
9	13.7	22.1	32.4	44.9	60.0	97.8	147.5	208.8	281.3
10	15.1	24.3	35.5	49.1	65.5	106.4	160.0	225.2	303.3
11	16.5	26.4	38.6	53.4	71.0	115.1	172.4	242.2	325.5
12	17.9	28.6	41.7	57.6	76.5	123.7	184.8	259.1	347.6
13	19.3	30.7	44.8	61.8	82.0	132.0	197.2	276.0	369.6
14	20.6	32.9	47.9	66.0	87.6	140.6	209.7	292.9	391.7
15	22.0	35.1	51.0	70.3	93.1	149.2	222.1	309.8	413.8
16	23.4	37.2	54.1	74.5	98.6	157.9	234.5	326.7	435.9
17	24.8	39.4	57.2	78.7	104.1	166.5	246.9	343.6	458.0
18	26.2	41.5	60.3	82.9	109.7	175.1	259.4	360.5	480.1
19	27.5	43.7	63.4	87.2	115.2	183.7	271.8	377.5	502.2
20	28.9	45.8	66.5	91.4	120.7	192.4	284.2	394.4	524.3
21	30.3	48.0	69.6	95.6	126.2	201.0	296.6	411.3	546.4
22	31.7	50.2	72.7	99.9	131.7	209.6	309.1	428.2	568.4
23	33.1	52.3	75.8	104.1	137.3	218.3	321.5	445.1	590.5
24	34.4	54.5	78.9	108.3	142.8	226.9	333.9	462.0	612.6
25	35.8	56.6	82.1	112.5	148.3	235.5	346.3	478.9	634.7
26	37.2	58.8	85.2	116.8	153.8	244.1	358.8	495.8	656.8
27	38.6	60.9	88.3	121.0	159.4	252.8	371.2	512.7	678.9
28	40.0	63.1	91.4	125.2	164.9	261.4	383.6	529.7	701.0
29	41.3	65.3	94.5	129.5	170.4	270.0	396.0	546.6	723.1
30	42.7	67.4	97.6	133.7	175.9	278.7	408.5	563.5	745.2

[1] See also table in Carnegie Pocket Companion.

Table 15a.—Tensile Strength of Bolts and Round Rods without Upset Ends

Diameter of rod	Diameter of root of thread	Weight per lin. ft.	Strength of rod: At 12,500 lb. per sq. in.	At 15,000 lb. per sq. in.	At 16,000 lb. per sq. in.	At 20,000 lb. per sq. in.
3/8	0.294	0.376	848	1,018	1,088	1,360
7/16	0.344	0.511	1,160	1,393	1,489	1,860
1/2	0.400	0.668	1,570	1,884	2,018	2,520
9/16	0.454	0.845	2,022	2,427	2,590	3,240
5/8	0.507	1.043	2,524	3,030	3,230	4,040
3/4	0.620	1.502	3,780	4,530	4,830	6,040
7/8	0.731	2.044	5,250	6,300	6,720	8,400
1	0.837	2.670	6,880	8,240	8,800	11,000
1 1/8	0.940	3.380	8,670	10,420	11,100	13,880
1 1/4	1.065	4.170	11,170	13,420	14,280	17,860
1 3/8	1.160	5.050	13,220	15,860	16,900	21,140
1 1/2	1.284	6.010	16,190	19,420	20,700	25,900
1 5/8	1.389	7.050	18,930	22,720	24,200	30,300
1 3/4	1.490	8.180	21,880	26,170	27,900	34,880
1 7/8	1.615	9.390	25,600	30,720	32,800	40,960
2	1.712	10.680	28,800	34,550	36,800	46,040
2 1/4	1.962	13.520	37,800	45,350	48,400	60,460
2 1/2	2.175	16.690	46,450	55,700	59,400	74,300
2 3/4	2.425	20.200	57,750	69,200	73,800	92,380
3	2.629	21.030	67,800	81,400	86,900	108,560

Table 15b.—Strength of Round Rods with Upset Ends

Diameter of rod	Diameter of upset	Weight per lin. ft.	Strength of rod: At 12,500 lb. per sq. in.	At 15,000 lb. per sq. in.	At 16,000 lb. per sq. in.	At 20,000 lb. per sq. in.
1/2	3/4	0.668	2,453	2,944	3,135	3,920
9/16	3/4	0.845	3,106	3,727	3,980	4,080
5/8	7/8	1.043	3,835	4,600	4,910	6,140
11/16	1	1.262	4,640	5,560	6,940	7,420
3/4	1	1.502	5,520	6,627	7,080	8,840
13/16	1 1/8	1.763	6,490	7,790	8,310	10,380
7/8	1 1/4	2.044	7,516	9,020	9,630	12,020
15/16	1 1/4	2.347	8,630	10,340	11,040	13,800
1	1 3/8	2.670	9,815	11,780	12,560	15,700
1 1/8	1 1/2	3.379	12,425	14,900	15,910	19,880
1 1/4	1 5/8	4.173	15,330	18,400	19,650	24,540
1 3/8	1 3/4	5.049	18,550	22,260	23,750	29,700
1 1/2	2	6.008	22,080	26,500	28,300	35,340
1 5/8	2 1/8	7.051	25,910	31,090	33,200	41,480
1 3/4	2 1/4	8.178	30,060	36,070	38,500	48,100
1 7/8	2 3/8	9.388	34,600	41,400	44,200	55,220
2	2 1/2	10.680	39,270	47,130	50,300	62,840
2 1/8	2 5/8	12.060	44,320	53,190	56,700	70,940
2 1/4	2 7/8	13.520	49,700	59,680	63,600	79,520
2 3/8	3	15.070	55,370	66,450	70,900	88,600
2 1/2	3 1/8	16.690	61,350	73,620	78,500	98,180
2 5/8	3 1/4	18.400	67,600	81,200	86,600	108,240
2 3/4	3 3/8	20.200	74,230	89,080	95,100	118,800

4. Lateral Resistance of Nails, Screws and Bolts.—When spikes, screws and bolts are subjected to lateral forces in a timber joint, shearing and bending stresses are produced in the spikes, screws, or bolts, and the timber in contact with the metal is subjected to pressure. In timber construction, joints of this nature are of common occurrence, and it is necessary to have safe working values for such details. The factors entering into a theoretical consideration of the stresses produced in such a joint are many and complex, and in the determination of safe working values, recourse must be had to the results of tests.

In the case of nails and screws a theoretical analysis of the stresses is not practical. Tests[1] have established fairly definitely the ultimate strength and elastic limits of such joints.

The safe working value for common wire nails or spikes for resistance to lateral forces in timber joints of yellow pine or Douglas fir may be taken at

$$p = 4{,}000d^2$$

where p = safe lateral resistance of one nail, and d = diameter of nail in inches.

The working values for the common sizes of nails in accordance with this formula are given in Table 16.

TABLE 16.—SAFE WORKING VALUE FOR LATERAL RESISTANCE OF ONE NAIL IN YELLOW PINE OR DOUGLAS FIR

Size of nail	6d	8d	10d	12d	16d	20d	30d	40d	50d	60d	80d
Strength in pounds	53	62	88	88	110	165	194	226	268	322	364

All tests made on nailed joints indicate that the strength of the joint is approximately the same whether the nail be driven so that the compression on the timber is against or across the grain. The resistance of the joint is, however, decreased from 25 to 33⅓ per cent if the nails are driven parallel to the fibers of the timber—for example, driving the nails into the ends of a stick of timber. A joint in which this condition exists is a header joint, frequently used in light joist construction.

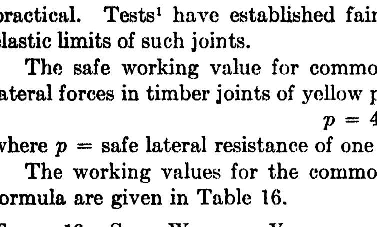


FIG. 1.—Typical load-slip curve of nailed joint, Bureau of Buildings, City of Portland.

When one piece of timber is spiked to another, the penetration of the nail into the second timber should not be less than one-half the length of the nail, and should preferably be in excess of this.

The slip of a nailed joint occurs at a comparatively small load, as may be seen from an inspection of the curve of Fig. 1, which is plotted from the published results of tests made by the Portland Bureau of Buildings.

The elastic limit of a nail in lateral resistance in air-dry long-leaf yellow pine occurs at a value of approximately $C = 7{,}000$ in the formula, $p = Cd^2$, and at an

[1] Tests for nails: Walker and Cross, *Jour.* Assn. Eng. Soc., vol. 19, Dec., 1897; Darrow and Buchanan, *Proc.* Ind. Eng. Soc., 1900; Morgan and Marish, Eng. Exp. Sta., Iowa State College, *Bul.* No. 2; Tests made for Bureau of Buildings, Portland, Ore., *Eng. News-Rec.*, vol. 79, No. 19, Nov. 8, 1917, also vol. 79, No. 26, Dec. 27, 1917; also "*The Timberman*," Portland, Ore., vol. 18, No. 12, Oct., 1917; "Tests Made to Determine Lateral Resistance of Wire Nails," Thomas R. C. Wilson, *Eng. News-Rec.*, vol. 75, No. 8, Feb. 14, 1917; Jacoby's "Structural Details," Dewell's "Timber Framing."

average slip of 0.028 in., as found by Wilson in the tests of the Forest Service (see reference in footnote, p. 404). The Portland tests show higher values for both elastic limit and slip at elastic limit.

5. Lateral Resistance of Wood Screws.—The lateral resistance of common wood screws was investigated as thesis work by Kolbirk and Birnbaum at Cornell University,[1] using timbers of cypress, yellow pine and red oak. From the results of these tests, the following formula for the safe lateral resistance may be used for yellow pine and Douglas fir:

$$p = 4{,}375d^2$$

Table 17 gives the safe working values in terms of gage numbers. In giving these values the assumption is made that the screw is imbedded in the second or main piece of timber approximately $\frac{9}{10}$ the length of the screw.

Table 17.—Safe Lateral Resistance of Common Wood Screws with Yellow Pine and Douglas Fir

Gage of screw	Diameter, inches	Safe lateral resistance, pounds
6	0.137	82
8	0.163	116
10	0.189	156
12	0.216	204
14	0.242	256
16	0.268	314
18	0.295	381
20	0.321	451
22	0.347	527
24	0.374	512
26	0.400	700

6. Lateral Resistance of Lag Screws.—Two typical cases of joints may be made: (1) Boards or planks screwed to a timber block, and (2) a metal plate screwed to a block of timber. The writer made a series of tests on both types of joint.[2] From the results of these tests, and also from a theoretical consideration of the probable distribution of pressures of lag screw against timber and resultant bending moments in the lag screw, the following values for lag screws in lateral shear and bending are recommended:

Safe Lateral Resistance of One Lag Screw

Metal plate lagged to timber	$\frac{3}{4}$- × $4\frac{1}{2}$-in. lag screw	1,030 lb.
	$\frac{7}{8}$- × 5-in. lag screw	1,200 lb.
Timber planking lagged to timber	$\frac{3}{4}$- × $4\frac{1}{2}$-in. lag screw	900 lb.
	$\frac{7}{8}$- × 5-in. lag screw	1,050 lb.

7. Lateral Resistance of Bolts.—In a typical detail of wooden joint, such as is illustrated in Fig. 2, a number of assumptions may be made as to the distribution of the bearing pressure of the bolt against the timber. Since as many different bending moments will obtain as assumptions of distribution of pressure are made, the resultant computed resistance of bolt to resist relative moment

[1] Abstract of results published in *Cornell Civil Engineer*, vol. 22, No. 2, Nov., 1913.
[2] *Eng. News*, vol. 76, No. 3, July 20; No. 4, July 27, and No. 17, Oct. 26, 1916.

of the timbers will vary accordingly. Two assumptions will be considered here: (1) a uniform distribution of bearing pressures, and (2) triangular distribution of bearing pressures.

1. *Uniform Distribution of Bearing Pressures.*—With this assumption, the bending moment in the bolt will be

$$M = \tfrac{1}{2}P(t'/2 + t''/4)$$

where t' = thickness of splice pad, and t'' = thickness of main timber. Under this assumption, the greater the thickness of side pieces t' (see Fig. 2), the larger diameter of bolt required. Table 18 gives the resisting moments of one bolt in flexure at various fiber stresses, varying from 12,000 to 24,000 lb. per sq. in.

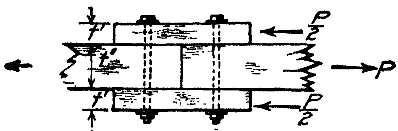


FIG. 2.—Typical bolted joint—bolts in "double shear."

The working values of bolts for typical timber joints, as found by this method are very low, especially for joints with thick splice pads. Hundreds of such joints are giving service in which the bolts are working at more than the ultimate stresses as computed by this method.

Bolts are usually driven with a tight fit in the holes and when such a condition exists, the pressure of the bolt on the timber is not uniform along the length of bolt, as has been determined by tests, and therefore the preceding value of bending moment on the bolt is incorrect.

TABLE 18.—RESISTING MOMENTS OF BOLTS

Size of bolt	Section modulus	Fiber stresses, pounds per square inch				
		12,000	16,000	20,000	22,500	24,000
5/8	0.0239	285	380	480	540	575
3/4	0.0414	495	660	830	930	995
7/8	0.0656	785	1,050	1,310	1,475	1,575
1	0.0982	1,180	1,570	1,960	2,205	2,360
1 1/8	0.140	1,680	2,240	2,800	3,150	3,360
1 1/4	0.191	2,290	3,055	3,820	4,300	4,585
1 3/8	0.255	3,060	4,080	5,100	5,735	6,120
1 1/2	0.331	3,970	5,295	6,620	7,445	7,945
1 5/8	0.421	5,050	6,735	8,420	9,470	10,105
1 3/4	0.525	6,300	8,400	10,500	11,810	12,600
1 7/8	0.646	7,750	10,335	12,920	14,535	15,505
2	0.785	9,420	12,560	15,700	17,660	18,840

The following method is proposed as offering a satisfactory method of computing the strength of such bolt joints:

2. *Triangular Distribution of Bearing Pressure on Bolts.*—The assumptions of this article are illustrated in Fig. 3 and are the result of a study of a series of tests of bolted joints made by the writer.[1] The theory of bearing pressures may be stated thus: It is assumed that the distribution of load on the bolt is triangular in shape; that the unit pressure (pounds per linear inch of bolt) is a

[1] See second footnote, p. 405.

maximum at the contact faces of the timbers, in amount equal to the strength of the timber in bearing,[1] and of approximately the distribution for the typical case, as shown in Fig. 3. It is also assumed that in the joint of Fig. 3, there is a definite minimum length "m," such that the moment resulting from the load on this length of bolt will just equal the flexural strength of the bolt. Further, it is assumed that in joints where the thickness of side timber is less than the limiting value "m" the pressure distribution diagram, while maintaining the general triangular shape, is modified in respect to the relative dimensions "a" and "b" (Fig. 3) within the limits $a = o$ and $a = t'/3$, and that the ratio a/t' remains such that the resulting bending moment in the bolt bears the same relation to the flexural strength of the bolt as the maximum intensity of pressure on the

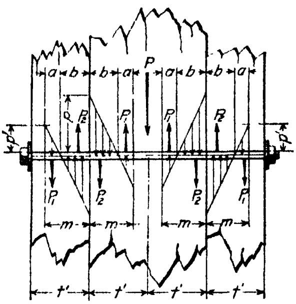


FIG. 3.

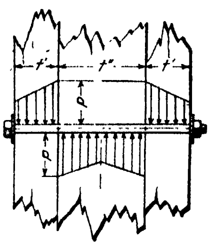


FIG. 4.

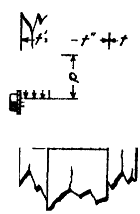


FIG. 5.

timber bears to the unit strength of the timber in compression. The above theory assumes that the ratio of thickness of timber to diameter of bolts is comparatively large. As the ratio of diameter of bolt to thickness of splice pad increases, the pressure distribution diagram on the length of bolt within the splice pad is assumed to change from a triangular shape (Fig. 3) through a trapezoidal shape (Fig. 4) until the limiting case is reached, with a short thick bolt of uniform distribution of pressure along the length of bolt (Fig. 5).

For the case illustrated in Fig. 3 there are two equal maximum bending moments in the bolt, occurring at points of zero shear. With the assumption that beyond a minimum value of t' or width of splice pad, the strength of joint is independent of the length of bolt, the length, for which the strength of the bolt in flexure is equal to the safe load on the bolt as determined from the compression on the timber, may be determined by equating the bending moment resulting from such load to the resisting moment of the bolt.

$$M = \frac{4}{9}P_1 m \qquad P_1 = \frac{pmd}{12}$$

whence

$$M = \frac{4}{9}\left(\frac{pmd}{12}\right)(m) = \frac{1}{27}(dpm^2) = \frac{\pi d^3 f}{32}$$

and

$$m = d\left(\frac{f\pi 27}{32p}\right)^{\frac{1}{2}}$$

[1] By strength is meant working strength.

where M = bending moment on bolt in inch pounds.
p = maximum allowable unit bearing stress of bolt against timber.
f = maximum allowable flexural unit stress in bolt.
t' = thickness of splice pad.
d = diameter of bolt in inches.
m = length of portion of bolt on which pressure exists.

Using the same notation, when m is less than t', the theory assumes that the ratio of the dimensions a and b changes, within the limits $a = o$ and $a = t'/3$, to the end that the greatest strength of joint is obtained with the provision that the capacity of the bolt in bending and the timber in compression is maintained simultaneously. For these cases the bending moment may be expressed by the general formula $M = Ct'^2$, and the total load on the joint by the general formula $P = Kt'$. In these formulas, M = moment on bolt in inch pounds, t' = width of splice pad in inches, and C and K are factors to be obtained from Diagram 1.

Table 19 shows the relation of C and K to varying ratios of a/t; for a bolt of 1-in. diameter, for the case of a triangular pressure diagram.

TABLE 19

Ratio a/t'	C	K
0	433	1,300
1/6	266	1,040
1/4	163	866
1/3	48	650

DIAGRAM 1.

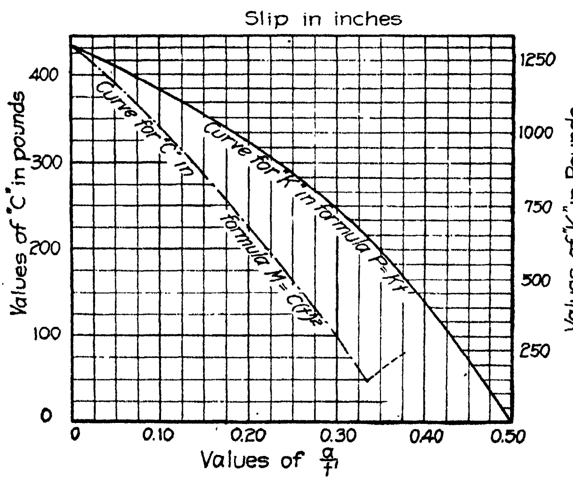


Diagram 1 shows the above variation of C and K with the ratios a/t', for a 1-in. bolt. By means of this diagram, the safe strength of a bolt in double shear for any thickness of splice pad may be found. The diagram is based on the values, p = 1,300 lb. per sq. in. for the safe pressure in end bearing of the diametral section of the bolt in timber, and f = 16,000 lb. per sq. in. for bolts.

Illustrative Problem.—Given a joint with 6-in. center timber, and two 3-in. splice pads, bolted with ⅞-in. bolts. What is the safe strength of one bolt, allowing a maximum unit compression against ends of fibers of timber and a maximum flexural stress of 16,000 lb. per sq. in. in the bolt.

From Table 18 the safe resisting moment of a ⅞-in. bolt at 16,000 lb. per sq. in. is 1,050 in.-lb. Since Diagram 1 is for a bolt of 1-in. diameter, the equivalent moment for entering the diagram is $\frac{1,050}{0.875} = 1,200$ in.-lb.

From the equation $M = Ct'^2$, $C = \frac{1,200}{9} = 133.3$.

Entering the diagram, a vertical line through the point on the dash and dot "C" curve for the value $C = 133.3$, intersects the full line "K" curve at a point giving $K = 810$ lb. Remembering that this value is for the case of a 1-in. bolt, the safe load for a ⅞-in. bolt is

$$P = \tfrac{7}{8}Kt' = (\tfrac{7}{8})(810)(3) = 2,130 \text{ lb.}$$

For the cases in which the pressure distribution on the bolt is trapezoidal, as in Fig. 4, Table 20 gives the values of C and K, in the formulas $M = C(t')^2$ and $P = Kt'$, respectively, for various ratios of the minimum unit pressure to the maximum unit pressures, all for a bolt of 1-in. diameter.

Table 20

Ratio p'/p	C	K
0	433	650
¼	650	812
½	867	975
¾	1,084	1,138
1	1,300	1,300

Diagram 2.

Diagram for Finding Safe Loads on a Bolted Joint—Bolt in "Double Shear." Diagram Drawn for 1-in. Bolt.

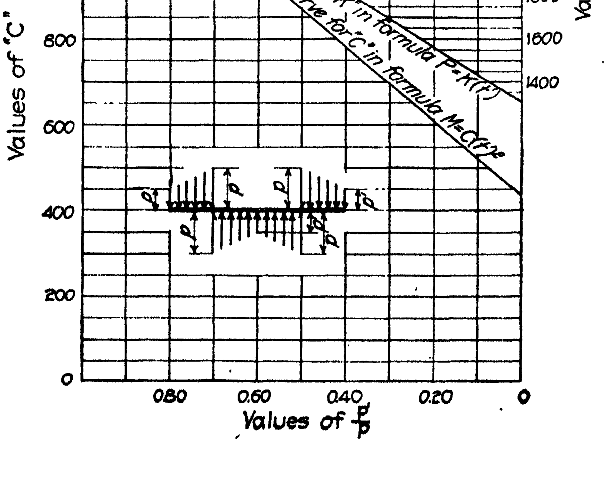


Diagram 2 gives the curves of these formulas for the trapezoidal distribution of pressure for a bolt 1 in. in diameter. These curves are to be used exactly as those of Diagram 1.

Illustrative Problem.—Given a joint of yellow pine timber with 5½-in. center, and two 2½-in. spliced pads, bolted with 1½-in. bolts. What is the safe strength of one bolt in lateral resistance?

From Table 18, the safe resisting moment of a 1½-in. bolt at 16,000 lb. per sq. in. is 5,295 in.-lb. To enter Diagram 2, which is drawn for a 1-in. bolt, the value of 5,295 must be divided by 1½. The equivalent moment is 5,295 × ⅔ = 3,530. From the equation $M = C(t')^2$, $C = \frac{3,530}{6.25} = 565$. From Diagram 2 the value of K in the curve $P = Kt'$, corresponding to $C = 565$, is 1,500 lb. This value is for a 1-in. bolt. Therefore, the safe load for a 1½-in. bolt is

$$P = (1,500)(2\tfrac{1}{2})(1\tfrac{1}{2}) = 5,625 \text{ lb.}$$

The values of Table 21 have been worked from the preceding theory by means of Diagrams 1 and 2.

TABLE 21.—VALUE OF ONE BOLT IN DOUBLE SHEAR

Bolt	Thickness side timbers, inches				
	2	3	4	5	6
	Thickness center timbers, inches				
	4	6	8	10	12
⅝	1,060	1,295	1,465	1,465	1,465
¾	1,440	1,685	1,990	2,100	2,100
⅞	1,925	2,135	2,475	2,845	2,850
1	2,480	2,655	3,000	3,380	3,700
1⅛	3,120	3,235	3,580	4,025	4,520
1¼	3,915	3,940	4,240	4,680	5,170
1⅜	4,840	4,690	4,955	5,415	5,970
1½	5,890	5,570	5,790	6,245	6,700

Maximum fiber stress in bolt in bending, 16,000 lb. per sq. in.

Maximum intensity of bearing pressure on wood, 1,950 lb. per sq. in.

Bearing on wood, average on diametral section of bolt, 1,300 lb. per sq. in.

Bolts in Single Shear.—The safe values of bolts acting in "single shear" may be taken at one-half the values of Table 21.

Bolts Bearing Across the Grain of Timber.—For "double shear" joints in which the bolts bear across the grain of the timber, the safe values may be taken at five-eighths the values of Table 21.

Metal Plates Bolted to Timber.—The values of Table 21 may be used for joints in which steel plates are bolted to timber; in other words, a steel fish plate joint, provided that the values of this table do not exceed the safe loads as determined by bearing of the plate on the bolt, or shear in the bolts.

8. Resistance to Withdrawal of Nails, Spikes, Screws, and Drift Bolts.—The resistance of nails, spikes, screws and drift bolts to withdrawal from timber is a function of the surface area of contact between metal and timber, and the unit resistance to withdrawal. Expressed algebraically,

$$P = AC$$

in which

P = total pounds required to move the spike, screw, or drift bolt.
A = surface of contact between metal and wood.
C = unit resistance to withdrawal.

The value of C depends upon the kind, quality, and condition of timber, condition of surface of nail, screw, or drift bolt, size of hole in which nail, screw, or bolt may have been driven or screwed, and direction of fibers of timber with reference to length of nail, spike, screw, or drift bolt. For practical purposes, C is a quantity determined solely by experiment. Ultimate values of C for wire and cut nails, boat spikes, and drift bolts are given in Table 22. These values are taken from a study of the numerous tests that have been made. The values for resistance to withdrawal as found by the tests vary so widely that, for safe working values, a safety factor of four should be used.

9. Washers.—For the more common timbers employed in building construction, the resistance to crushing across the grain of the timber is much smaller than resistance to end crushing. For this reason it is necessary to use washers under heads and nuts of bolts in timber construction to prevent the nuts and head from crushing into the timber when the nuts are tightened, and also when the bolts take their assumed stresses.

There are five types of washers used in timber construction: (1) Cast-iron O.G. washers, (2) cast-iron ribbed washers, (3) malleable iron washers, (4) circular pressed steel washers, and (5) square plate washers.

TABLE 22.—ULTIMATE RESISTANCE TO WITHDRAWAL OF WIRE AND CUT NAILS, WOOD SCREWS, LAG SCREWS, BOAT SPIKES AND DRIFT BOLTS

(All Quantities Expressed in Pounds per Square Inch of Contact between Metal and Timber)

	Yellow pine	Douglas fir	White pine	White oak	Redwood
Cut nails[1]	500	500	300	1,200	300
Cut nails[2]	300	300	275	1,000	150
Wire nails[1]	300	300	170	900	300
Wire nails[2]	250	250	100	800	200
Wood screws	1,500	1,500	900	2,200	900
Lag screws	800	800	500	1,200	
Boat spikes[3]	500	500	270	1,000	
Boat spikes[4]	370	370	200	750	
Drift bolts[5]	400	400	240	600	
Drift bolts[5]	200	200	120	300	

[1] Driven perpendicular to grain of timber.
[2] Driven parallel to grain of timber.
[3] Edge of point parallel to grain of timber.
[4] Edge of point across grain of timber.
[5] Driven in holes $\frac{1}{16}$ to $\frac{1}{8}$ in. less in diameter than drift bolt.

For cases in which the axis of bolt is inclined to the bearing surface of the timber, bevelled cast-iron washers may be employed (see Fig. 11 and Table 28). The five types of washers mentioned are illustrated in Figs. 6 to 10 inclusive and Tables 23 to 27 inclusive give detailed dimensions.

FIG. 6.—O.G. cast-iron washer.

In the case of bolts acting wholly in tension there can be no question of the necessity of washers. Washers should be properly designed, both for strength and stiffness, and of proper size to limit the bearing pressure on the timber to the safe working value. For Douglas fir or yellow pine either the square plate washers, ribbed cast-iron, or cast-iron O.G. washers of equivalent area should be used. Attention

is called to the fact that in the malleable washer, the full area of the base of washer is not available for bearing. For example, the ¾-in. malleable washer has an actual bearing area of about 4 sq. in., or an actual efficiency of approximately 60 per cent of its nominal area. Even the cast-iron O.G. washers of Table 23

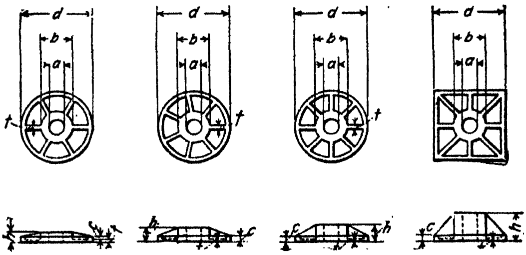


Fig. 7.—Cast-iron ribbed washers.

stress the timber to approximately 750 lb. per sq. in., for a unit stress of 16,000 lb. per sq. in. in the rod.

When the bolt acts wholly in shear and bending, smaller washers, such as the malleable washers, are permissible, though not necessarily advisable. In

Fig. 8.—Malleable iron washer.

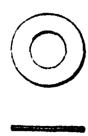


Fig. 9.—Circular pressed steel washer.

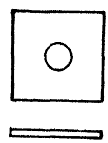


Fig. 10.—Square steel plate washer.

such instances it is often practically certain that the timber will shrink, and that the washers will never be tightened, and for this reason the use of malleable washers may be justified, in order to save expense. On the other hand, when there is a chance that some maintenance work may be counted upon in the shape of washer tightening, good construction will prescribe either a special cast-iron washer or a square plate washer, sufficient in size to meet the capacity of the bolt in tension.

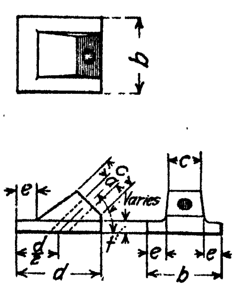


Fig. 11.—Bevelled cast-iron washer.

In order to avoid special washers, malleable washers of larger size than the nominal size for the bolt used are sometimes specified. Such a procedure is unwise for two reasons: (1) The holes in the larger washer are of such diameter with respect to the diameter of the head and nut of the bolt, that a poor bearing between head or nut and washer results; and (2) the carpenter will invariably put stock sizes of washers and bolts together if there is a chance to do so.

The circular cut or pressed steel washer should never be used in timber construction, except between metal and metal.

The selection of a washer as between a special size O.G., ribbed cast iron, or a square steel plate washer, will depend on the relative prices of cast iron and steel, availab lity of foundry and steel shops, and size of jobs. When large size washers are required and the job is a small one, the square plate washer will usually be found cheapest.

No square plate washer should have a thickness less than one-half the diameter of bolt. A good rule is to add 1/16 in. to the thickness thus found.

When the center line of bolt or rod is not normal to the bearing face of the timber, the timber must be notched, or a bevelled washer used. If the section of timber is ample, a notch is the cheapest detail. The pressure of the washer

TABLE 23.—WASHERS—O.G. CAST-IRON

Size of bolt, inches	Weight per 100 lb.	Diameter, inches	Thickness, inches	Area, square inches
1/2	35	2 1/4	1/2	3.78
5/8	75	3	5/8	6.76
3/4	100	3 1/4	3/4	7.86
7/8	145	3 1/2	13/16	9.02
1	185	4	7/8	11.79
1 1/8	285	4 1/2	1 1/8	14.91
1 1/4	375	5	1 1/4	18.41
1 1/2	600	6	1 1/2	26.50

TABLE 24.—WASHERS—CAST-IRON RIBBED

(See Fig. 7)

Size bolt	Size upset	*a*	*b*	*c*	*d*	*h*	*t*	Shape base	No. ribs	Weight
5/8	Not upset	3/4	1 9/16	1/8	3 1/8	3/4	1/8	C	6	0.56
3/4	Not upset	7/8	1 3/4	3/16	4	1	1/4	C	6	1.10
7/8	Not upset	1	2 1/4	3/16	4 1/2	1 1/8	1/4	C	6	1.80
3/4	1	1 1/8	2 3/8	1/4	5 1/4	1 1/4	1/4	C	6	2.79
13/16	1 1/8	1 1/4	2 5/8	1/4	5 5/8	1 5/16	1/4	C	6	3.29
7/8	1 1/4	1 3/8	3	5/16	6 1/2	1 7/16	5/16	C	7	5.30
1	1 3/8	1 1/2	3 1/4	5/16	7	1 9/16	5/16	C	7	6.34
1 1/8	1 1/2	1 5/8	3 1/2	3/8	7 7/8	1 11/16	3/8	C	7	9.04
1 1/4	1 5/8	1 3/4	3 3/4	3/8	8 5/8	1 15/16	3/8	C	7	11.30
1 3/8	1 3/4	1 7/8	4	3/8	9 1/8	2 1/4	3/8	C	7	13.59
1 7/16	1 7/8	2	4 3/8	7/16	10	2 3/8	7/16	C	8	18.66
1 1/2	2	2 1/8	4 5/8	7/16	10 3/8	2 7/16	7/16	C	8	20.39
1 5/8	2 1/8	2 1/4	4 3/4	1/2	11 1/4	2 5/8	1/2	C	8	25.99
1 3/4	2 1/4	2 3/8	5 1/8	1/2	12 1/8	2 3/4	1/2	C	8	30.62
1 7/8	2 3/8	2 1/2	5 1/8	5/8	11 1/4	4 1/4	5/8	Sq.	8	48.23
2	2 1/2	2 5/8	5 5/8	3/4	12	5 1/8	3/4	Sq.	8	69.32

against the timber is then inclined to the direction of fibers, and, consequently, a higher unit bearing pressure may be used, in accordance with the formula and values of Art. **11.**

For the larger size of bolts and rods, notching the timber sufficiently to provide the required area for bearing may cut the stick beyond the safe limit. In such a case, either a combination of a flat washer with a smaller cast-iron bevelled washer may be used, or a special cast-iron bevelled washer may be designed. The latter solution is much the better of the two. If this washer be made square or rectangular, the component of the stress in the rod parallel to the face of the timber may be taken care of by setting the washer into the timber. In the former case, this component will produce bending in the rod or bolt.

TABLE 25.—WASHERS—MALLEABLE IRON

Size of bolt, inches	Weight per 100 washers	Diameter, inches	Thickness, inches
1/2	15	2 1/2	1/4
5/8	22	2 3/4	5/16
3/4	33	3	7/16
7/8	50	3 1/2	7/16
1	68	4	1/2
1 1/8	87	4 1/2	1/2
1 1/4	150		5/8
1 1/2	190		3/4

TABLE 26.—WASHERS—WROUGHT-IRON

Size of bolt, inches	No. in 100 lb.	Diameter, inches	Size of hole, inches	Gage	Thickness, inches
3/16	39,400	9/16	1/4	18	0.05
1/4	15,600	3/4	5/16	16	0.063
5/16	11,250	7/8	3/8	16	0.063
3/8	6,800	1	7/16	14	0.078
7/16	4,300	1 1/4	1/2	14	0.078
1/2	2,600	1 3/8	9/16	12	0.125
9/16	2,250	1 1/2	5/8	12	0.125
5/8	1,300	1 3/4	11/16	10	0.125
3/4	970	2	13/16	9	0.156
7/8	828	2 1/4	15/16	8	0.172
1	600	2 1/2	1 1/16	8	0.172
1 1/8	500	2 3/4	1 1/4	8	0.172
1 1/4	384	3	1 3/8	8	0.172
1 3/8	288	3 1/4	1 1/2	7	0.189
1 1/2	267	3 1/2	1 5/8	7	0.189
1 5/8	230	3 3/4	1 3/4	7	0.189
1 3/4	206	4	1 7/8	7	0.189
1 7/8	182	4 1/4	2	7	0.189
2	168	4 1/2	2 1/8	7	0.189
2 1/4	122	4 3/4	2 3/8	5	0.219
2 1/2	106	5	2 5/8	4	0.234

TABLE 27.—WASHERS—SQUARE STEEL PLATE

Unit Bearing Pressure—350 lb. per sq. in.
Unit Tension in Bolt or Rod—16,000 lb. per sq. in.

Diameter of bolt or rod	Diameter of upset	Side of square washer	Thickness of washer
5/8	Not upset	3 1/4	3/8
3/4	Not upset	4	7/16
7/8	Not upset	4 1/2	1/2
3/4	1 in.	4 3/4	9/16
1 3/16	1 1/8	5	5/8
7/8	1 1/4	5 1/2	1 1/16
1	1 3/8	6 1/4	3/4
1 1/8	1 1/2	7	1 3/16
1 1/4	1 5/8	7 3/4	7/8
1 3/8	1 3/4	8 1/2	1 5/16
1 1/2	2	9 1/4	1 1/16

TABLE 28.—WASHERS—CAST-IRON BEVELED

Size rod	a	b	c	d	t	e
3/4	7/8	3 1/2	1 3/4	4	5/8	3/4
7/8	1	4 1/4	2	4 1/2	3/4	1
1	1 1/8	4 3/4	2 1/4	5 1/4	7/8	1 1/8
1 1/8	1 1/4	5 1/4	2 1/2	6	1	1 1/4
1 1/4	1 3/8	6 1/4	2 3/4	6 1/2	1	1 1/2

10. Resistance of Timber to Pressure from a Cylindrical Metal Pin.—When a pin, bolt, etc. of circular cross-section bears against the ends of the fibers, the load on the pin is resisted by pressure of the timber against the metal, and such differential pressures are always normal to the surface of the pin. The differential pressures may be supposed to be replaced, for practical purposes, by two resultant reactions, one parallel and the other perpendicular to the line of action of the applied force. The second of these resultant reactions tends to split the timber, since it produces tension across the fibers of the timber. Consequently, for the case in hand, the usual permissible unit bearing pressure against the ends of the fibers must be reduced. Also the particular detail must be investigated to make sure that the tension across the fibers due to the cross pressure is within the safe unit stress for the timber in question.

Tests and theoretical considerations indicate that for a round pin or bolt bearing against the ends of timber, the safe average unit bearing pressure to be applied to the diametral plane of the pin may be taken at 2/3 the usual allowable compression against the ends of timber. The resultant secondary pressure across the fibers may be taken at 1/10 the applied load. When the direction of the applied load is perpendicular to the direction of the fibers, the safe average diametral pressure may be taken at 4/10 of the permissible unit compression across the fibers.

For the case of pins and bolts in tight fitting holes in dense Southern pine and Douglas fir, the values of 1,300 lb. per sq. in. for end bearing and 800 lb. per sq. in. in cross bearing may be used.

Illustrative Problem.—What is the safe load on a $1\frac{1}{4}$-in. bolt, bearing against the ends of the fibers of a 6- × 6-in. block of Douglas fir, and what is the force tending to split the block of timber?

The safe load is $1\frac{1}{4} \times 6 \times 1{,}300 = 1{,}950$ in.-lb. The force tending to split the timber is $1{,}950 \times 0.1 = 195$ lb.

11. Compression on Surfaces Inclined to the Direction of Fibers.—The *allowable intensity of pressure* on timber, when the direction of pressure is neither parallel nor perpendicular to the direction of fibers, was investigated by Prof. M. A. Howe on specimens of yellow pine, white pine, cypress, white oak, and redwood.[1] On the basis of these tests, Prof. Howe recommends the formula:

$$r = q + (p - q)(\theta/90°)^2$$

where

r = allowable normal unit stress on inclined surface.
p = allowable unit stress against ends of fibers.
q = allowable unit stress normal to direction of fibers.

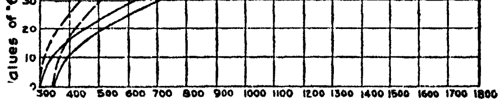
Using the same notation, Prof. Jacoby in "Structural Details" develops the formula:

$$r = p \sin^2 \theta + q \cos^2 \theta.$$

Mr. Russell Simpson of the University of California, has recently made a series of tests, as thesis work, on the bearing values for inclined surfaces of Douglas fir and California white pine. He finds that Jacoby's formula gives results closely approximating the test values at the elastic limit, while Howe's formula holds for a constant indentation of 0.03 in. Diagram 3 gives the curves of the formulas of Howe and Jacoby for values of p = 1,800 lb. per sq. in., q = 350 lb. per sq. in.; and p = 1,600 lb. per sq. in., q = 300 lb. per sq. in.

DIAGRAM 3.

DIAGRAM FOR SAFE BEARING PRESSURE ON TIMBER SURFACES INCLINED TO DIRECTION OF FIBER.



Solid line curves - Formula: $r = p \sin^2\theta + q \sin^2\theta$
Broken line curves - Formula: $r = q + (p-q)\left(\frac{\theta}{90}\right)^2$

Working values for actual design of timber joints involving bearing on surfaces inclined to the direction of fibers should be based on the elastic limit. The full line curves of Jacoby's formula are therefore recommended for design.

12. Tension Splices.—The tension splice in timber building construction occurs usually in the lower chord of a roof truss. This detail is probably the

[1] *Eng. News*, vol. 68, No. 5, and vol. 68, No. 10.

most troublesome to design and frame efficiently of all timber joints. A detail that is efficient on paper is often very unsatisfactory when viewed in the field. Any detail that depends for its action on the simultaneous bearing of more than two contact faces is to be avoided if possible, although it is often impracticable to so limit the design. Again, that detail which is so designed that the bearing faces of splicing members and the bearing faces of the spliced or main timbers may be pu lled together in the field after the joint is framed, has a very decided advanta geover any other type of tension splice. The ideal splice, just described, will be found to give a low efficiency when measured in terms of effective area of main timbers for resisting tension. However, in many cases, such inefficiency may well be allowed, in order to secure certain definite action of splice joint. Importance of the connection, cost of materials, quality of workmanship to be anticipated, possibility of only occasional or no inspection after completion, are all factors that should be carefully considered before deciding upon the particular type of tension splice to be adopted.

The following types of tension splices will be considered and a detail joint of each type developed for a typical example:

(1) Bolted wooden fish plate splice, (2) Modified wooden fish plate splice, (3) Bolted steel fish plate splice, (4) Tabled fish plate splice, (5) Steel tabled fish plate splice, (6) Tenon bar splice, and (7) Shear pin splice.

It will be assumed that a 6- × 8-in. Douglas fir stick must be spliced to safely stand a total stress of 40,000 lb. Specifications of steel structures often call for the detail of splice to be of sufficient strength to develop the strength of the members. The same specification may be applied to the timber joint, although it is customary to design the splice for the computed stress in the member.

For the case under discussion the safe working stress in the timber for tension will be taken at 1,500 lb. per sq. in. The required net area for tension is therefore $\frac{40,000}{1,500} = 26.7$ sq. in.

12a. Bolted Fish Plate Splice.—The bolted fish plate splice is shown in Fig. 12. The size of bolts will be computed in accordance with the formula

$$M = \tfrac{1}{2}P(t'/2 + t''/4)$$

where P is the total load on on bolt; t' is the thickness of splice pad, or fish plate; and t'' is the thickness of main timber (see Art. **7**). This formula assumes the load on each bolt to be uniformly distributed along its length.

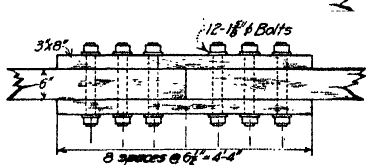


Fig. 12.—Bolted wooden fish plate splice.

Assume 1¾-in. bolts, and splice plates 3 × 8 in. With bolts spaced in pairs, the net width of splice plate will then be 8 − (2) (1¾) = 4½ in. The required thickness of one plate is then $\frac{26.7}{9} = 2.97$, showing that a 3-in. thickness is sufficient. Assume 6 bolts required. The load on one bolt is then 40,000/6 = 6,667 lb. The bending moment on one bolt is (6,667/2)(½ × 3 + ¼ × 6) = 10,000 in.-lb. With a flexural stress of 24,000

lb. per sq. in., the required section modulus of one bolt = 10,000/24,000 = 0.416 in., and the required diameter of bolt = $\sqrt[3]{0.416/0.098} = \sqrt[3]{4.26}$ = 1.62 in.

The unit bearing pressure on the diametral section of bolt = $\frac{6,667}{(1.625)(6)}$ = 685 lb. per sq. in., which is about one-half the amount allowed. The minimum distance between bolts must next be computed. This distance will be taken as the sum of (*a*) computed distance necessary for shearing along the grain of the timber, (*b*) computed distance giving required area for transverse tension, and (*c*) diameter of bolt.

Total shearing area required................ $= \frac{6,667}{150}$ = 44.44 sq. in.

or distance (*a*).................... $= \frac{44.44}{12}$ = ... 3.7 in.

Area required for transverse tension........... $= \frac{(6,667)(0.1)}{150}$ = 4.44 sq. in.

or distance (*b*).................... $= \frac{4.44}{6}$ = ... 0.74 in.

Diameter of bolt (*c*).. 1.63 in.

Minimum spacing of bolts.. 6.07 in.

The spacing of bolts will be made 6½ in.

12*b*. Modified Wooden Fish Plate Splice.—In the modified wooden fish plate splice, the size of bolts will be reduced to 1 in., and the value of each bolt taken at 2,655 lb., in accordance with the values of Table 21, p. 410.

The number of bolts required is $\frac{40,000}{2,655}$ = 15.

14 1-in. bolts will be used, giving a load of 2,857 lb. per bolt.

Spacing of bolts:

(*a*) Distance required for shear $\frac{2,857}{(150)(12)}$ = 1.58 in.

(*b*) Distance required for transverse tension = $\frac{(2,857)(0.1)}{(150)(6)}$ = 0.32 in.

(*c*) Distance of bolt.............................. = 1.00 in.

2.90 in.

Spacing of bolts will be made 3 in. The detail is shown in Fig. 13.

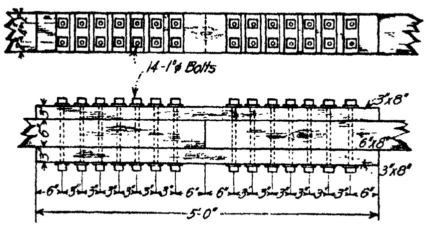


FIG. 13.—Modified wooden fish plate splice.

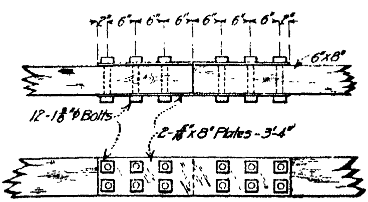


FIG. 14.—Bolted steel fish plate splice.

12*c*. Bolted Steel Fish Plate Splice.—Figure 14 shows a bolted steel fish plate splice. The bending in the bolts is reduced from that in the first type, due to the smaller lever arm. The section of steel plate must be sufficient for tension, and for bearing on the bolts. Otherwise, the computations are similar to those of the bolted fish plate splice.

Net section of steel plate = $\frac{40,000}{15,000}$ = 2.67 sq. in.

Assume two 1½-in. bolts in pairs. Then net width = (2)(1$\frac{9}{16}$) = 4.875 in., and required thickness is $\frac{2.67}{(2)(4.875)}$ = 0.28 in., requiring a $\frac{5}{16}$-in. plate. Assume six bolts.

As before, each bolt must take 6,667 lb. The minimum diameter of bolt required with a $\frac{5}{16}$-in. plate at 15,000 lb. per sq. in. in bearing is $\frac{3}{4}$ in. Assuming a uniform distribution of pressure along the length of bolt, the bending on bolt $= \left(\frac{6,667}{2}\right)(\frac{1}{2} \times \frac{5}{16} + \frac{1}{4} \times 6) =$ 5,520 in.-lb. At 24,000 lb. per sq. in., the required diameter of bolt from Table 18 is seen to be $1\frac{3}{8}$ in.

The unit pressure of the bolt on the ends of the fibers is $\frac{6,667}{(1.375)(6)} = 810$ lb. per sq. in. The spacing of bolts may be figured as before, and will be less than that computed in the detail of the bolted fish plate splice by the difference in diameter of the bolts. The spacing will be made 6 in.

12d. Tabled Wooden Fish Plate Splice.—The detail of a tabled wooden fish plate splice is shown in Fig. 15. The points to be investigated in this detail are: (1) Net section of main timber and splice pad; (2) bearing between splice pad and main timber; (3) length of table of fish plate for shear; (4) tension in bolts; and (5) possibility of bending on splice pads if bolts become loose because of shrinkage of timbers.

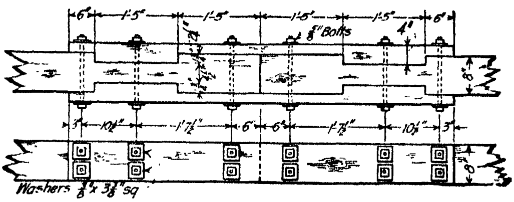


FIG. 15.—Tabled wooden fish plate splice.

Net section of main timber required, as before, 26.7 sq. in.

Net section of fish plate required, as before, $\frac{40,000}{(2)(1,500)} = 13.4$ sq. in.

Allowing for two $\frac{3}{4}$-in. bolts, net depth of fish plate $= \frac{13.4}{(8 - 1\frac{1}{2})} = 2.06$ in.

Total bearing area required between fish plate and main timber $= \frac{40,000}{1,600} = 25$ sq. in.

Depth of cut into main timber $= \frac{25}{(8)(2)} = 1.57$ in. Depth will be made $1\frac{3}{4}$ in. It will be necessary to use an 8- × 8-in. timber, instead of a 6- × 8-in. stick, with 4- × 8-in. fish plates.

Total net depth of fish plate $2\frac{1}{4}$ in.

Shearing area required for table of fish plate $= \frac{40,000}{(2)(150)} = 133$ sq. in. Length of table $= \frac{133}{8} = 17$ in.

The action of this joint produces a bending moment in the fish plate which must be resisted by the bolts. The resultant stress in the fish plate acts at the center of the uncut portion, while the resultant of the pressure between fish plate and main timber is at the center of the table. This couple produces a moment, in this case, of

$$(20,000)(\tfrac{1}{2})(2\tfrac{1}{4} + 1\tfrac{3}{4}) = 40,000 \text{ in.-lb.}$$

The lever arm of the bolts in the center of the table about the end of table is $8\frac{1}{2}$ in. Using two bolts, the stress in each bolt is $\frac{40,000}{(2)(8\frac{1}{2})} = 2,353$ lb. A $\frac{1}{2}$-in. bolt is sufficient for his stress, but bolts less than $\frac{5}{8}$-in. diameter are not advisable in a timber joint. The required area of washers is $\frac{2,353}{350} = 6.72$ sq. in., which area would be supplied by a 3-in. circular washer. The washers shown are square steel $\frac{3}{8} \times 3\frac{3}{8} \times 3\frac{3}{8}$ in.

If the timber should shrink and the bolts remain loose, each fish plate would be subjected to the full bending of 40,000 in.-lb., except as the friction of the ends of the table against the main timber might reduce such bending. The section modulus of the net section of fish plate is $(\frac{1}{6})(8)(2\frac{1}{4})^2 = 6.75$ (correct for two bolts). The extreme fiber stress due to bending would then be $\frac{40,000}{6.75} = 5,926$ lb. per sq. in. To this stress must be added the uniform tensile stress, which is $\frac{20,000}{(8)(2\frac{1}{4})} = 1,110$ lb. The maximum fiber stress would therefore be 7,036 lb. per sq. in., an amount nearly equal to the ultimate strength of the timber. For this reason, the joint should be well spiked together, and in particular the fish plate should extend at either end beyond the table, to allow a number of spikes to be driven here. If the cut at the ends of the tables be made with a bevel towards the center of the joint, the same result will be obtained.

12c. Steel-tabled Fish Plate Splice.—The most economical and practical detail of the steel-tabled fish plate splice consists of steel splice plates with steel tables riveted to the plates, as shown in Fig. 16. The points to be investigated are: (1) Necessary net area of plate to resist tension; (2) required thickness of tables to keep the bearing of tables against the ends of the fibers of the timber within the safe working stresses; (3) number of rivets between tables and fish plate; (4) distance between table, limited by longitudinal shear in the timber; and (5) bolts required to hold tables in the notches in the timber.

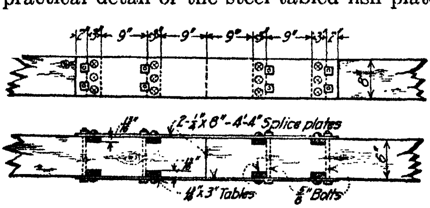


FIG. 16.—Steel-tabled fish plate splice.

The 6- × 8-in. main timber will be sufficient for this type of splice.

Net area of steel plates $= \frac{40,000}{15,000} = 2.67$ sq. in.

Assume 3 rivets in one row. Then net width of plate is $8 - (3)(\frac{3}{4}) = 5.75$ in., and required thickness of plate is $\frac{2.67}{(2)(5.75)} = 0.23$ in. A ¼-in. plate will be sufficient for tensile strength. Bearing area required for tables $= \frac{40,000}{1,600} = 25$ sq. in.

Assume 4 tables on each fish plate. Required total thickness of tables is $\frac{25}{(4)(8)} = 0.78$ in. Make the depth $\frac{13}{16}$ in. = 0.815 in.

Rivets required in each table, limiting value of one ¾-in. rivet in bearing at 20,000 lb. per sq. in. on ¼-in. plate being 3,750 lb. $= \frac{40,000}{(4)(3,750)} = 2.67$.

Use three rivets and make table $\frac{13}{16} \times 3$ in.

The distance between end of main timber and first table, and the distance between tables, must be sufficient for longitudinal shear in the timber. Total shearing area required $= \frac{40,000}{150} = 267$ sq. in. Distance between tables $= \frac{267}{(4)(8)} = 8.35$ in. Call this distance 9 in., making the distance center to center of tables 12 in.

As in the case of the wooden fish plate splice, the bending moment to be resisted by bolts is the load transmitted by one table times one-half the combined thickness of fish plate and table, or

$$M = (10,000)(\tfrac{1}{2})(\tfrac{13}{16} + \tfrac{1}{4}) = 5,300 \text{ in.-lb.}$$

Two bolts will be placed against the outer edge of table, making the lever arm of the bolts $3\frac{1}{2}$ in. The stress in one bolt is then $\frac{5,300}{(3\frac{1}{2})(2)} = 760$ lb. Two $\frac{5}{8}$-in. bolts will be used for each table.

12*f*. Tenon Bar Splice.—The tenon bar splice is one of the oldest splices used, though not seen so frequently today as formerly. It is probably the simplest and most effective tension splice that can be made. The detail is shown in Fig. 17. The points to be computed are (1) size of rod for tension; (2) width of bar for proper bearing against the timber, and also for the hole for the rod passing through the ends; (3) depth of bar for bending; (4) distance of bar from end of timber to provide sufficient bearing area;and (5) net section of timber. To give general stiffness to this joint, Fig. 17 shows the addition of two 2- × 8-in. splice pads bolted with $\frac{3}{4}$-in. bolts.

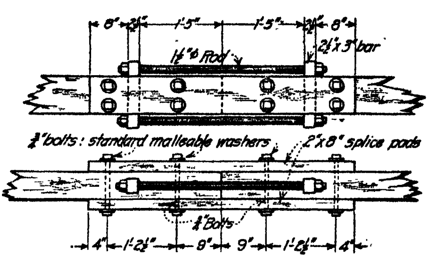


FIG. 17.—Tenon-bar splice.

An 8- × 8-in. main timber will be assumed. Size of rod area required $= \frac{40,000}{(2)(16,000)} = 1.25$ sq. in. A $1\frac{1}{2}$-in. rod has an area of 1.295 sq. in. at the root of thread, and this size rod will be used. Since the rod must be placed at such a distance from the timber that the nuts may be tightened, and since it is desirable to keep the length of the bar as small as possible, hexagonal nuts will be used. (It is obvious that the bending moment on the bar increases with the distance between center lines of rods.) The long diameter of a $1\frac{1}{2}$-in. hexagonal nut is $2\frac{3}{4}$ in., hence the distance from the side of timber to the center line of rod will be made $1\frac{1}{2}$ in.

Size of bar required: The pressure of the timber against the bar will be assumed to be uniform. Hence the bending moment on the bar will be $(20,000)(1\frac{1}{2} + \frac{1}{4} \times 8) = (20,000)(3\frac{1}{2}) = 70,000$ in.-lb. Using a fiber stress of 24,000 lb. per sq. in. in bending, since the bar is a short beam, the required section modulus is $\frac{70,000}{24,000} = 2.92$ in.

The bearing area required is $\frac{40,000}{1,600} = 25$ sq. in. The required width of bar is therefore $\frac{25}{8} = 3.13$ in. Since a 3-in. bar is a stock size, a width of 3 in. will be used. This width will give a full bearing for the hexagonal nut, and will allow $1\frac{3}{16}$ in. of metal on each side of the hole. If a 6- × 8-in. timber were used, the required width of bar would be $4\frac{1}{4}$ in., which would reduce the section of timber below the allowable.

The depth of bar must now be computed. The section modulus $\frac{1}{6}bd^2 = 2.92$ in., when $d = \sqrt{\frac{(2.92)(6)}{b}} = \sqrt{\frac{(2.92)(6)}{3}} = \sqrt{5.84} = 2.4$ in. The bar size will be taken at $2\frac{1}{2} \times 3 \times 14$ in.

The shearing area required between the bar and end of timber is $\frac{40,000}{150} = 267$ sq. in. The distance required between the bar and end of timber is therefore $\frac{267}{(2)(8)} = 16.8$ in., say 17 in.

12*g*. Shear Pin Splice.—In the shear pin splice, the 6- × 8-in. main timber will be sufficient. This splice is shown in Fig. 18. The stress is trans-

mitted across the joint by means of the circular pins of hardwood or steel. These pins are driven in a bored hole with a driving fit for the pins. The joint is a comparatively easy one to frame. The bolts take some tension, due to the couple of the forces acting on the pins. The working values for the pins are taken from Art. **10**.

The splice pads in this detail are 3- × 8-in. timbers. The pins are 2 in. in diameter, of extra heavy steel pipe. The total net section of splice pads is then 4 × 8 = 32 sq. in., giving a unit stress in tension of $\frac{40,000}{32} = 1,250$ lb. Using the working value of 800 lb. per lin. in. of pin, the safe value of a 2- × 8-in. pin is 6,400 lb. The number of pins required is then $\frac{40,000}{6,400} = 6.25$. Six pins will be used.

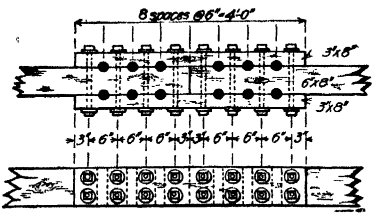


FIG. 18.—Shear pin splice.

The tension in the bolts will be taken at one-half the total tensile stress, or 20,000 lb. Eight ⅝-in. bolts will be used, giving a working value of 2,500 lb. per bolt. The bolts will be placed in pairs, endways between the pins. The pins will be placed 6-in. centers.

13. General Comparison of Tension Splices.—The tenon bar splice, when it can be used, is to be recommended. It is direct in its action; shrinkage of the timber cannot destroy its effectiveness; there being but one bearing surface, the splice will surely act as designed; the two sections of timber can be drawn tightly together in the field; and the splice is almost fool-proof.

The wooden tabled fish plate splice is also effective where there is but one table in each splice pad either side of the joint. In those joints where more tables are necessary, however, there enters at once the possibility, and even the probability, that all the contact faces will not act simultaneously. In other words, the effectiveness of the splice in such a case depends wholly on the skill and care in workmanship. In this detail, also, shrinkage of the timber adds an uncertainty as to the strength of the joint.

The bolted steel fish plate splice makes a neat appearing splice for exposed work, and is much in favor on that account. For a moderate stress in the timber to be spliced, it is fairly economical.

The steel tabled fish plate splice is open to the same objection as the wooden tabled splice. The bearing surfaces of the steel tables are very likely to be uneven, making a close fit between steel and timber almost impossible. On paper, the joint is neat and effective and adaptable to almost any case. Unless rigid inspection in the shop and field is maintained, the actual joint is likely to be disappointing. The bearing edges of all tables should be milled; the holes in the tables should be drilled, and tight riveting secured. Careless and inferior workmanship in the steel shop on the metal splice plates is to be expected.

The shear pin splice is effective and simple; its greatest drawback is the effect of shrinkage in the timber which will allow the pins to become loosened. This splice should not be used with unseasoned or partially seasoned timber, unless it is absolutely certain that the bolts will be kept tight as the timber seasons.

The bolted wooden splice is effective, but cumbersome, and unsuited for large stresses, due to the unusual size of bolts.

The modified wooden bolted splice is satisfactory for comparatively small stresses and when rigid inspection can be counted upon to see that the bolts are driven in close fitting holes. For large stresses, the required number of bolts will be excessive.

Architectural appearances may prohibit certain types of splices as being unsightly. The bolted steel fish plate splice and the tabled steel fish plate splice are the neatest in appearance, and for this reason are extensively used in exposed work.

14. Compression Splices.—Compression splices naturally divide into two divisions: (1) Those joints which take only uniform compression at all times, and (2) those joints which, while compression is the principal stress, may be called upon at some time to take either flexure, or tension, or a combination of both.

Some of the compression splices used in construction are shown in Fig. 19. These joints, in the order lettered, are (*a*) the butt joint, (*b*) the half lap, and (*c*) the oblique scarf.

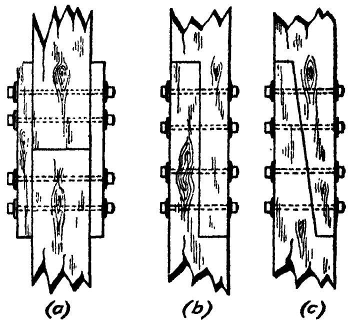


Fig. 19.—Compression splices.

The butt joint differs from all the other joints in that it has but one surface of contact. For this reason, it is superior to all the others, where uniform compression alone is to be transmitted. The efficiency of all the other joints depends wholly upon the skill and care of the carpenter who frames the joint. In other words the butt joint for the condition named is the simplest, and therefore the best. Indeed, the splice plates, if bolted, or bolted and keyed, may make the butt joint suitable for carrying both tension and flexure.

The oblique scarfed splice is stronger in flexure than the half lap. In the half lap joint, however, there is more timber in straight end bearing than in the oblique scarf.

In constructing compression joints in timbers which are vertical in position, the bolts through one end of the splice pads, if such exist, should be placed after the upper timber has come to a bearing on the lower timber; otherwise the bolts may receive a heavy load before the timbers come to a full bearing.

15. Connections between Joists and Girders.—When possible, joists should rest upon the tops of girders, and not frame into the sides of the girders. The former construction, however, involves a loss in head room in a building, increased height of building walls and columns. It also involves more shrinkage, since the shrinkage is directly proportional to the depth of timber. In the case of a building with masonry walls and timber interior, the construction of joists resting upon the girders will, with green or unseasoned timber, result in unequal settlement of the floors. The inner ends of the outer floor bays will settle the amount of shrinkage of joist plus girder, while the outer ends will settle only the amount of shrinkage of the joists, since the joists frame directly into the masonry. The considerations of equal settlement and gain in building height will usually

dictate the use of joist hangers in a building with heavy masonry walls. Hangers resting on top of girders will not reduce shrinkage effect.

In a building of the mill-building type with wall posts and girders, and corrugated steel or wooden sheathed walls, the increased height due to framing the joists on top of the girders will be offset by the saving in the cost of joist hangers.

The joists should extend over the full width of girder, and be toenailed into the girders. When the joists break over the girders they should lap at least 12 in. and be well spiked together. Solid bridging of a depth equal to the depth of the joists, and of a width not less than 2 in., is usually placed between the joists, and directly over the center of girder. Such bridging holds the joists firmly in position, and also acts as a fire stop. This construction is shown in Fig. 20.

15a. Joists Framed into Girders.—In very light construction the joists, when framed into the sides of a girder, are sometimes only toenailed. In other cases, especially when the joists frame into only one side of the girders, such

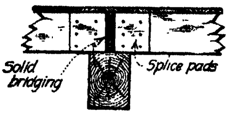


Fig. 20.

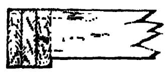


Fig. 21.

Fig. 22.

girder built up of several vertical pieces, the outer piece is spiked into the ends of the joists, as in Fig. 21. All such joints are makeshifts, and extremely unreliable. As has been pointed out in a previous article (see Art. 4), nails driven into the ends of timbers—i.e., parallel to the direction of fibers—have a low strength. Further, there is always the danger of the nails thus driven causing the joists to split.

Sometimes a strip is nailed or bolted to the sides of the girder, upon which the joists rest, as in Fig. 22. If properly designed, such strips will be not less than 4 in. wide and 4 in. deep, bolted, not nailed to the girder. The bolts should be sufficient in number to take the reaction of the joists, and should be not less than $2\frac{1}{2}$ in. from the bottom of girder.

Illustrative Problem.—Given a floorbay 14 × 16 ft.; live load of 60 lb. per sq. ft.; girders spanning the shorter side of the floor bay. Assume double thickness of flooring 1-in. *T* and *G* finished floor over 1-in. rough floor. Working fiber stress is flexure 1,600 lb. per sq. in.; working unit stress in longitudinal shear 150 lb. per sq. in; working unit stress in cross bearing 300 lb. per sq. in.

Weight of floor construction, exclusive of girders:

Flooring	6
Joists	5
Bridging	1
Total dead load	12
Live load	60
Total load	72 lb. per sq. ft.

With joists 16-in. centers, and counting the clear span for joists as 15 ft., the following figures result:

Total load on one joist = $(15)(1\frac{1}{3})(72) = 1{,}440$ lb.

Bending moment = $(\frac{1}{8})(1{,}440)(15)(12) = 32{,}400$ in.-lb.

Required section modulus = $\dfrac{32{,}400}{1{,}600} = 20.$

Assume joist 2 × 10 in., actual section $1\frac{5}{8} \times 9\frac{1}{2}$, actual section modulus 24.44.

For a 15-ft. span, this size is the minimum for deflection. In the computation for girder size, the live load may be reduced 20 per cent, making total load 60 lb. per sq. ft.

Load = (14)(16)(60) = 13,440 lb. $M = (\frac{1}{8})(13{,}440)(14)(12) = 282{,}000$ in.-lb.

$$\text{Required section modulus} = S = \frac{282{,}000}{1{,}600} = 176.$$

An 8- × 14-in., finished section $7\frac{1}{2} \times 13\frac{1}{2}$, has a section modulus of 227.8. An 8- × 12-in. girder, finished size $7\frac{1}{2} \times 11\frac{1}{2}$, would have a section modulus of 165 under the required amount. The reaction of one joist is 720 lb., requiring a bearing area of $\frac{720}{300}$ = 2.4 sq. in. The bolting strip will be 4 × 4 in. $\frac{5}{8}$-in. bolts will be used, and the working load per bolt will be taken at 900 lb.[1] Since the load per linear foot of girder is 16 × 60 = 960 lb., the bolts must be spaced $\frac{900}{960}$ (12) = 11-in. centers, or 13 bolts per girder.

In the above illustrative problem, the depth of joist plus the depth of bolting strip just equals the depth of girder. This relation does not always hold, as girder depth is often but little more than the depth of joist. To avoid having the bottom of joists lower than the girder, joists are often notched as shown in Fig. 23. Such construction is not good, since the strength of the joists is greatly reduced by notching. The joists tend to split in the corner of the notch, due to the difference in stiffness on either side of the vertical cut.

Fig. 23.

In some cases, the ends of the joists are framed with tenons fitting into sockets or recesses cut into the girder. This type of framing is to be condemned on account of the serious weakening of both joist and girder.

15b. Joist Hangers.—The most satisfactory manner of framing joists into the sides of girders is by the use of joist hangers. There are many stock types of these, among which may be named the Duplex, Van Dorn, Ideal, Lane, National, and Falls. Some of these different types are shown in Figs. 24 to 27 inclusive. A stock joist hanger should not be used without investigating carefully its strength and the amount of bearing given to the joist. Referring to the figures illustrating the different types, the fact should be noted that the Duplex hanger will result

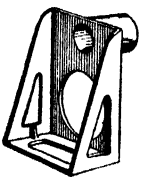


Fig. 24.—Duplex joist hanger.

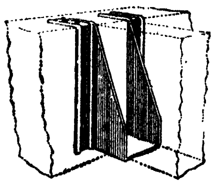


Fig. 25.—Van Dorn patented steel joist hanger.

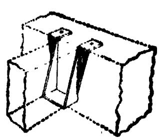


Fig. 26.—"Ideal" single hanger.

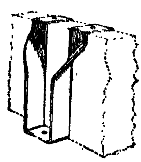


Fig. 27.—"Falls" joist hanger.

in less settlement of floor than any of the other types, since the connection of this hanger, unlike all the others, is on the side of the girder, and, hence, is affected by the shrinkage of one-half instead of the whole depth of girder. The published tests of joist hangers, as given in the various manufacturers' catalogs, will bear close scrutiny. Often in the effort to prove the merits of the particular hanger, the exact loads carried by one hanger are not always clear. Sometimes, also, hard-

[1] From Table 21, p. 410, $\frac{5}{8}$-in. bolt "double shear" with 4- and 8-in. timbers, good for 1,465 lb. in end bearing. For side bearing, safe load = $\frac{5}{8}$ × 1,465 = 915 lb.

wood is employed in the tests, in order to avoid failure of the joist by crushing of the fibers. The Duplex hanger unquestionably has many advantages over other hangers. It is practically certain that all the other hangers will fail by the hooks over the girder crushing the fibers of the timber on the corner of the girder and then straightening out.

15c. Connection of Joist to Steel Girder.—When steel girders are used with timber floor joists, the types of connection are similar to those discussed for wooden girders, i.e., the joists may frame on top of the steel girder (usually an I-beam) or into the side of the girder.

Buildings with this combination construction, in which the joists simply rest on top of the I-beams, without any attachment whatever, are sometimes seen.

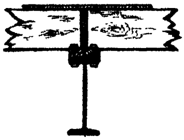


Fig. 28.

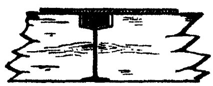


Fig. 29.

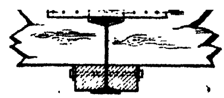


Fig. 30.

In such cases, the I-beam is supported laterally only by friction between the timber and steel. This practice is to be avoided. To secure a definite connection between the joists and girder, a wooden strip may be bolted to the top flange of the I-beam, and the joists toenailed to this wooden strip, as in Fig. 28. The principal objection to this construction is the weakening of the I-beams from the holes punched through the flange.

When the joists frame into the sides of the I-beams, they are often, for light loads, supported by the lower flanges of the I-beam, as in Fig. 29. Obviously the weak point of this detail is the small bearing of the joist on the steel. To overcome the difficulty, timbers may be cut to rest snugly against the flange and web, and bolted through the web. The joists may then be nailed into these timber

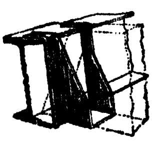


Fig. 31.—Van Dorn I-beam hanger.

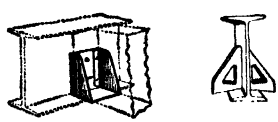


Fig. 32.—Duplex I-beam hanger.

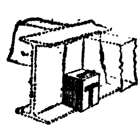


Fig. 33.—Duplex I-beam box.

strips, as illustrated in Fig. 30. The supporting timber should be of sufficient width to extend under and beyond the vertical cut of the notch in the joist for the upper flange.

A serious difficulty in constructions of this nature is the problem of supporting the flooring over the upper flange of the I-beam. If such flooring rests on the joists and the upper flange of the I-beam, the shrinkage of the joists will produce a high place in the floor over all the steel beams. To overcome this difficulty small strips, say of $1\frac{1}{2} \times 2$-in. timber, may be spiked to the sides of the joists to carry the floor over the girder.

Joist hangers, notably the Duplex and Van Dorn hangers, may be obtained for connection between timber joists and steel girders (see Figs. 31, 32, and 33).

The method of support shown in Fig. 30, however, will be found very satisfactory and generally cheaper than the joist hangers.

16. Connections between Columns and Girders.—The connection between timber columns and girders involves consideration, not only of strength of columns and of supports for the girders, but also of general stiffness of the building, since the posts and girders are generally counted upon to form the structural frames for resisting lateral forces, as wind and vibration of machinery. Columns always splice at or near the floor lines, hence the connection of girder to column includes the consideration of column splice. Continuity of the columns is always to be sought, both from the standpoint of stiffness and reduction of shrinkage. In total, the objects to be gained in the connection of girders and post are: (1) Continuity of column for stiffness and reduction of shrinkage; (2) reduction of column area from a lower story to an upper story as determined by floor load; (3) sufficient bearing area for girders on the supports; (4) continuity of girders at the column for stiffness; and (5) provision for girders releasing from column, in event of a serious fire, without pulling the column down. All these provisions are not attainable in every case, and the nature of the building may not warrant the expense of securing all these objects.

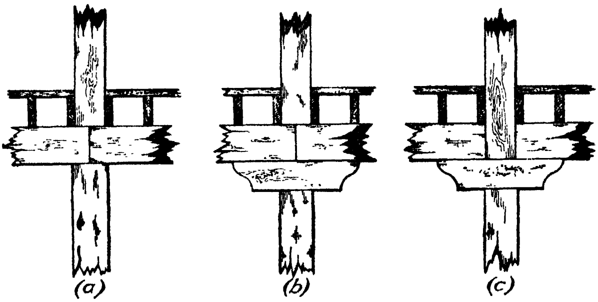


FIG. 34. Defective details of column and girder connections.

In the discussion of this subject, a distinction must be made between the ordinary building, including both frame buildings and buildings with masonry walls, or corrugated steel walls, and the special type of building known as "mill construction" or "slow-burning construction." The first class consists of those buildings which have the ordinary joist and girder construction, either with or without plastered ceilings and interior columns encased with lath and plaster. This class will be treated in the following paragraphs.

For the purpose of illustrating these principles, some details of connection of columns and girders will be briefly discussed. Figure 34 shows three defective details, which, nevertheless, are often seen. It is almost certain that in Fig. 34*a* the girders have not sufficient bearing across the fibers, and that with full load, crushing will result. In *b* the bottom of the upper post will crush the fibers of the upper side of the girder, and a worse condition will prevail under the bolster, unless the latter is hardwood. Even then, if the posts are not working at a very low unit stress, crushing of the bolster will result. The shrinkage in both *a* and *b* will be considerable, and nearly double in *b* what it will be in *a*. The

detail of c with the upper post resting on a hardwood bolster is the best of the three details, although shrinkage has not been eliminated.

For many buildings, the details shown in Fig. 35 will provide satisfactory connections. All of the desirable conditions enumerated previously are fulfilled, with the exception of release of girders in case of fire. The vertical bolster blocks are set into the lower post and bolted, or bolted and keyed to the sides of the column with circular pins or with the rectangular iron keys. In each of the three details, the girders may be given sufficient end bearing by properly proportioning the thickness of bolster block; the bolster has end bearing on the post, and no timber in cross bearing intervenes between the two sections of post. Partial continuity of post, sufficient for general stiffness of building, is secured by means of timber splice pads in detail c without sacrificing the girder ties. The splice plates of the girder across column may be of steel. This will avoid the use of wooden fillers under the girder splice pads. A further modification of these details to allow the girders to release in case of fire may be made by using dog-irons instead of the girder splice pads.

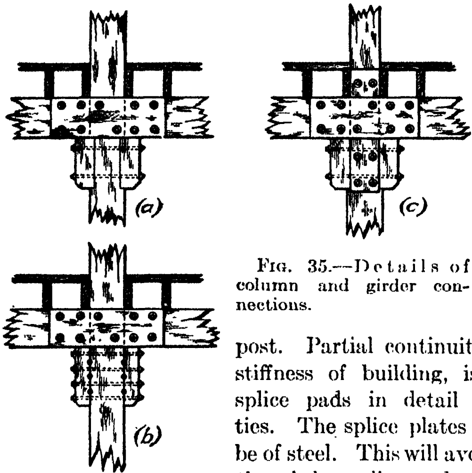


FIG. 35.—Details of column and girder connections.

The section of bolster is to be determined by requirements of girder bearing; the amount the bolster is set into the post by computations for end bearing; its length should be not less than 12 in., and preferably not less than 16 in. The size of bolts may be determined by taking moments about the center of the bearing on the post. The keyed and bolted bolster is proportioned as for the shear-pin tension splice.

Illustrative Problem.—Assume the problem of Art. **15a**. Floor bay 14 × 16 ft., girders 8 × 14 in., joists 2 × 10 in., first story height 16 ft. Assume the detail to occur at the second floor of a four story building. The load in the upper column will be taken at 30,500 lb., the first story column will then take 30,500 lb. plus the second floor load. The live load will be 60 per cent of 60 = 36 lb. per sq. ft., which, with a dead load of 12 lb. per sq. ft. will give a total unit load of 48 lb. per sq. ft., and a total increment of column load for the second floor of 10,800 lb. The first story column load will then be 41,300 lb. The upper column section will be made an 8- × 8-in., and the lower section a 10- × 10-in. The girder reaction is 6,720 lb. (For design of girder and its connections, live load is 80 per cent. (60) = 48 lb. per sq. ft.) At 300 lb. per sq. in. the required bearing and thickness of bolster must be 22.5/7.5 = 3 in. The bolster size will be made $5\frac{1}{2} \times 9\frac{1}{2} \times 1$ ft. 4 in.

The required area in end bearing is $\frac{6{,}720}{1{,}600} = 4.2$, or with a width of $9\frac{1}{2}$ in. the bolster must be set into the post 4.2/9.5 = 0.44 in. Actually the dap will be made $\frac{3}{4}$ in. The upper bolts will be placed 3 in. below bottom of girder. Taking moments about the center of bearing of the bolster on the dap, and neglecting the lower bolts, $M = (6{,}720)(2\frac{3}{4}) =$ 18,500 in.-lb. This overturning moment will be resisted by compression of the lower portion of the bolster against the post, and tension in the two upper bolts. This pair of bolts

is 13 in. above the seat of the bolster in the post, and the effective lever arm of these bolts may be taken at $\frac{3}{4}$ of their height above the bolster seat. The tension in either of the two bolts is then

$$p = \frac{18{,}500}{(2)(13)(\frac{3}{4})} = 950 \text{ lb.}$$

The maximum intensity of pressure between the bolster and post need not be investigated, as it will be very small with the length of bolster used.

Attention is called to the details of Fig. 35, in that the normal spacing of the joists has been modified at the posts, to bring a joist either side of the post. When these joists are either spiked or bolted to the post, and in addition a short piece of joist is spliced across the butt joint of the joists where such joint occurs at the post, a simple and inexpensive construction is secured which gives considerable stiffness to the building frame.

16a. Post and Girder Cap Connections.—The bolster connections above discussed are usually impractical to employ, if ceilings exist, as the bolster will project beneath the ceiling line. In such cases, and in other cases where the

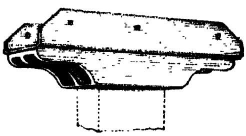


Fig. 36. Duplex malleable iron and steel combination cap.

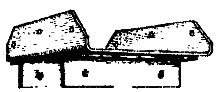


Fig. 37.—Ideal steel post cap, No. 3.

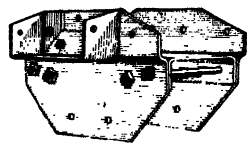


Fig. 38.—Duplex steel post cap.

above construction may be deemed unsightly, metal post-caps of cast iron, wrought iron, or steel are used. Standard post-caps, usually of pressed steel, are made by the manufacturers of joist hangers, and may be purchased in stock sizes. Typical details of girder and post connections, using standard post-caps, are given in Figs. 36, 37, and 38 taken from manufacturers' catalogs. The prices of these caps based on the unit cost per pound of steel are rather high, and it may often be possible to build up structural post-caps that will give satisfaction at a lower cost. Sometimes short pieces of I-beams or heavy channels, unsuited on account of length for any other purpose, may be purchased cheaply, and used for post-caps for cases in which it is only necessary to frame girders into two opposite sides of the posts; in other words, in the use of a two-way connection.

A four-way post-cap is one which provides for beams on four sides of the posts. Four-way post-caps with joist and girder construction always result in unequal settlement of the floor. The joists, being supported on or by the girders, will settle an amount equal to the shrinkage in the depth of the girder, while the joists framing into the post and resting on the post-cap will not settle. The use of joist hangers between joist and girder will not do away with this settlement, although the use of that type of hanger which connects into the approximate center of the girder will reduce the settlement to that due to the shrinkage of one-half the depth of girder.

Cast-iron post-caps must be carefully designed to take care of the flexural stresses. A typical cast-iron post-cap is shown in Fig. 39.

Illustrative Problem.—Assume girder 12 × 16 in. on a 14-ft. span, upper story post 12 × 12 in. and lower story post 14 × 14 in. The actual section of sixed girder will be

$11\frac{1}{2} \times 15\frac{1}{2}$. Using a working stress of 1,800 lb. per sq. in., the safe load is 39,469 lb., say 40,000. The reaction is then 20,000 lb. At 300 lb. per sq. in., the required bearing area is $\frac{20,000}{300} = 67$ sq. in. With a width of $11\frac{1}{2}$ in., the cap must have a seat $\frac{67}{11.5} = 5.8$ in. long, say 6 in., and will project 5 in. over the face of the 14- × 14-in. post. The moment on the post-cap may be assumed to be a maximum at the edge of the upper story post, with a value $M = (20,000)(3) = 60,000$ in.-lb. For cast iron, the working unit stress in flexure will be taken at 4,000 lb. per sq. in. The required section modulus of cap must therefore be $\frac{60,000}{4,000} = 15$. The sides of cap form two beams of rectangular section resisting this moment. Assuming a thickness of metal of 1 in., the depth of side must be $d = \sqrt{(7\frac{1}{2})(6)} = 6\frac{3}{4}$ in. The thickness of seat must now be computed. With a uniform bearing, the seat may be computed as a beam with fixed ends, or $M = (\frac{1}{12})(Wl)$; the pro-

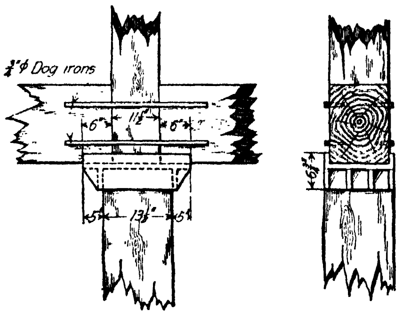


FIG. 39.—Details of column and girder connection with special cast-iron post cap.

jecting width of plate is 5 in. The load on this portion is $\frac{5}{6} \times 20,000 = 16,667$ lb. The length will be taken at $12\frac{1}{2}$ in., or between the centers of sides. Therefore $M = (\frac{1}{12})(16,667)(12\frac{1}{2}) = 17,360$ in.-lb. The section modulus required is $\frac{16,667}{4,000} = 4.34$. The width being 5 in., the depth must be $d = \sqrt{\frac{6}{5}(4.34)} = 2.28$ in. The base must therefore be supported by ribs. Two ribs will be introduced. The bearing plate will now be assumed to take only one-half of the bending, one-half the load being transmitted by the rib to the vertical collar around the post. The thickness of base and collar must then be sufficient for each to sustain 6,650 in.-lb. Since both the projecting seat and the collar are fixed along one edge, the allowable unit stress in bending will be increased 50 per cent.

$$\frac{8,333}{6,000}$$

thickness is 1.29. A thickness of $1\frac{1}{4}$ in. will be used.

SECTION 6

DESIGN OF REINFORCED CONCRETE MEMBERS

BY ARTHUR R. LORD AND W. STUART TAIT

Reinforced concrete members are sharply distinguished from wooden or steel members in that, in the typical concrete structure, the various members are necessarily built monolithically and react upon each other. The determination of the proper external moment acting upon any member is therefore more difficult in reinforced concrete than in any other type. In this volume, however, we assume in treating of the design and detailing of reinforced concrete members that the loads, reactions, moments and all restraints acting upon each member are perfectly known and that our problem is limited to the proper design of members under such fully determined conditions. In practical designing work the designer must commonly determine the precise conditions governing each member as a part in a monolithic structure. This much more intricate problem is treated in the volume entited "Reinforced Concrete and Masonry Structures."

Even the design of the fully conditioned member of reinforced concrete is more complicated than that of any member of homogeneous material. The combination of two unlike materials in a single member makes necessary several computations in design which are not required with wooden or steel members. The use of proper diagrams and tables will greatly facilitate the computations and may be said to be practically indispensable. The designer should, however, become thoroughly familiar with the design of reinforced concrete members from the formulas alone and in the examples in this section we have frequently shown the complete design as well as the abbreviated design in which tables and diagrams are used to save work. A few of the more fundamental tables and diagrams in continual use are given in this volume and these have been selected or prepared with especial care. Diagrams involving several consecutive computations (and in which several intermediate scales are necessarily suppressed) have been avoided. In practical design several simple diagrams, used in conjunction with a design summary sheet, have been found much preferable to the complex diagrams. In actual structures members of one kind will commonly be so nearly of a size that certain design steps will be identical and the use of the simple diagrams as compared with the more complex reduces the labor and reduces as well the danger of making mistakes.

The principal reinforced concrete members, the elements, as it were, of which reinforced concrete structures are composed, are the following:

(*a*) The rectangular beam in flexure (of which the solid one-way slab is a special case) resting upon aligned supports and reinforced for tension only.

(*b*) The T-beam in flexure (of which the ribbed slab is a special case) resting upon aligned supports and reinforced for tension only.

(c) Same as (a) or (b) but reinforced for compression as well as tension.

(d) The slab in flexure supported along its four edges and reinforced in two directions.

(e) The flat slab in flexure supported directly upon columns.

(f) Members subject to direct axial compression.

(g) Members subject to direct axial tension.

(h) Members subject to direct stress and flexure combined.

(i) Balanced cantilever slabs, or footings.

Each of these classes of members will be treated separately with the necessary information for design given in compact form together with some examples of design computations.

RECTANGULAR BEAMS AND ONE WAY SOLID SLABS

A rectangular beam as a structural member may be a separate beam simply supported or partially fixed at its ends, a single span of a continuous beam, a single span in a rigid frame structure, or a cantilever projection. In order to design the beam completely, the moment and shear must be known or readily obtainable for any section along its length.

1. Formulas. – For this case the commonly used *straight-line* formulas have already been developed in Sect. 1 in the chapter on "Simple and Cantilever Beams." The symbols used in these formulas are given in Appendix A with the following exception:

K = factor expressing the ratio of the resisting moment of a beam or slab to bd^2. If followed by the subscript c it is based upon the resisting moment as limited by the compression in the concrete. If followed by the subscript s it is based upon the resisting moment as limited by the steel in tension. Without subscript, it commonly represents the condition of balanced reinforcement when $K_c = K_s$. (Subscripts for M have the same significance as noted above.)

The ormulas, as developed in Sec. 1, or deduced therefrom, are as follows:

$$k = \frac{1}{1 + \dfrac{f_s}{nf_c}} \tag{1}$$

$$f_c = \frac{f_s k}{n(1-k)} \tag{2}$$

$$f_s = \frac{nf_c(1-k)}{k} \tag{3}$$

Formulas (1), (2) and (3) apply to any type of reinforced concrete beam in which the values of f_c, f_s and n are known since they depend solely upon the straight line theory of stress variation. They are derived from the similar triangles of Fig. 1.

The remaining formulas given below apply to rectangular beams with tensile reinforcement only.

Position of the neutral axis

$$k = \sqrt{2pn + (pn)^2} - pn \tag{4}$$

Arm of resisting couple

$$j = 1 - \frac{k}{3} \tag{5}$$

Steel ratio for balanced reinforcement (when both f_c and f_s are used at their full allowable values).

$$p = \frac{0.5}{\frac{f_s}{f_c}\left(\frac{f_s}{nf_c} + 1\right)} \tag{6}$$

Steel ratio in general

$$p = \frac{A_s}{bd} = \frac{f_c k}{2f_s} \tag{7}$$

For over-reinforced beams

$$M = M_c = 0.5 f_c k j b d^2 \tag{8}$$

$$K = K_c = \frac{M_c}{bd^2} = 0.5\, f_c k j \quad (9) \qquad bd^2 = \frac{2M}{f_c k j} = \frac{M}{K_c} \quad (10)$$

For under-reinforced beams

$$M = M_s = p f_s j b d^2 = f_s A_s j d \tag{11}$$

$$K = K_s = \frac{M_s}{bd^2} = p f_s j \quad (12) \qquad bd^2 = \frac{M}{p f_s j} = \frac{M}{K_s} \quad (13)$$

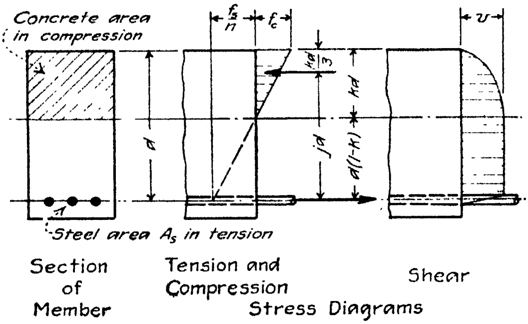


FIG. 1.—Stress distribution in rectangular beams.

Compressive unit stress in extreme fiber of concrete

$$f_c = \frac{2M}{kjbd^2} = \frac{2pf_s}{k} \tag{14}$$

Tensile unit stress in longitudinal reinforcement

$$f_s = \frac{M}{A_s jd} = \frac{M}{pjbd^2} \tag{15}$$

Area of tensile reinforcement

$$A_s = \frac{M}{f_s jd} = pbd \tag{16}$$

Bond unit stress on longitudinal bars

$$u = \frac{V}{\Sigma o j d} \tag{17}$$

Shearing unit stress (as a measure of diagonal tension)

$$v = \frac{V}{bjd} \qquad (18) \qquad\qquad bd = \frac{V}{jv} \qquad (19)$$

Tensile unit stress in web reinforcement consisting of series of bars

$$f_v = \frac{V'a}{A_vjd} = \frac{V's \sin \alpha}{A_vjd} \qquad (20) \qquad\qquad A_v = \frac{V'a}{f_vjd} = \frac{V's \sin \alpha}{f_vjd} \qquad (21)$$

Tensile unit stress in web bars bent up in single plane

$$f_v = \frac{V'}{A_v \sin \alpha} \qquad (22) \qquad\qquad A_v = \frac{V'}{f_v \sin \alpha} \qquad (23)$$

In Fig. 1 the distribution of tensile, compressive and shearing stress over the cross-section of a rectangular beam is indicated and some of the symbols used in the above formulas illustrated. Table 1, p. 466, gives values of p, K, k, j, kj, $\frac{f_s}{nf_c}$ and $\frac{k}{n(1-k)}$ for balanced reinforcement with various combinations of f_c and f_s and considering that $n = 15$, 12 and 10. Diagram 1 gives values of K, k, j, and p for various values of f_c and f_s, considering that $n = 15$.

2. Span Length.—The span length is commonly considered as the *clear* span between supports when the beam is cast monolithic with substantial supports, or the center to center distance between supports (but not to exceed the clear span plus the depth of the beam) when the beam is not monolithic with its supports. Span lengths are somewhat arbitrarily fixed by city ordinances and vary considerably, but the best practice is given in the 1921 Joint Committee report.

3. Shear and Moment Diagrams.—Where the load on a beam is other than uniformly distributed throughout its length, it is always desirable to make a load diagram, and the graphical representation (to scale) of the shear and moment at all sections expedites design and aids in securing accurate results. Figure 2, which combines the load, shear, and moment diagrams for a single beam, is an example. Certain relations should be noted in this figure. The algebraic sum of the loads on either side of any section x-x (or between any two sections) gives the change in the ordinate to the shear diagram between the reaction and the section (or between the two sections). Also, the algebraic sum of the areas under the shear curve between any two sections is equal to the change in ordinate to the moment diagram between the two sections. From these relations the shear diagram is readily constructed by summing up the loads across the beam, and in like manner the moment diagram is easily constructed by summing up areas under the shear diagram across the beam. The reactions (the end ordinates of the shear diagram) and the moments at the beam ends (the end ordinates of the moment diagram) depend upon external forces and restraints and must be known as a condition prerequisite to design of the member. As an aid to design, it should be noted also that the moment passes through a maximum or minimum value at sections where the shear passes through zero. The moment diagram locates accurately the points of inflection ($M = 0$) and since the steel area A_s is proportional to the moment, it indicates the limiting points for bending or terminating bars. The shear diagram indicates the distance over which web reinforce-

ment may be required, and a simple graphical construction for locating the spacing and position of stirrups may be superimposed upon either the shear or moment diagram.

For the most common case of all—namely, that of beams under uniformly distributed load—the load-shear-moment graph is not generally drawn, since the

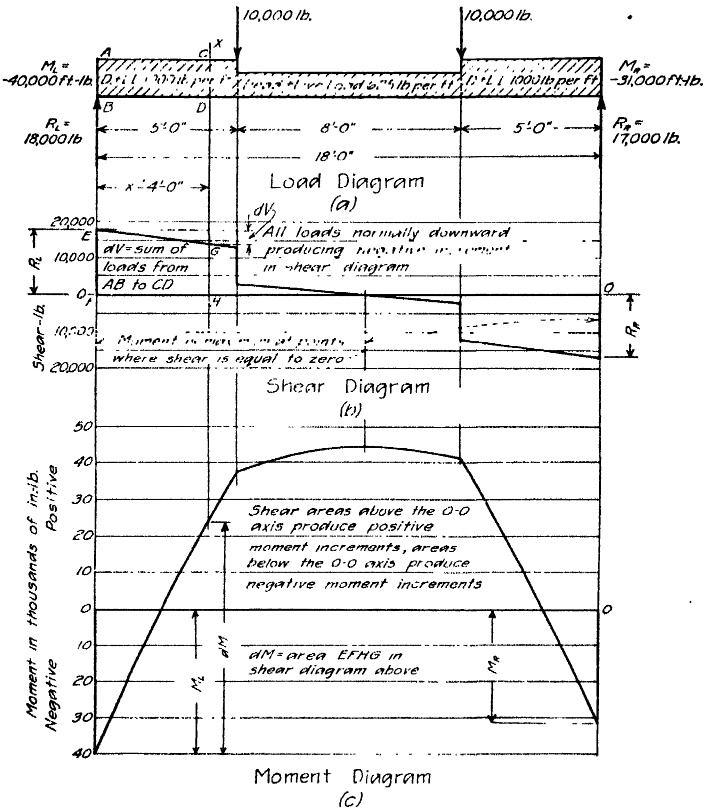


FIG. 2.—Load-shear-moment graph.

design formulas for this case are simple and the limits have been tabulated and diagrams prepared that facilitate design to an even greater degree.

4. Selection of Working Stresses.—The working stresses to be used in design are determined by the building ordinances where the building is to be erected, by the owner's specifications, or by the designer's judgment, as the case may be. Commonly used standards are given in the accompanying table.

DESIGN STRESSES BY VARIOUS SPECIFICATIONS

	Working stresses permitted			
	A. C. I.	Chicago	New York	1921 J. C.[4]
f_c—ordinary fiber stress	$0.375f'_c$	$0.35\ f'_c$	$0.325f'_c$	$0.40f'_c$
f_c—adjoining supports of continuous beams	$0.41\ f'_c$	$0.41\ f'_c$	$0.375f'_c$	$0.45f'_c$
f_c—direct compression	$0.25\ f'_c$[1]	$0.02\ f'_c$[1]	$0.025f'_c$	$0.02f'_c$[2]
u—bond unit stress:				
plain bars	$0.04\ f'_c$	$0.035f'_c$	$0.04\ f'_c$	$0.04f'_c$
deformed bars	$0.05\ f'_c$	$0.05\ f'_c$	$0.05\ f'_c$	$0.05f'_c$
v—shearing unit stress:				
no web steel (long. not anchored.)	$0.02\ f'_c$	$0.02\ f'_c$	$0.02\ f'_c$	$0.02f'_c$
no web steel (long. anchored.)	$0.03\ f'_c$	....	....	$0.03f'_c$
limit with web reinf. and long. anchored	$0.12\ f'_c$	$0.067f'_c$	$0.075f'_c$[3]	$0.12f'_c$
f_s—unit stress in steel:				
structural grade	16,000	16,000	16,000	16,000
intermediate grade	16,000	16,000	16,000	18,000
hard grade	18,000	18,000	16,000	18,000
cold drawn wire	18,000	18,000	20,000	18,000

[1] A. C. I. and Chicago allow 50 per cent increase in stress in case not over 50 per cent of total area is under load.

[2] For J. C. allowance on partially loaded columns, see Formula (80), p. 499.

[3] With web reinforcement to take entire shear.

[4] The somewhat higher stresses of the 1921 J. C. Report are based on test values of the strength of the concrete in compression and not on assumed strengths.

At the support of continuous or restrained beams an increase in the compressive unit stress is commonly permitted, for the amount of which various specifications should be consulted. This allowance is permitted because the extra high stress intensity occurs over an extremely short length of beam only, due to the very rapid decrease in the negative moment at the ends of beams. Under such conditions the less highly stressed concrete on either side restrains these fibers and enables them to carry much higher stresses than they could normally sustain.

5. Assumption of Beam Weight.—In designing a rectangular reinforced concrete beam the size and hence the weight of the beam itself is always an unknown quantity and must first be assumed. With a little experience a very close guess can be made. The depth of the beam will ordinarily be limited by architectural considerations. The width of a rectangular beam is ordinarily made from ½ to ¾ of d. The assumed dead weight of member must be added to the given loads, its reactions to the given reactions, and its probable end moments to those given.

6. Steps to be Taken in Design.—The remaining steps in the design of a rectangular beam are as follows:

(*a*) From the maximum end shear (= reaction) calculate the value of bd by Formula (19). It is generally advisable to use two values of v, covering a range within which the designer wishes the final value to lie.

(*b*) From the maximum moments, positive and negative, determine the value of bd^2 by Formula (10) taking K from Table 1, p. 466, for the stresses specified. Now study the values of bd and bd^2 secured above and select values of b and d to suit both. Check the assumed beam weight against this selection. It may be

necessary at this point to repeat the work thus far a number of times to obtain consistent results, but experience will soon reduce this to a single trial.

(*c*) Find the area A_s of tensile steel required for positive and negative moment, using Formula (16). Either part of the formula may be used, taking the value of j or of p from Table 1. j is very commonly taken as $\frac{7}{8}$ for rectangular beams and slabs, without material error. The number and size of bars may be taken from Tables 5, 6, and 7, pp. 472, and 473, respectively.

(*d*) Compute the bond unit stress by Formula (17). The two critical sections will commonly be: (1) At the face of the support for the tension steel at the top of the beam, and (2) at the point of inflection for the tension steel at the bottom of the beam. If the bond stress is too high it may be decreased by the use of smaller rods or the allowable bond increased by hooking the ends of the rods.

(*e*) Compute the shearing unit stress and design the web reinforcement. This is discussed in Art. **7**.

(*f*) Determine the total depth of the beam by adding in the necessary thickness of protective concrete. This is discussed in Art. **8**.

7. Design of Web Reinforcement.—The most economical web reinforcement consists of that portion of the longitudinal steel which lies in the web in passing from the upper to the lower plane. Where a considerable number of bars are bent, a very effective system of web reinforcement may be provided by proper detailing. The stress in this steel may be computed by Formula (20) or (22), depending upon whether the rods are bent in several sets or all in one plane. To resist the negative moment properly this steel must ordinarily reach the upper level at some distance out from the face of the support (see Fig. 7, p. 455, and Art. **25**), and this leaves a portion of the beam immediately adjacent to the support without web reinforcement. There may also be another portion, if shearing stresses are high, between the bent bars and the center that will need web reinforcement. To cover these portions, vertical or inclined stirrups are usually introduced, the stress being computed by Formula (20). Inclined stirrups and bent bars are more logical web reinforcement than vertical stirrups, since they take tensile stress from the first loading of the beam, whereas vertical stirrups become effective only when the concrete begins to show fine tensile cracks. In other words, vertical stirrups only jump to the rescue when the concrete starts to fail in diagonal tension. However, they do effectively prevent that failure and hold the cracks to much the same minute size as the tension cracks which are similarly restrained by the longitudinal reinforcement. Vertical stirrups have the preference in the field over inclined stirrups on account of the ease with which they may be placed and the support they afford the longitudinal steel before the concrete is placed. Tests show that the ultimate strength of beams is not affected by the direction of the stirrups within the limits of 45 deg. and somewhat beyond 90 deg. to the longitudinal steel. The deflection will be somewhat greater with vertical stirrups.[1]

The shearing unit stress permitted on the plain concrete, v_c, is commonly specified as $0.02f'_c$ to $0.025f'_c$ (but not to exceed 40 to 50 lb. per sq. in.). The total shear carried by the concrete at any section will be v_cbjd, while that remaining to be taken up by the web reinforcement will be $V' = V - v_cbjd$. Some specifications call for two-thirds of the total shear to be taken by the web rein-

[1] Vertical stirrups are of importance in design for torsion also.

forcement which would make $V' = \frac{2}{3}V$. The design of stirrups by formulas is laborious, and graphical methods are usually resorted to. For the case of uniformly distributed load, Diagram 5, pp. 476 and 477, gives the number and spacing of stirrups, either vertical or inclined, with a minimum of effort. This diagram is based on a working stress in the concrete, v_c, of 40 lb. per sq. in.

The total number of stirrups required to reinforce the web of any beam is a function of the area of its shear diagram and hence also a function of the ordinates

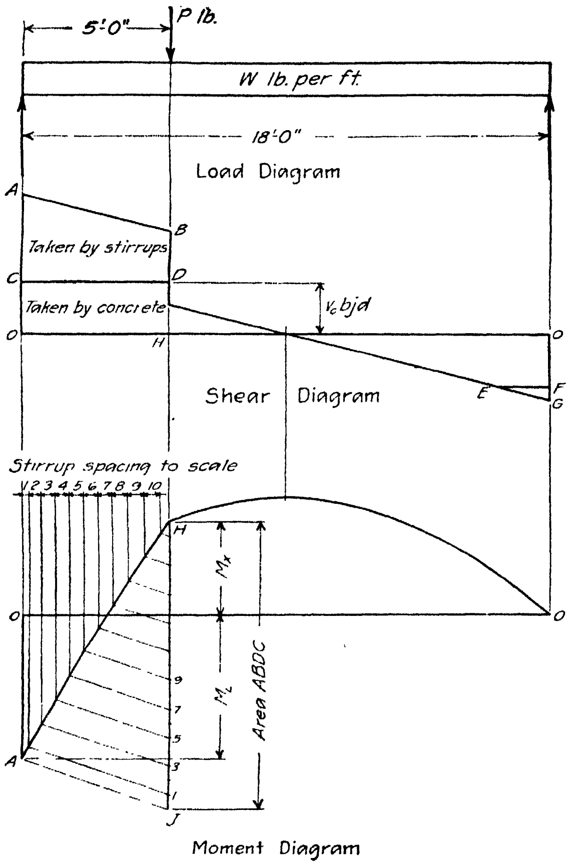


FIG. 3.—Graphical solution for stirrups.

to the moment diagram. In Fig. 3 a load-shear-moment graph for a beam has been drawn and on the shear diagram that portion of the shear accounted for by the concrete at a unit stress v_c is shown. At the left end there remains an area $ABDC$ which must be carried on web steel. By Formula (20) the area $V's$ carried by a single stirrup is $V's = \frac{f_v A_v jd}{\sin \alpha}$. The total number of stirrups required

at the left-hand end will therefore be $\frac{\text{Area } ABDC}{f_vA_vjd} \sin \alpha$. For vertical stirrups, $\sin \alpha$ becomes one and drops out. The number at the right end is found in the same way from the triangular area FFG. Since the area $V's$ will commonly be expressed in inch pounds, the distances along the shear diagrams must be reduced to inches to avoid confusion. The stirrup section A_v will represent the area of two legs in the common U-type of stirrup. On the moment diagram the area of the shear curve $ABHO$ is the sum of the ordinates M_x and M_L. The area $CDHO = (v_cbjd)x$, where x is the length of beam at that end requiring web reinforcement. The area $ABDC$ is the difference, and the formula for the total number of stirrups at either end may be written

$$N_s = \frac{(M_x) \pm (M_L) - (v_cbjd)x}{f_vA_vjd} \sin \alpha \tag{24}$$

The $\pm$ sign must be so selected as to give an arithmetical sum of the moments.

The spacing of stirrups can be determined graphically from the moment diagram. This method of determining stirrup spacing is exact when the stress v_c on the concrete is taken as one-third or any other fixed proportion of the total shearing unit stress. For the usual assumption of a fixed value of v_c independent of the shearing unit stress on the member, the results are approximate but the error is small. For concentrated loads, the error becomes negligible, and it is for this case that this method is chiefly used, since the case of uniform load is more easily designed from Diagram 5, pp. 476 and 477. To determine the stirrup spacing from the moment diagram, compute the area $ABDC$ and plot it to any scale, as the line HJ, with H in the moment curve, and draw the closing line AJ. Now divide HJ into a number of equal parts twice as great as the number of stirrups required at the left end of the member. From the first, third, fifth, etc., of these points draw lines parallel to AJ and their intersection with the moment curve locates the position of the stirrups which must be scaled parallel to the axis of the beam. The moment curve must be plotted accurately to permit of this determination. The number of stirrups found above is the minimum that may be used, except that certain ones may be omitted in portions of the beam fully reinforced against shear by bent-up longitudinal bars. Additional stirrups are often required to comply with regulations as to maximum spacing, etc.

Some designers strongly favor the use of bent-up bars exclusively as web reinforcement and such an arrangement is welcome in the field. Generally this would involve the addition of at least one bar to the steel area over the support, since a single bar can hardly be placed so as to be effective in tension and at the same time properly reinforce the end of the beam against diagonal tension. It is somewhat difficult to specify the right position for this first or end shear bar on account of lack of adequate test data on this type.

The 1921 J. C. specification is perhaps the best at the present time for bars bent up in a single plane. This specification provides that the upper point of the bend shall lie over the face of the support or over the support itself, while the lower part of the bend shall lie at that section on which the unit stress v_c equals the limiting value for plain concrete. These two points thus determined, the angle α is fixed, and the area of the sloping steel is computed by Formula (23).

The design of web reinforcement consisting of bent-up longitudinal bars is commonly made as follows: Determine the maximum shear at the face of the

support and also the distance from the face of the support to a vertical section on which the shearing unit stress, v, by Formula (18) does not exceed the allowable value for plain concrete webs. This distance is the "run" of the first bent bar and the "rise" is known from the beam depth and the embedment of the bar top and bottom. Determine the ratio of run to rise from the upper part of Diagram 4, p. 475, or otherwise. With this ratio enter the lower part of Diagram 4 and read the total shear carried by the available bar (or bars) at this slope, based on Formula (23). If $V - (v_c bjd)$ exceeds the allowable shear for the available bars, as just determined from Diagram 4, additional bars must be bent up or other web reinforcement provided to take the remaining shear. If other inclined bars are added, the distance from the face of the support to the section where the shear is fully provided for by the web steel and concrete already designed, becomes the new "run" and the process is repeated to select the proper size of bar at the new slope that will make up the deficiency in the total shearing resistance. If stirrups are added, they may be figured by Formula (21) taking the shear on the stirrups V' as the deficiency at the stirrup location. Stirrups should provide web reinforcement over the length between the support and the section where the deficiency in the bent-up bars disappears.

Diagram 7, p. 479, may be used for determining the length of the sloping portion of bent bars.

8. Depth of Concrete below Horizontal Reinforcement.—The thickness of the protective coating of concrete over the reinforcement varies for different purposes. The minimum requirement is that to prevent the admission of moisture to cause corrosion and this depends upon the size of reinforcement used. With large bars and with large percentages of steel, cracks become larger. For slabs, therefore, a protective cover of ½ to ¾ in. clear over the bar is ample while for beams with large concentrations of heavy bars 1½ in. is the minimum. Very commonly some degree of fire protection is also desired and for ordinary exposures the cover over slab bars should be not less than ¾ in., and over girder bars not less than 1½ in. For severe fire exposures these figures should be increased to 1 and 2 in. respectively. Furthermore, certain aggregates[1] used in concrete manufacture go to pieces under rapid temperature changes. Proper fire protection where such aggregates (granite is a good example) are used involves the addition of an extra inch of cover reinforced with a light mesh placed 1 in. under the surface to prevent spalling. It is not necessary to make any deduction in the width of beam, b, for fireproofing, but the bars should be kept away from the surface at least 1½ in.

For concrete members exposed to the weather at all times not less than 1½ in. of cover is required on walls and slabs and not less than 2 in. on beams and columns.

9. Designing of Solid One-way Slabs.—Solid one-way slabs are designed in exactly the same manner as beams, a strip of slab 1 ft. wide (b = 12 in.) being treated as a beam. The steps are identical with those given for beams, except that A_s represents the steel area in a 12 in. width of slab due to a certain size of bar at a certain spacing, instead of the steel area due to a certain number of bars. If A_1 is the area of one bar, the spacing in inches to give any total area A_s in a

[1] Quartz-bearing minerals especially, also sandstone.

width of 12 in. will be $\frac{12A_1}{A_s}$. The minimum protective cover of ½ in. of concrete should be used only when positive devices are specified to insure that this clear cover is obtained.

10. Bar Spacing and Supporting Devices—Bar Sizes.—The specification of adequate spacing and supporting devices to hold the reinforcing bars in their designed position while the concrete is being placed is properly a part of the design and should be shown on the details.

Present day practice has largely standardized ten bar sizes for use as concrete reinforcement and many of the largest steel bar manufacturers will furnish only these sizes. To avoid the necessity of substitution it is best to use only standard sizes in design and in the tables in this volume only such sizes are given—see Tables 5, 6 and 7. For column ties and stirrups an eleventh size, namely ¼″φ, is also readily obtainable.

11. Minimum Spacing of Bars in Beams.—The particular ordinance or specification under which the design is made will invariably contain many limitations on the details of the design and these vary widely in different cases. Some city ordinances have not been revised adequately for periods of 10 to 20 years. The practice of the building department is often more liberal than the letter of the code. The minimum spacing of bars in each layer is commonly specified. This is governed properly by the size of the aggregate, the size of the bars, and the bond stresses. The aggregate must pass readily between the bars and not form pockets to destroy the possibility of bond development. This establishes a minimum clear spacing generally specified as 1¼ times the largest aggregate used, but never less than 1 in., and never less than 1½ times the greatest cross-sectional dimension of the bars. Beyond these minimums the concrete between the bars must have sufficient area to transmit, in horizontal shear, the increment (or decrement) in the tension received from the bars through bond stress.

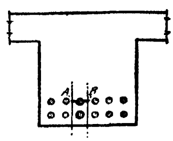


Fig. 4.—Spacing of longitudinal bars.

In Fig. 4 the concrete on the section AB (of unit length) must be able to resist a horizontal shear equal to the sum of the bond stresses on the perimeter of the lower bar and the semi-perimeter of the upper bar. This consideration is the basis of the allowable clear spacing worked out in the accompanying table, in which the rapid increase in the spacing with many layers of bars is well shown.

Clear Spacing Between Bars of Size Shown—(Inches)

Layer	⅝ in. φ or less	¾ in. φ	⅞ in. φ	1 in. φ	1 in. □	1⅛ in. □	1¼ in. □
Outer	1.00	1.12	1.31	1.50	2.12	2.39	2.66
2nd	1.48	1.77	2.06	2.36	3.00	3.38	3.75
3rd	2.46	2.94	3.44	3.93	5.00	5.62	6.25
4th	3.43	4.11	4.80	5.50	7.00	7.86	8.75

The above values are based upon a bond unit stress of 100 lb. per sq. in. and upon a horizontal shearing unit stress of 200 lb. per sq. in. If the actual maximum bond unit stress is greater or less than 100 lb. per sq. in., the clear spacing will vary in proportion, except that the minimum spacing based on size of aggregate and on size of bar must be maintained in all cases. This table contemplates the use of ¾ in. sone as the maximum size.

The bending up of some of the bars before the point of inflection is reached will generally transfer the critical section for the lower steel from the point of inflection to the point of bending. Since the shear is much lower here than at the face of the support, the spacing may be less than the values in the table, as noted.

If both ends of the bars are hooked, the clear spacing may be reduced to the greatest cross-sectional dimension of the bars, the other limitations remaining as before. Bars, both ends of which are bent up and carried into the adjoining spans, may properly be considered as equal to bars with hooked ends in this respect, insofar as the portion of the bar taking positive moment is concerned.

For rectangular beams the critical section as to bar spacing is at the face of the support for the steel taking negative moment and at the point of inflection for the steel taking positive moment. If the top of the beam at the support is monolithic with the floor slab, forming a T-beam, only the section at the point of inflection remains critical.

12. Bars Carried Through to Support.—The design sometimes results in a large number of bars for either positive or negative moment but not for both, so that some of the bars may be terminated short of the support. Every beam should have a number of bars carried straight through in the bottom to the support. The minimum is generally considered to be one-fourth of the bars required for positive moment at the center, but it is much more common to see one-half of the bars carried through in this manner. The bond at the point of inflection is the determining factor, formula (17) being the proper one to use to figure the number of bars required at the allowable bond stress. The length of these straight bottom bars is commonly taken as the distance center to center of supports.

13. Points Where Horizontal Reinforcement May be Bent.—Having determined the minimum number of bars which must continue in the bottom, the points at which bars may be bent up are determined from the moment diagrams remembering that the steel area required is directly proportional to the moment. By dividing the maximum positive moment ordinate into as many equal parts as there are bars required at the center and projecting these parts horizontally to the moment curve, the points at which successive bars may be bent is readily determined. A similar construction upon the negative moment ordinate shows where bars may be bent down and completely fixes the best bending of the bars. Diagram 6 gives the proportion of the total steel area which may be bent up or terminated (considering moment only) at any point on the span for several moment conditions. At least two bars and at least one-fourth of the total number of bars are commonly carried straight in the top from the support to the quarter point of the clear span (or to the point of inflection if farther from the support). When the reinforcement for negative moment cannot be carried into the adjoining span, in the usual manner, special anchorage must be provided for it at the end of the beam, generally taking the form of hooked ends with the diameter of the bend not less than eight times the diameter of the bar. In frames the bars are bent down into the columns.

14. Size of Web Reinforcement Bars.—Web reinforcement bars in general and vertical stirrups in particular are required to develop very high tensile stresses in very short embedded lengths. This means high bond unit stress and

severe anchorage requirements and requires the use of small rods in ordinary beam depths. It is a very common fault to use too large stirrups, and some of the objection to vertical stirrups is really objection to this abuse. A ½-in. vertical stirrup in a beam of 12 in. depth is no more effective than a ⅜-in. stirrup and the proper size to use in this beam would be ¼-in. The stirrup of large diameter simply slips. The 1921 J. C. report contains the rule, based upon tests, that the size of web bar should be such as to develop not less than 20 per cent of its total design tension by bond in a depth equal to $0.4d$. For a plain round vertical stirrup in 2,000-lb. concrete, this limits the maximum stirrup diameter to $d/50$, while a deformed bar could be $d/40$ in diameter, considering that f_v is taken as 16,000 lb. per sq. in. in both cases. Many designers advocate lower stresses in web reinforcement, even as low as 10,000 lb. per sq. in. in extreme cases. This requires much more web steel but it may properly be used in the form of somewhat larger bars. The use of 16,000 lb. per sq. in. is fully warranted by tests results, with the size limits stated. In Diagram 5, p. 477, at the bottom, the minimum values of d, for which stirrups of various size may properly be used, are stated.

15. Anchorage of Web Reinforcement Bars.—Each end of each web reinforcement bar should be provided with end anchorage in one of three ways:

(*a*) By continuity of the web bar with the longitudinal bar.

(*b*) By carrying the web bar about at least two sides (a 90-deg. angle of bend) of a longitudinal bar (the web bar continuing).

(*c*) By making a semi-circular hook of the end of the web bar, thus engaging an adequate portion of concrete to prevent its pulling out.

Using (*a*) the bends should be made to a radius not less than four bar diameters to avoid excessive stress in the concrete locally at points of bend. Using (*b*) or (*c*) the web bar should come as close to the top and bottom of the beam as the protective cover requirement will permit, and the end of the bar should be not nearer than eight times its own diameter from its high or low point. The radius of bend in (*c*) should be not less than four bar diameters. Figure 5 shows properly detailed stirrups for rectangular beams. It is not important in a rectangular beam which end of a stirrup, bent as shown, is uppermost if both ends are adequately anchored. Type "a" is the common form of vertical stirrup. Type "b" is required in some cities when the steel in the compression face is designed to take compression. Type "c" is more commonly used for inclined stirrups, while type "d," with the hooks turned outward, is the common stirrup for T-beams. Sharp bends should be avoided to prevent overstress in the concrete with resultant crushing locally at such points. Inclined stirrups, kept within the size limits given above, will not need to be mechanically attached to the longitudinal steel, as tests indicate that slipping occurs only with bars of larger size.

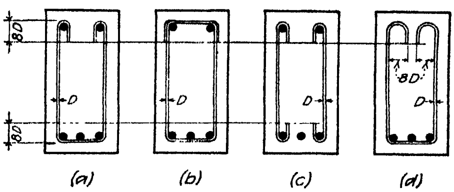


FIG. 5.—Stirrup bending details.

Some steel manufacturers have placed upon the market unit frames in which the stirrups are made of continuous wires or small rods, looped about the top and

bottom bars, and spaced to meet the design. These are excellent details but for some reason have not come into general use.

16. Rules for Stirrup Spacing.—The spacing of stirrups is governed by rules developed from test data which set limits of an arbitrary nature in addition to the limits arising from the diagonal tension on the member. The Joint Committee report of 1921 is representative of the best present-day practice. The spacing a (see notation, Appendix A) between inclined stirrups should not exceed $\frac{3}{4}d$ at sections where stirrups are required by the shear, while with vertical stirrups a maximum spacing of $\frac{1}{2}d$ is recommended. For very high shearing stresses the Joint Committee reduces these limits to $\frac{1}{2}d$ for inclined stirrups and to $\frac{1}{3}d$ for vertical stirrups. Common practice does not live up to these standards in general but allows about 50 per cent greater spacings. With very high shearing stresses, say in excess of 120 lb. per sq. in., no liberties should be taken with Joint Committee specifications. The maximum distance from the face of the support to the first stirrup should not exceed $\frac{1}{4}d$ (Joint Committee) measured on the axis of the member. Common practice doubles this allowance at times. Stirrups inclined less than 30 deg. to the vertical are classed as vertical stirrups in applying these rules.

17. Beam Sizes Influenced by Formwork.—The dimensions of concrete beams are frequently influenced in practical design by the formwork. The beam bottom form is made up of plank which must be cut to fit the beam width, unless this width is made to fit the plank. Widths of 5⅝ in., 7⅝ in., 9⅝ in. and 11⅝ in. permit the use of one-piece plank bottoms without stripping or filling, while widths of 13¼ in., 15¼ in., etc., are suitable for two-piece bottoms for wider beams. Adherence to such a program will also avoid the common error of making an immense number of different beam sizes. A reasonable number of different depths and widths of beams will be economical even though some concrete is used at less than its allowable compressive unit stress. Forms frequently cost more than either the steel or the concrete.

18. Camber in Forms.—Beams deflect due to shrinkage of the concrete in hardening and due to plastic (inelastic and hence permanent) deformation in the concrete under long continued loading, as well as due to the load on the beam at any given instant. Forms should therefore be cambered and the amount of camber should take into account all of the causes of deflection. Measurements of completed floors show that the usual allowances are too small. This may be said to be a matter for the contractor to handle rather than the designer, but it frequently happens that the designer is the only person connected with the job who is competent to compute the proper camber and that he should take it upon himself to see that it is done.

19. Construction Loads.—When designing members for very light loads, it is sometimes advisable to investigate the effect of such loading as is liable to come upon them during the construction of the building of which they are a part. The engineer must at times protect himself against the carelessness or ignorance of some man on the job who is looking for a space to tuck away a couple of carloads of gypsum or partition tile. The safe load, in some easily understandable units, should be plainly marked on the drawings at least.

20. Temperature Reinforcement.—Individual members are rarely of such length and proportions as to require temperature reinforcement and this subject

is treated in the volume on "Reinforced Concrete and Masonry Structures." A partial treatment will be found under the design of walls in Art. **63**, p. 504. As parts of a long building, members may require temperature design and the designer must not overlook this possibility.

Illustrative Problem.—To design a rectangular beam, freely supported, of 20-ft. span resting on 12-in. brick walls, carrying a uniformly-distributed load of 1,850 lb. per ft. The total depth of beam is limited to 36 in. A 2-in. protective cover over the principal reinforcement is to be provided. $f_c = 650$, $f_s = 16{,}000$, and $n = 15$.

For solution in detail see Design Sheet 1. The following notes refer to that design sheet: (*a*) Deductions for effective depth are 2 in. (protective cover) plus ½ in. (half of bar dimension); (*b*) instead of revising the moment computation above, the revised load is introduced in this way as a correction; (*c*) selected from Table 5, p. 472; (*d*) clear span is 20 ft. less 1 ft. (wall) = 19 ft.; (*e*) distance *a* in Diagram 5 is distance from the face of support to point where $v = v_c = 40$ lb. per sq. in.

Illustrative Problem.—To design a rectangular beam to carry two concentrated loads of 10,000 lb. each, applied 5 ft. from either end of 18-ft. clear span, and also to carry uniformly distributed loads amounting to 375 lb. per ft. between the concentrated loads and 750 lb. per ft. between the concentrated loads and ends of span. The left reaction from this loading is fixed (by adjoining span conditions) as 15,750 lb. and the right reaction as 14,750 lb. A bending moment of −33,250 ft.-lb. acts upon the left end of member and one of −24,250 ft.-lb. acts upon the right end. Use 1921 J. C. stresses, assuming a 2,000-lb. concrete.[1]

For solution see Design Sheet 2. The following notes apply to that sheet: (*a*) Based on usual assumption of $wL^2/12$; (*b*) weight of reinforced concrete is commonly assumed to be 150 lb. per cu. ft.; (*c*) spacing of bars taken from Table 6; (*d*) sign of moment is neglected—reinforcement is in top of member; (*e*) this steel area will be designed when steel available from adjoining span is known; (*f*) by inspection, bond is higher at right support and only one computation is necessary; (*g*) in evaluating Formula (24), p. 439, the length x is here given in feet because the moments are stated in foot-pounds, and the whole then multiplied by 12 to reduce to inch-pounds.

Illustrative Problem.—Check the design of a rectangular beam, as given below, and determine magnitude of all principal stresses, using 1921 J. C. specifications and assuming a 2,000-lb. concrete.[1] The beam is 12 in. wide and 24 in. deep, with 2 in. cover over the main bars. The reinforcement consists of four ¾-in. round deformed bars of medium grade steel of which two are bent up at each end. There are 8 ⅜-in. round vertical stirrups at each end spaced 4 in., 2 at 6 in., 2 at 8 in., 2 at 10 in., and 12 in. This is one of a series of spans, all alike, making up a continuous beam. The clear span is 18 ft. and the superimposed load 1,500 lb. per ft.

For solution see Design Sheet 3. The following notes apply to that sheet: (*a*) Some specifications require $wL^2/12$, but this is not a controlling item in this design; (*b*) this computation indicates that very low stirrup stresses have been used in this design—number of stirrups could be reduced; (*c*) $v = 72$ (approx.) at 1st stirrup less ($v_c =$) 40 leaves 32 lb. per sq. in., to be taken by stirrup.

Illustrative Problem.—To design a fully continuous floor slab for a superimposed load of 150 lb. per sq. ft. The span is 8 ft. Use A. C. I. stresses.[1] Assume a 2,000-lb. concrete and hard grade plain bars.

For solution see Design Sheet 4. The following notes apply to that sheet: (*a*) See discussion of critical section for bond stress under beams; (*b*) the ratio 10/12 is introduced in Formula (17) to give the shear for one straight bar, these bars being spaced 10 in. on centers in the usual arrangement where alternate bars are bent up; (*c*) compressive stress in the concrete and diagonal tension are practically never critical in solid slabs like this and are rarely figured.

[1] See table on p. 436.

DESIGN SHEET 1

Assume dead weight of beam as $450^{\#/'}$

Total dead and live load $= 450 + 1{,}850 = 2{,}300^{\#/'}$

$$M = \frac{wL^2}{8} = (2{,}300)(20)^2\left(\frac{12}{8}\right) = 1{,}380{,}000''\#$$

For $f_c = 650$, $f_s = 16{,}000$ and $n = 15$ (from Table 1)

$$p = 0.0077;\ j = 0.874,\ \text{say } \tfrac{7}{8};\ k = 0.379;\ \text{and } K = 107.5$$

By (19) $$bd = \frac{(2{,}300)(10)}{(80 \text{ to } 120)(\frac{7}{8})} = 219 \text{ to } 329''^2$$

At center, by (10) $$bd^2 = \frac{1{,}380{,}000}{107.5} = 12{,}830''^3$$

$d = 36 - 2\tfrac{1}{2} = 33.5''^{(a)}$. By (10) $b = \dfrac{12{,}830}{(33.5)^2} = 11.45''$, say $11\tfrac{5}{8}''$

Weight of beam $= (11\tfrac{5}{8})(36)\left(\dfrac{150}{144}\right) = 436^{\#/'}$. $D + LL = 2{,}286^{\#/'}$

At center, by (16) $A_s = \dfrac{2{,}286^{(b)}}{2{,}300} \cdot \dfrac{1{,}380{,}000}{(16{,}000)(\frac{7}{8})(33.5)} = 2.94^{\square''}$

$= 2\text{-}1''\square$ and $2\text{-}\tfrac{3}{4}\ \phi^{(c)}$

At face of support, the number of bars required in bottom,

by (17), $$\Sigma o = \frac{V}{ujd} = \frac{(2{,}286)(9.5)^{(d)}}{(80)(\frac{7}{8})(33.5)} = 9.26^{\square''}$$

Carry 2-1″□ and 1-¾″ϕ($\Sigma o = 10.35\square''$) straight through in bottom.
Bend up 1-¾″ϕ to top of member of each end.

By (18) at end $$v = \frac{(2{,}286)(9.5)}{(11.62)(\frac{7}{8})(33.5)} = 64^{\#/\square''}$$

From Diagram 4, $a^{(e)} = (0.19)(19)(12) = 43''$

$A_v = (0.34)(2.94) = 1.00^{\square''} = 5\text{-}\tfrac{3}{8}''\ \phi$ U-stirrups. Use 8.

Size of stirrups $= \dfrac{33.5}{50} = 0.67'' = \tfrac{5}{8}''\phi$ maximum

Stirrup spacing in terms of $a = 0.065a,\ 0.145a,\ 0.178a,\ 0.258a$
Stirrup spacing in inches $= 2\tfrac{1}{2}'',\ 6'',\ 7\tfrac{1}{2}'',\ 11''$
Total depth of beam $= 33.5 + \tfrac{1}{2} + 2 = 36''$

DESIGN SHEET 2

Assume dead weight of beam $= 250^{\#/'}$
Correction for dead weight.

Item	Superimposed loads	Dead weight	Total
Left reaction	15,750	2,250	18,000
Right reaction	14,750	2,250	17,000
Left moment	−33,250	−6,750(*a*)	−40,000
Right moment	−24,250	−6,750(*a*)	−31,000

Figure 2 is the load-shear-moment graph for this member. From this figure maximum moment at center is 44,700′#.

For $f_s = 18{,}000$, $f_c = 800$ and $n = 15$ (from Table 1)

$$p = 0.0089;\ j = 0.867;\ k = 0.4;\ \text{and } K = 138.7$$

By (19) $\qquad bd = \dfrac{18{,}000}{(80 \text{ to } 120)0.867} = 173 \text{ to } 259''^2$

At center, by (10) $\qquad bd^2 = \dfrac{(44{,}700)(12)}{138.7} = 3{,}870$

Try $b = 11\tfrac{5}{8}''$. By (10) $d^2 = \dfrac{3{,}870}{11.62} = 332 \qquad d = 18.25''$, say 19″

$bd = 11.62 \times 19 = 221$. O.K. by (19) above.

Beam weight $= (11.62)(21)\left(\dfrac{150}{144}\right)^{(b)} = 254^{\#/'}$. Assumption O.K.

At center, by (16) $\qquad A_s = \dfrac{(44{,}700)(12)}{(18{,}000)(0.867)(19)} = 1.81''^2 = 6-\tfrac{5}{8}''\phi$ **bars.**

Width of beam $= (5 \times 1\tfrac{5}{8})^{(c)} + \tfrac{5}{8} + 3 = 11\tfrac{3}{4}''$ $\quad 11\tfrac{5}{8}''$ O.K.

At left end, by (16), $\qquad A_s = \dfrac{(40{,}000)(12)}{(18{,}000)(0.867)(19)} = 1.63''^{2(e)}$

At right end, by (16) $\qquad A_s = \dfrac{(31{,}000)^{(d)}(12)}{(18{,}000)(0.867)(19)} = 1.26''^{2(e)} = 4-\tfrac{5}{8}''\phi$ bars

At face of right support$^{(f)}$, by (17), $\qquad u = \dfrac{17{,}000}{4(1.96)\,(0.867)\,(19)} = 132^{\#/\square''}$

At left point of inflection, by (17), $\qquad u = \dfrac{16{,}200}{3(1.96)(0.867)\,(19)} = 168^{\#/\square''}$

$u = 132$ is O.K. with deformed bars with hooked ends.
$u = 168$ is too high. Carry $4-\tfrac{5}{8}''\phi$ bars through in bottom, $u = 126$
This is O.K. with deformed bars with hooked ends.

At left end, by (18) $\qquad v = \dfrac{18{,}000}{(11.62)(0.867)(19)} = 94^{\#/\square''}$ O.K.

Max. stirrup size $= \tfrac{19}{50} = 0.38'' = \tfrac{3}{8}''\phi$ $A_v = 0.2208^{\square''}$ for $1-\tfrac{3}{8}''\phi$ U

No. of $\tfrac{3}{8}''\phi$ **U-stirrups at left end, by** $(24)^{(g)} =$

$$\frac{37{,}500 - (-40{,}000) - (40)(11.62)(0.867)(19)(5)}{(16{,}000)(0.867)(19)(0.2208)} \times 12 = 8$$

No. of $\tfrac{3}{8}''\phi$ **U-stirrups at right end, by (24)** $=$

$$\frac{41{,}500 - (-31{,}000) - (40)(11.62)(0.867)(19)(5)}{(16{,}000)(0.867)(19)(0.2208)} \times 12 = 7$$

Total depth of beams $= 19 + \tfrac{5}{16} + 1\tfrac{1}{2} = 20\tfrac{13}{16}''$, say 21″.

DESIGN SHEET 3

Live load per foot of span $= 1{,}500^{\#}$

Weight of beam $= 12 \times 24 \times \dfrac{150}{144} = \dfrac{300^{\#}}{1{,}800^{\#/\prime}}$

$$b = 12''\ d = 24 - 2 - \tfrac{3}{8} = 21.62''\ A_s = 4 \times 0.44 = 1.76^{\square''}$$

Moment at center $= \dfrac{wL^2(a)}{16} = \left(\dfrac{12}{16}\right)(1{,}800)(18)^2 = 438{,}000^{\prime\prime\#}$

Moment at ends $= \dfrac{wL^2}{12} = \left(\dfrac{12}{12}\right)(1{,}800)(18)^2 = 584{,}000^{\prime\prime\#}$

Approx. steel stress, by (15) $\quad f_s = \dfrac{584{,}000}{(1.76)(\tfrac{7}{8})(21.62)} = 17{,}500^{\#/\square''}$

At end, by (9) $\quad K = \dfrac{584{,}000}{(12)(21.62)^2} = 104$

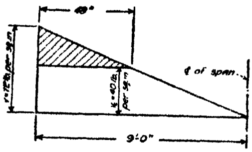
By (7) $\quad p = \dfrac{1.76}{(12)(21.62)} = 0.0068$

From Diagram 1, with $p = 0.0068$ and $K = 104$, $f_c = 650^{\#/\square''}$, $f_s = 17{,}400^{\#/\square''}$, $j = 0.88$.

At face of support by (17) $\quad u = \dfrac{(1{,}800)(9)}{(9.42)(0.88)(21.62)} = 90^{\#/\square''}$

At $\tfrac{4}{10}$ point of span by (17) $\quad u = \dfrac{(1{,}800)(0.4)(18)}{(4.71)(0.88)(21.62)} = 144^{\#/\square''}$

By (18) $\quad v = \dfrac{(1{,}800)(9)}{(12)(0.88)(21.62)} = 72$



By (24) and preceding text,

$$\text{Shaded area} = (32)(12)(0.88)(21.62)\left(\frac{48}{2}\right) = 175{,}000^{\prime\prime\#}$$

No. of $\tfrac{3}{8}''\ \phi$ U-stirrups $= \dfrac{175{,}000}{(16{,}000)(0.2208)(0.88)(21.62)} = 2.6$ each end.

By (20) $f_v = \dfrac{(v - v_c)bjds}{A_v jd} = \dfrac{(30)(12)(0.88)(21.62)(7)}{(0.2208)(0.88)(21.62)} = 11{,}400^{\#/\square''}$ in stirrup next to support.

DESIGN SHEET 4

For A. C. I. stresses, $f_c = 750$, $f_s = 18{,}000$ and $n = 15$, Table 1 gives $j = 0.872$, $k = 0.3846$ and $K = 125.7$.

Assume a 4-in. slab, weight $= 40^{\#}/\square'$

Superimposed load on slab $= \dfrac{150^{\#}/\square'}{200^{\#}/\square'}$

Moment at center (or supports) $= \dfrac{wL^2}{12} = \left(\dfrac{12}{12}\right)(200)(8)^2 = 12{,}800''^{\#}$

$b = 12''$. By (10) $d^2 = \dfrac{12{,}800}{(12)(125.7)} = 8.5''^2$ $d = 2.9''$ 4'' slab O.K.

Use $d = 3.05''$. By (16) $A_s = \dfrac{12{,}800}{(18{,}000)(\frac{7}{8})(3.05)} = 0.267\square''/'$

Spacing of $\frac{3}{8}''$ round bars $= \dfrac{(12)(0.1104)}{0.267} = 5''$ on centers

Same moment and same reinforcement required at supports.
Try bending up alternate rods and lapping to quarter points.

At $\frac{1}{10}$ point of clear span, by $(17)^{(a)}$ $u = \dfrac{(200)(0.4)(8)(10)^{(b)}}{(1.18)(\frac{7}{8})(3.05)(12)} = 169^{\#}/\square'$

This bond is higher than ordinances allow, but is commonly allowed with well anchored bars. To conform to the ordinances all the bars would be carried through in the bottom and additional $\frac{3}{8}''$ round bars at 5'' centers provided in the top of the slab over supports.

At face of support, by $(18)^{(c)}$ $v = \dfrac{(200)(4)}{(12)(\frac{7}{8})(3.05)} = 25^{\#}/\square''$

By (14), compression in concrete,$^{(c)}$ $f_c = \dfrac{(2)(12{,}800)}{(0.385)(0.875)(12)(3.05)^2} = 68^{\#}/\square''$

Total slab thickness $= 3.05 + 0.19 + 0.75 = 3.99''$

T-BEAMS AND RIBBED ONE-WAY SLABS

In reinforced concrete structures the slabs and beams are poured in one operation and the two are in fact monolithic. The compression occurring in the top of the beam near the center of the span will be resisted by the slab directly over the beaim and for a considerable distance on either side. Thus the effective section resisting positive moment of the vast majority of beams in actual structures is a T-section and such beams are called T-beams. In ordinary *tile-and-concrete-joist* construction and in *open-joist* construction the same co-action of slab and rib occurs and these types must be designed as T-beams.

A T-beam as a structural member may be a separate beam, a single span in a continuous beam or in a framed structure, or a cantilever projection as in footings. Furthermore all T-beams with end restraint must be treated as rectangular beams over some part of their length. In order to present a T-section in flexure for both positive and negative moment, a beam must have an I-shape. Such beams do occasionally occur but the usual case of a T-beam member involves design as a T-beam only for positive moment near the center and design as a rectangular beam for negative moment at each end.

21. Formulas.—Certain additional notation becomes necessary in designing T-beams. This notation is given in Appendix A.

Figure 6 shows the distribution of stress over a T-beam section and illustrates some of the symbols used in the formulas. In this figure the portion *adhe* is called the *flange*, the portion *fgnm* is called the *stem*, and the portion *bcnm* is called the *web*.

If the slab is thick and the beam shallow, kd may be less than t and the neutral axis lies within the flange. This condition is commonly designated as Case I in T-beam analysis. Since the concrete in tension is neglected in flexure the stress distribution and formulas for positive moment are identical with those for rectangular beams given in Art. **1**, p. 432, except the following:

$$v = \frac{V}{b'jd} \qquad (25) \qquad\qquad b'd = \frac{V}{jv} \qquad (26)$$

For negative moment b' must be substituted for b in *all* formulas and the design made for rectangular beams at the support where there is no flange available for compression. Very commonly this negative moment section will have compressive reinforcement and in this case the design becomes that of a doubly-reinforced rectangular beam.

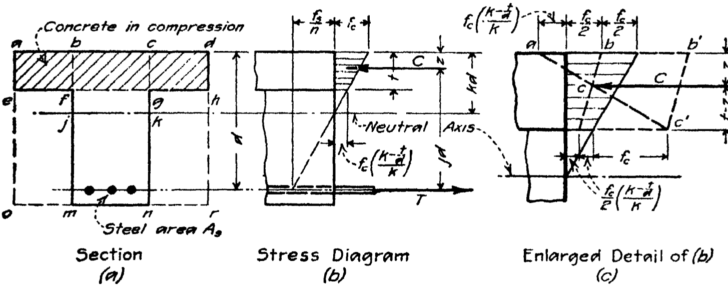


Fig. 6.—Stress distribution in T-beams (Case II).

If the slab is sufficiently thin so that kd is greater than t and the neutral axis lies below the flange entirely, as illustrated in Fig. 6, the area of concrete available to take compression is less than in a rectangular beam of size bd and new formulas must be developed. Two cases, (Case II and Case III) arise under this condition. In Case II the small amount of compression in the stem between the neutral axis and the under side of the flange is neglected. The stress in this portion is low as shown in the stress diagram, Fig. 6, decreasing to zero at the neutral axis, and the width affected is only the width of the stem. The principal formulas for design are derived in the same manner as for rectangular beams.

$$\text{Position of the neutral axis } k = \frac{pn + 0.5\left(\frac{t}{d}\right)^2}{pn + \frac{t}{d}} \qquad (27)^1$$

For balanced reinforcement values of k may be taken from Table 1

$$\text{Arm of resisting couple } j = 1 - \frac{t}{d} \cdot \frac{3k - 2\left(\frac{t}{d}\right)}{6k - 3\left(\frac{t}{d}\right)} \qquad (28)^1$$

[1] See footnote on p. 451

Percentage of reinforcing steel $p = \frac{t}{d} \cdot \frac{f_c}{f_s}\left(1 - \frac{\frac{t}{d}}{2k}\right)$ (29)[1]

For neutral axis at lower face of flange with balanced reinforcement

$$d = t\left(\frac{f_s}{nf_c} + 1\right) \tag{30}$$

For over-reinforced beams $M = M_c = f_c j \left(\frac{t}{d}\right) bd^2 \left[1 - \frac{\frac{t}{d}}{2k}\right]$ (31)[1]

$$K = K_c = f_c j\left(\frac{t}{d}\right)\left[1 - \frac{\frac{t}{d}}{2k}\right] \tag{32}$$

For under-reinforced beams $M = M_s = A_s f_s jd$ (33)

$$K = K_s = \frac{M_s}{bd^2} = f_s pj \tag{34}$$

$$A_s = \frac{M}{f_s jd} \tag{35}$$

Formulas (1), (2), (3), (40), (43) and (44) also apply to Case II. Values of K, j and p for T-beams with various ratios of t to d may be taken directly from Table 2, p. 468.

[1] Formulas (27), (28), (29) and (31) are derived as follows from Fig. 6(c):

From similar triangles abc and $ab'c'$

$$\frac{ab}{z} = \frac{ab'}{t} \text{ or } z = t\left(\frac{ab}{ab'}\right)$$

Substituting values of k, f_c and t/d, we have

$$z = t\left(\frac{\frac{f_c}{2} + \frac{f_c}{k}\left((k - \frac{t}{d}\right)}{\frac{3f_c}{2} + \frac{3f_c}{2k}\left(k - \frac{t}{d}\right)}\right) = t\left(\frac{3k - 2\frac{t}{d}}{6k - 3\frac{t}{d}}\right)$$

$jd = d - z$ Hence $j = 1 - \frac{z}{d}$

$$= 1 - \frac{t}{d} \cdot \frac{3k - 2\left(\frac{t}{d}\right)}{6k - 3\left(\frac{t}{d}\right)} \tag{28}$$

Equating the total tension to the total compression, we have

$$pf_s bd = \frac{f_c + f_c\left(\frac{k - \frac{t}{d}}{k}\right)}{2} bt$$

which reduces to

$$p = \frac{f_c}{f_s} \cdot \frac{t}{d}\left[1 - \frac{\frac{t}{d}}{2k}\right] \tag{29}$$

Substituting $\frac{f_c}{f_s} = \frac{k}{n(1 - k)}$ in (29),

$$k = \frac{pn + 0.5\left(\frac{t}{d}\right)^2}{pn + \frac{t}{d}} \tag{27}$$

$$M_c = f_c btjd\left[\frac{1}{2} + \left(\frac{k - \frac{t}{d}}{2k}\right)\right]$$

or

$$M_c = f_c j\left(\frac{t}{d}\right)bd^2\left(1 - \frac{\frac{t}{d}}{2k}\right) \tag{31}$$

A third case is distinguished when the compression in the web below the flange is taken into account (Case III). The Joint Committee gives formulas for this case as follows:

Position of neutral axis

$$kd = \sqrt{\frac{2ndA_s + (b - b')t^2}{b'} + \frac{[nA_s + (b - b')t]^2}{(b')^2}} - \frac{nA_s + (b - b')t}{b'} \qquad (36)$$

Position of resultant compression

$$z = \frac{(kdt^2 - \frac{2}{3}t^3)b + [(kd - t)^2(t + \frac{1}{3}(kd - t))]b'}{t(2kd - t)b + (kd - t)^2b'} \qquad (37)$$

Arm of resisting couple $jd = d - z$ (38)

Compressive unit stress in extreme fiber of concrete

$$f_c = \frac{2Mkd}{[(2kd - t)bt + (kd - t)^2b']jd} \qquad (39)$$

Tensile unit stress in longitudinal reinforcement

$$f_s = \frac{M}{A_s jd} \qquad (40)$$

These formulas are very cumbersome to use. Equal accuracy may be obtained by considering the T-beam, Fig. 6, as made up of a rectangular beam *bcnm* plus a T-beam (made up of the sum of *abmo* and *cdrn*). The two parts will affect the value of p and K (for balanced reinforcement) in proportion to their widths. In other words, if K_r and p_r are the values from Table 1 for rectangular beams and K_t and p_t the values for a T-beam (Case II) taken from Table 2 (for the proper value of t/d), then, for the combination

$$K = K_r\left(\frac{b'}{b}\right) + K_t\left(\frac{b - b'}{b}\right) \qquad (41)$$

and

$$p = p_r\left(\frac{b'}{b}\right) + p_t\left(\frac{b - b'}{b}\right) \qquad (42)$$

Also

$$bd^2 = \frac{M}{K} \qquad (43)$$

and

$$A_s = pbd \qquad (44)$$

Approximate formulas for Case II may be obtained by considering that the average value of the compressive unit stress over the flange can never be less than $\frac{1}{2}f_c$, and the depth z to the resultant of the compressive stress can never be greater than $\frac{t}{2}$. The arm of the resisting couple will always be greater than $(d - \frac{t}{2})$. Approximate equations result:

$$M_s = A_sf_s\left(d - \frac{t}{2}\right) \qquad (45)$$

$$A_s = \frac{M_s}{f_s\left(d - \frac{t}{2}\right)} \qquad (46)$$

$$M_c = \frac{1}{2}f_cbt\left(d - \frac{t}{2}\right) \qquad (47)$$

$$f_c = \frac{2M}{bt\left(d - \frac{t}{2}\right)} \qquad (48)$$

These formulas are used in rapid estimating work but should not be used for design. In the case when $\frac{t}{d}$ is very small the error is considerable, as a considerable portion of web concrete is then neglected in compression and the average stress is greater than $\frac{1}{2} f_c$. When the neutral axis lies in the lower surface of the flange, the true value of z is $\frac{t}{3}$ instead of $\frac{t}{2}$ and the formulas become wasteful.

22. Steps in the Design of a T-beam.—The steps in the design of a T-beam are much the same as in the design of a rectangular beam, the principal difference being in the numerical values of k and p. The weight of beam to be assumed, however, is only the weight of the stem, as the flange weight is commonly in the superimposed load. The steps are:

(*a*) Determine the value of $b'd$ by Formula (26).

(*b*) Determine whether or not the design of the beam comes under Case I. The value of t will have been determined previously in the design of the slab. (It may, of course, be changed, if found desirable, to aid the beam design.) The value of d will generally depend on architectural considerations and is therefore known or can be readily assumed. Use Formula (30) to determine the limiting value of d for Case I. If d by (30) is equal to or greater than the value indicated by external conditions, the beam is Case I, and the design may proceed exactly like a rectangular beam except that Formulas (25) and (26) replace Formulas (18) and (19). If d by (30) is less than the value indicated by the external conditions, it is a Case II or Case III T-beam and the design proceeds as follows:

(*c*) Determine the value of bd^2 by Formula (43). For Case III the value of K is found using Formula (41) in a cut-and-try process in which experience creates skill. For Case II the value of K and p may be taken from Table 2.

(*d*) Determine p from Formula (42) for Case III, or from Table 2 for Case II, and determine A_s from Formula (44). Decide on the number and size of bars and check to see if they can be placed in the width b'. If two layers are necessary, the value of d may be reduced and a re-design may be required.

(*e*) Investigate the bond stress on the longitudinal steel. Article **25** takes up this matter in more detail.

(*d*) Design the web reinforcement exactly as for a rectangular beam of width b'.

(*g*) The protective cover for the reinforcement in the stem of T-beams should be the same as for rectangular beams. In the upper portion of a T-beam, the protective cover may be reduced to that in a slab, remembering that this cover will ordinarily be above the *slab steel* which rests upon the beam steel. Where plaster is applied directly to the concrete, the fire proofing cover may be reduced, according to some ordinances by one-half the thickness of the plaster but not more than ½ in.

23. Brackets and Haunches.—The preceding outline of steps to be taken in design applies to the *T*-section under positive moment at midspan in which the wide flange is in compression and the stem in tension. The two ends of the T-beam where the stresses are reversed must now be designed for negative moment as rectangular beams. If brackets or haunches are not permissible, this design may fix the values of b' and d for the entire beam. Brackets are made by deep-

ening the beam from the point where the ordinary depth is sufficient to take the compression, increasing to the column. Haunches are made by increasing b' from the same section to the column. The use of either bracket or haunch decreases the shearing stress and permits of decreased web-reinforcement. Either one adds to the expense of the formwork considerably, and where a reasonable amount of compressive reinforcement will serve the same purpose it is generally used in preference to brackets or haunches.

24. Use of T-beam Diagram.—The best T-beam design and review diagram for Case II with which we are familiar is Diagram 2, p. 469. In using this diagram the usual calculations for shear and bond must be made to complete the design. The proper proceeding in reviewing designs using this diagram is as follows:

The values of $p, t/d, K\left(=\frac{M}{bd^2}\right)$ and k are established in any review problem.

(*a*) Enter upper part of diagram with p, t/d and K and determine f_s.

(*b*) Enter lower part of diagram with p, t/d and f_s and determine f_c.

To use Diagram 2 for design, proceed as follows:

(*a*) Assume depth and weight of member and correct given moments and shears to include effect of the weight. The assumption of depth establishes d and t/d very closely, the slab thickness being known.

(*b*) Enter lower part of diagram with f_s, f_c and t/d and determine p.

(*c*) Enter upper part of diagram with t/d, p and f_s and determine K.

(*d*) From the relation $K = \frac{M}{bd^2}$ compute b.

(*e*) From Formula (26) determine b' and design the web reinforcement.

(*f*) By Formula (17) investigate the bond at the point of inflection.

(*g*) Check the assumed weight and revise design if necessary to correct same or to secure proper proportions of member.

25. Discussion of Stresses in a T-beam.—Let Fig. 7 represent a T-beam member with the longitudinal reinforcement as shown in (*a*), the uniform load as in (*b*), the shear curve as in (*c*) and the moment curve as in (*d*). The moment curve shown is drawn in accordance with the almost universal specification that for a continuous beam of equal spans uniformly loaded the maximum positive moment at center of intermediate spans shall be taken as $\frac{WL}{12}$ and the maximum negative moment at the corresponding supports at the same value. (The 1921 J. C. report recognizes a *positive* moment of $\frac{WL}{16}$ for this case which is more reasonable.) These two values are not for any one condition of load on the adjoining spans but represent maximums at the two principal sections under all possible loadings. The moment curve is therefore composed of two portions, one representing positive moments at their maximum values and the other representing negative moments at their maximum values. The shear curve is the same for both cases. The point of inflection is seen to have possible positions as far from the support as point 5 and as near to it as point 3. Assume that the depths are so selected as to give eight bars at the center and eight at the support. Since A_s is proportional to M, a graphical construction may be made

showing the moment carried by each set of two bars and this has been done. Starting at the center of the beam it is seen that *considering moment only* two bars of the bottom reinforcement can be bent up or terminated at point 6, two

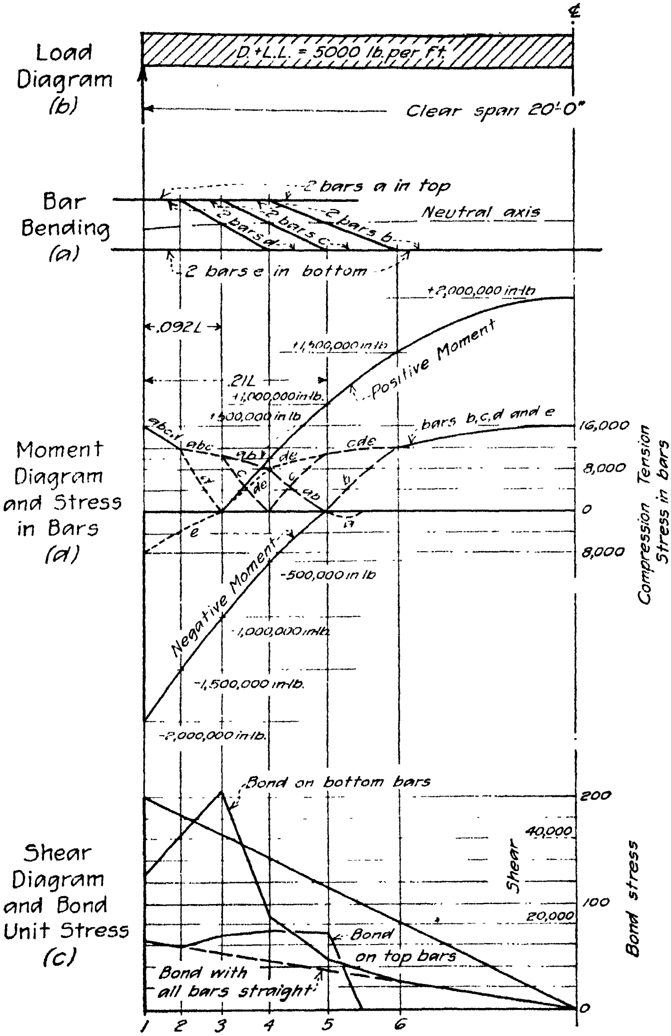


FIG. 7.—Study of stresses in beam.

more at point 5, two more at point 4, and the last two at point 3 where the positive moment becomes zero even in the extreme case of loading on adjacent spans.

Similarly considering the upper reinforcement from the support towards the center, and taking account of *moment requirements only*, the eight bars required at the end can be reduced to six at point 2, to four at point 3, to two at point 4 and entirely cut off at point 5 where the negative moment becomes zero at the other extreme. As has been mentioned, (Art. **12**) good practice requires that at least one-fourth of the bars required at the center be carried through in the bottom of the beam to the support and bond considerations frequently raise this proportion. These are the two bottom bars that otherwise would be terminated at point 3. At the top, also, practice has standardized very largely on carrying at least one-fourth of the bars to the quarter point of the clear span instead of stopping at point 5 which is $0.21L$ from the face of the support. The intervening bars could be bent (for moment) as shown in Fig. 7*b* and would provide an efficient arrangement of web reinforcement except for sections close to the support and possibly, if the shearing stress is high, beyond the outer bent rod.

In Fig. 7*d* curves have been drawn showing quite closely the tensile unit stress in the various pairs of rods as it varies from the center to the support figured by Formula (15), and in Fig. 7*c* the bond unit stress variation is shown in a similar manner, figured by Formula (17). Note that by applying Formula (17) to sections between the point of inflection and the support the bond stress on the tensile (upper) reinforcement is found. The lower reinforcement is in compression and its bond stress is lower. A study of Fig. 7*c* indicates that the critical section for bond in intermediate spans of continuous beams under the usual moment specifications will commonly be found at a section at a distance of $\frac{1}{10}$ of the clear span from the face of the support.

26. Bond Considerations.—At the end of simple beams, and at the outer end of beams continuous at one end only, the section at the face of the wall support may be critical for bond and very commonly the bottom rods will need to be hooked over such supports. The top steel will almost always require hooks over the end support. It is a safe and at present practically universal rule to provide tension steel in the top at the ends of all concrete members thus avoiding the possibility of cracks forming at that point and destroying the shearing resistance of the beam at its critical section.

In Fig. 7 all bars have been bent up and used for negative reinforcement as far as practicable. It sometimes happens that a portion of the reinforcement at either the center or the support is not needed at the other section of maximum moment value. Such bars are terminated where they are no longer needed for tension. Large bari should always end in a hook bent to a diameter not less than eight times the bar diameter to minimize the slipping. Even an elementary consideration will show the designer that exceedingly high bond stresses must occur at such rod ends and that some slip is bound to occur. It is also true in most cases that where a bar is bent from the lower to the upper plane, the bond stress at the neutral axis will be so high as to cause some slipping and the actual tension in such rods will rarely decrease to zero as shown in Fig. 7*d*. In this case the opposed tensions above and below the neutral axis tend to balance each other so that high bond stresses are justified in design. Their magnitude is never figured since the anchorage is exceptionally good. The bending of such bars, however, should be gradual to avoid high stresses in local compression at the points of bend. The radius of bend should be not less than four bar diameter.

27. Lateral Spacing of Longitudinal Bars.—The lateral spacing of the longitudinal bars for positive moment is governed by the same condition set forth in Art. 11 for rectangular beams. For negative moment, however, the minimum spacing can be used in the bars in all layers since there is ample concrete in the flanges of a T-beam to take the horizontal shear.

28. Critical Sections.—For T-beams the critical section in horizontal shear is across the stem at the under face of the flange and adjoining the support. The intensity is the same as the intensity of the vertical shear and is figured by Formula (25). The same web reinforcement will provide against both vertical and horizontal shearing stresses in all usual cases. Other sections where caution is sometimes urged in T-beam design are the two sections through the flange in the planes of the sides of the stem. In the writer's opinion these sections are almost never critical, since the shear is very low at the center where the tee is used and at the support the T-beam becomes a rectangular beam. Reinforcement must be provided to take the cross bending of the flange as a slab across the stem, where this is not provided as a part of the slab design. Such a case occurs where beams of T-shape are separate on one or both sides, the flange being provided solely to increase the compression area.

29. Ribbed Slabs of the One-way Type.—Ribbed slabs are T-beams with or without fillers of such a character as to increase their strength. Figure 8 shows several common types. Hollow clay tile greatly increase the strength in compression and shear and also the rigidity. (Where the tile are laid with staggered joints, the thickness of one web may properly be added to the width of the concrete joist as effective in shear, and one half of the thickness of the top slab of the tile may be added to the concrete as effective in compression.) Gypsum tile are similarly effective to a less degree. Metal domes are commonly considered as forms only, even if they are left in place permanently. The design of ribbed slabs is precisely the same as that of T-beams as outlined above. Diagram 8 gives information of value in selecting trial dimensions.

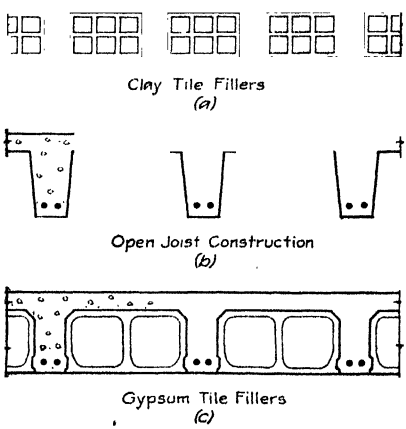


FIG. 8.—Types of ribbed slab construction.

30. General Proportions of T-beams.—Tests have shown conclusively that the slab is effective as a compressive flange in T-beams to a very great distance from the stem. City codes and the ideas of those who sometimes write specifications have become "set" at various stages in our developing knowledge of this matter and vary widely. The latest Joint Committee report permits the use of the adjoining slab for a distance on either side of the stem equal to 8 times the slab thickness. Of course, not more than one-half the clear distance to the adjoining T-beam is available and a limitation that b shall not exceed $\frac{1}{4}$ to $\frac{1}{3}$ of the span length of the beam is commonly recognized in design. For beams

having a tee on one side only the effective width of slab acting with the beam should be reduced to four to six times the slab thickness, depending on the depth of the beam and the ratio of t to d. The important considerations are: (1) the *moment* produced on the unsymmetrical beam section when the center of action of the compressive stresses is moved from its usual position directly above the center of action of the tensile stresses, and (2) the *torsion* due to unbalanced external loading. A rational design is very difficult and is seldom if ever made, but the designer remains responsible for keeping these moments to reasonable amounts. The slab thickness should bear a sensible relation to the effective depth of the beam. Table 2 includes the range of $\frac{t}{d}$ properly used for design. For ribbed slabs without fillers some ordinances limit the width of flange, b, to $\frac{3}{4}$ of the distance center to center of joists as a maximum, but if the slab is properly reinforced and of proper thickness no special limitations are necessary.

In ribbed construction the stem will commonly be wider at the support (to resist diagonal tension) than at the center. This is accomplished in clay tile construction by using smaller tile at the ends of rows or in gypsum block construction by sawing off standard blocks to give the necessary joist flare. In metal forms the flare is provided by the shape of the forms themselves. The required flare at the ends of joists should be called for on the details in all cases.

In hospitals and hotels it frequently becomes necessary to limit the depth of the main girders to that of the ribbed slab between. In such cases the ribbed slab must be made unusually deep unless very expensive girders with heavy compressive reinforcement are resorted to.

31. Reinforcement for Shrinkage Stresses.—Most materials in hardening are subject to shrinkage and if the section is irregular, as in a T-beam, severe shrinkage stresses may be present at the three planes of juncture of flange and stem, and flange and web. Reinforcement should always be provided across these sections. The slab steel and the stirrups or ties for compression steel are generally present in adequate amount, but where a designer in some special case dispenses with either of these reinforcements he must provide a suitable system of ties across such sections. The proper amount is a matter of judgment depending on the relative proportions of the parts. Certain proportions of parts, as in cast iron, reduce the destructive effect of shrinkage to a minimum.

Illustrative Problem.—The beams in a floor are spaced 4 ft. on centers and the slab is 6 in. thick. The total superimposed load, including slab, is 1,800 lb. per ft. The clear span is 18 ft. 6 in. and the moment coefficient at the center is fixed by ordinance as $wL^2/12$. The total depth of beam is limited to 24 in. Design one beam as a T-beam at the center, using $f_c = 700$, $f_s = 18{,}000$ and $n = 15$.

For solution see Design Sheet 5. The following notes apply to that sheet: (*a*) Since 16.25 is less than 21.5, the neutral axis falls below the flange; (*b*) this is the more accurate but much more laborious solution but is followed when a number of beams are to be designed at one time, all having the same value for t and d—it is also used when the slab is very thin and the stem very wide, a case in which the other method is quite wasteful; (*c*) if this value had not checked that assumed, a new assumption and re-computation would be necessary; (*d*) average of minimum spacing for $\frac{7}{8}$- and $\frac{3}{4}$-in. round bars; (*e*) this is the solution commonly used, and is quite accurate for usual proportions of stem and flange; (*f*) when pinched for flange width a solution by Case III will reduce this width somewhat.

Illustrative Problem.—A T-beam is subject to a bending moment at the center of 3,000,000 in.-lb., including the moment of its own weight. The stem width is 12 in. and the total depth is 30 in. The flange thickness is $5\frac{1}{2}$ in. and the total available width is

56 in. Reinforcement consists of four 1⅛-in. square bars in one layer, centered 3 in. above the bottom of the beam, and two 1¼-in. square bars in a second layer, centered 7 in. above the bottom of the beam. The reinforcement is bent up at the ends. Find the stresses in concrete and steel.

For solution see Design Sheet 6.

Illustrative Problem.—Design an *open joist* (ribbed slab without fillers) floor for a superimposed load of 75 lb. per sq. ft. on a clear span of 20 ft., using A.C.I. specifications for 2,000 lb. concrete and structural grade steel.[1] Joists are to be on 24-in. centers. This floor is for the center bay of a building five bays wide.

DESIGN SHEET 5

Assume weight of stem, below flange, as $200^{\#/\prime}$

Superimposed load $= 1{,}800^{\#/\prime}$

$2{,}000^{\#/\prime}$

With one layer of bars, d will be approx. $24 - 2\tfrac{1}{2} = 21.5''$ $\dfrac{t}{d} = \dfrac{6}{21.5} = 0.28$

By (26) $\quad b'a = \dfrac{(2{,}000)(9.25)}{(100)(0.86)} = 216''^2$. Assume $b' = 10''$, say $9\tfrac{5}{8}''$

By (30) $\quad d = 6\left(\dfrac{18{,}000}{(15)(700)} + 1\right) = 16.25''^{(a)}$ *Not* Case I.

Moment at center $= \left(\dfrac{12}{12}\right)(2{,}000)(18.5)^2 = 685{,}000''^{\#}$

Solution as Case III T-beam$^{(b)}$

For $f_c = 700$, $f_s = 18{,}000$, $n = 15$ and $\dfrac{t}{d} = 0.28$, from Table 2, K lies between 107.9 and 113.1, and p lies between 0.00675 and 0.00716.

Try $K = 111$. By (43) $b = \dfrac{685{,}000}{(111)(21.5)^2} = 13.3''$

By (14) $\quad K = (113.1)\left(\dfrac{9.6}{13.3}\right) + (107.9)\left(\dfrac{3.7}{13.3}\right) = 111.0^{(c)}$

By (42) $\quad p = (0.00716)\left(\dfrac{9.6}{13.3}\right) + (0.00675)\left(\dfrac{3.7}{13.3}\right) = 0.00705$

By (44) $\quad A_s = (0.00705)(13.3)(21.5) = 2.02^{\square''} = 2 - \tfrac{7}{8}''\phi$ and $2 - \tfrac{3}{4}''\phi$

Width of stem $= \left(3 \times 2^{(d)}\right) + \tfrac{3}{4} + 3'' = 9\tfrac{3}{4}''$ $\quad 9\tfrac{5}{8}''$O.K.

Weight of stem, below flange $= (9.62)(18)\left(\dfrac{150}{144}\right) = 180$ O.K.

At $\tfrac{1}{10}$ point of span, by (17) $\quad u = \dfrac{(2{,}000)(0.4)(18.5)}{(5.1)(0.86)(21.5)} = 157^{\#/\square''}$

At face of support, by (18) $\quad v = \dfrac{(2{,}000)(9.25)}{(9.62)(0.86)(21.5)} = 104^{\#/\square''}$

From Diagram 4, $a = (0.31)(13.5)(12) = 69''$

$A_v = (1.28)(2.02) = 2.59^{\square''} = 12 - \tfrac{3}{8}''\phi$U-stirrups

Max. stirrup $= \dfrac{21.5}{50} = 0.43'' = \tfrac{3}{8}''\ \phi$

Stirrup spacing in terms of a = 0.043; 3 @ 0.104; 0.145; 0.212
in inches = 3″, 3 @ 7¼″, 10″, 14½″

Solution as Case II T-beam$^{(e)}$

For $\dfrac{t}{d} = 0.28$ $\quad K = 107.9$ $\quad p = 0.00675$ (From Table 2.)

By (43) $\quad b = \dfrac{685{,}000}{(107.9)(21.5)^2} = 13.8''^{(f)}$

By (44) $A_s = (0.00675)(13.8)(21.5) = 2.00^{\square''} = 2 - \tfrac{7}{8}''\phi$ and $2 - \tfrac{3}{4}''\phi$

The balance of the design is the same as Case III above.

[1] See table on p. 466.

DESIGN SHEET 6

For this steel arrangement d = (approx.) 25″

$$t = 5.5'' \quad \frac{t}{d} = \frac{5.5}{25} = 0.22 \quad A_s = (4)(1.256) + (2)(1.5625) = 8.19^{\square''}$$

$$p = \frac{8.19}{(56)(25)} = 0.00585$$

$$K = \frac{M}{bd^2} = \frac{3,000,000}{(56)(25)^2} = 86$$

Enter upper half of Diagram 2 with these values of K, p and $\frac{t}{d}$

$$f_s = 16,600^{/\#\square''}$$

Enter lower half of Diagram 2 with p, $\frac{t}{d}$ and f_s

$$f_c = 625^{\#/\square''}$$

Spacing of bars in bottom layer $= \frac{12 - 3 - 4.5}{3} = 1\frac{1}{2}''$ clear

Steel should be re-arranged.

DESIGN SHEET 7

A.C.I. Specifications: $+M = -M = \frac{wL^2}{12}$, $f_c = 750$, $f_s = 16,000$, $n = 15$, $k = 0.413$ $= 40$

Assume weight of slab as	$60^{\#/\square'}$
Superimposed load	$75^{\#/\square'}$
	$135^{\#/\square'}$

Load per joist = (2)(135) = $270^{\#/'}$

Try 8″ joist and 2″ slab. $d = 10 - 1\frac{1}{2} - \frac{3}{8} = 8.12''$ $\quad \frac{t}{d} = \frac{2}{8.12} = 0.246$

From Table 2, $K = 115.7$, $p = 0.0082$, $j = 0.894$.

$$M = \left(\frac{12}{12}\right)(270)(20)^2 = 108,000''^{\#}$$

By (10) $\quad b = \frac{108,000}{(115.7)(8.12)^2} = 14.15$

By (44) $\quad A_s = (0.0082)(14.15)(8.12) = 0.94^{\square''} = 2\text{-}\frac{3}{4}''\phi$ bars per joist

By (17)[(a)] $\quad u = \frac{(270)(0.4)(20)}{(2.356)(0.894)(8.12)} = 101^{\#/\square''}$. Use deformed bars

By Table 5 width of beam $= b' = (2)(1\frac{1}{2}) + (2)(\frac{3}{4}) + 1.12 = 5.62''$ at center

By (26)[(b)] $b' = \frac{(270)(10)}{(40)(0.894)(8.12)} = 9.3''$ required width of joist at support without stirrups

Weight of floor $= (2)(12) + \frac{8}{2}[(\frac{3}{4})(6) + (\frac{1}{4})(9.3)] = 51^{\#/\square'}$[(d)] O.K.

At support $-M = -108,000\left(\frac{252}{270}\right) = -101,000''^{\#}$, $A_s = 0.84^{\square''} = 2\text{-}\frac{3}{4}''\phi$

Rectangular section $K = 133.5$. By (10) $b' = \frac{-101,000'}{(133.5)(8.62)^2} = 10.2''$

Make joist 6″ wide from center to quarter point flaring to 10″ wide at [(c)] support.

For solution see Design Sheet 7. The following notes apply to that sheet: (*a*) Assuming one ¾-in. square bar carried through straight in bottom; (*b*) stirrups are rarely used in ribbed slabs, but joists are flared from the quarter point to end of span; (*c*) $d = 8.62$, as only the cover for a slab is required for top steel; (*d*) this difference of 18 lb. per ft. between actual and assumed weight should be taken into account—in this case it permits the steel to be reduced by using two ¾-in. round bars (area = 0.88 sq. in.) where the uncorrected value of A_s was 0.94 sq. in. (joist width computation is corrected).

BEAMS AND SLABS REINFORCED FOR COMPRESSION

Cement is cheaper than steel for reinforcing concrete against compression and the designer should realize that compressive reinforcement of beams and slabs always exacts its penalty of added cost, both in the construction of the work and in the labor of design. There are many cases where the use of compressive reinforcement is justified, however. In the design of T-beams at the support (really rectangular beams as has been pointed out) it may be less expensive to put in compression steel than to build forms for brackets or haunches. This is commonly true when the necessary amount of compressive reinforcement can be secured by simply lapping across the support the usual straight bottom bars from the beams on either side.

32. Formulas.—Additional symbols are required for the formulas used in the calculation of double-reinforced beams. The symbols used in the formulas that follow are as given in Appendix *A* with the following additions:

K_1 = factor expressing ratio of resisting moment of member with balanced reinforcement (but without compressive steel) to bd^2.

K_2 = factor expressing ratio of resisting moment of couple formed of C' and its balancing tension to bd^2.

K = factor expressing ratio of external moment to $bd^2 = K_1 + K_2$.

p_1 = steel ratio for member with balanced reinforcement.

p_2 = steel ratio for the tensile steel added to balance the compression steel.

p = total tensile steel ratio = $p_1 + p_2$.

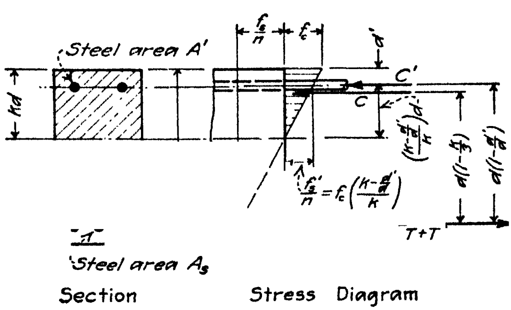


FIG. 9.—Stress distribution in doubly reinforced beams.

In Fig. 9 some of the above symbols are illustrated, and relationships are shown permitting the ready derivation of Formulas (57) to (61) inclusive, which are commonly used in design.

The usual formulas expressing the relationships in rectangular beams reinforced for compression are given by the Joint Committee as follows:

Depth to the neutral axis $k = \sqrt{2n\left(p + p'\frac{d'}{d}\right) + n^2(p + p')^2} - n(p + p')$ (49)

For balanced reinforcement k may be taken from Table 1.

Depth to resultant compression

$$z = \frac{\frac{1}{3}k^3d + 2p'nd'\left(k - \frac{d'}{d}\right)}{k^2 + 2p'n\left(k - \frac{d'}{d}\right)} \tag{50}$$

Arm of resisting couple $jd = d - z$ (51)

This is commonly assumed as $j = 0.86$ in design.

Compressive unit stress in extreme fiber of concrete

$$f_c = \frac{6M}{bd^2\left[3k - k^2 + \frac{6p'n}{k}\left(k - \frac{d'}{d}\right)\left(1 - \frac{d'}{d}\right)\right]} \tag{52}$$

Compressive unit stress in longitudinal reinforcement

$$f_s' = nf_c\frac{k - \frac{d'}{d}}{k} \tag{53}$$

Tensile unit stress i longitudinal reinforcement

$$f_s = \frac{M}{pjbd^2} = \frac{M}{A_sjd} \tag{54}$$

Formulas (17) to (23), inclusive, for bond stress and web reinforcement apply to double reinforced beams the same as to ordinary beams.

These formulas have been re-written[1] so as to facilitate their use in design, and Tables 3 and 4 are based upon this treatment.

Percentage of tensile reinforcement

$$p = \frac{\frac{K}{f_s} + 2n(1 - k)\left(\frac{k}{3} - \frac{d'}{d}\right)}{1 - \frac{d'}{d}} \tag{55}$$

Percentage of compressive reinforcement

$$p' = \frac{\frac{K}{f_c} - \frac{k}{2}\left(1 - \frac{k}{3}\right)}{\frac{n}{k}\left(k - \frac{d'}{d}\right)\left(1 - \frac{d'}{d}\right)} \tag{56}$$

Tables 3 and 4, pp. 470 and 471 respectively, give values of p and p' for various values of $\frac{K}{f_s}$ and $\frac{K}{f_c}$, and for $\frac{d'}{d}$ equal to 0.1 and 0.2.

A simpler set of formulas is based on the combination of a flexural member with balanced reinforcement and a couple composed of the compression in the compressive reinforcement and the tension in added tensile reinforcement to

[1] By Mr. Ralph R. Leffler of Chicago.

balance. On this basis the formulas take very useable form and are applicable to T-beams as well as to rectangular beams.[1]

Factor for external moment $K = \dfrac{M}{bd^2}$ (57)

Factor for compression reinforcement couple $K_2 = K - K_1$ (58)

Percentage of added tensile reinforcement $p_2 = \dfrac{K_2}{f_s\left(1 - \dfrac{d'}{d}\right)}$ (59)

Total percentage of tensile reinforcement $p = p_1 + p_2$ (60)

Percentage of compression reinforcement $p' = p_2 \dfrac{1 - k}{k - \dfrac{d'}{d}}$ (61)

Values of k, p_1, j and K for rectangular beams may be taken directly from Table 1, p. 466, or for T-beams from Table 2. These are the formulas to use in design whenever the actual condition does not fall within the limited range of Tables 3 and 4.

33. Steps to be Taken in Design. In the design of a rectangular beam with compressive reinforcement the values of b and d are limited and known; otherwise compressive reinforcement would not be used. The steps in design are as follows:

(*a*) Determining b' from the shearing unit stress by Formula (18) taking $j = 0.86$.

(*b*) Compute K by Formula (57) and also K/f_s and K/f_c.

(*c*) Check to see that n and d'/d are within the limits of Tables 3 and 4. If so, determine the value of p and p' by interpolation from these tables.

(*d*) Compute A_s $(= pbd)$ and A' $(= p'bd)$ and determine the number and size of bars.

(*e*) Check the bond unit stress on the tensile reinforcement by Formula (17).

(*f*) Complete the design of web reinforcement as in a rectangular beam.

In case the values of n or of d'/d, found under (*c*) above, are outside the range of Tables 3 and 4, the design can be made by solving equations (55) and (56). For this purpose the value of k is taken from Table 1.

If a T-beam is to be designed with compressive reinforcement, or if a rectangular beam is beyond the range of Tables 3 and 4, the steps in design using Formulas (57) to (61) are as follows:

[1] Formulas (59) and (61) are derived as follows from Fig. 9, considering only the couple composed of C' and its balancing tension T':

$$M_2 = K_2bd^2 = np'f_cbd^2\left(\frac{k - \frac{d'}{d}}{k}\right)\left(1 - \frac{d'}{d}\right) \qquad (a)$$

$$= p_2f_sbd^2\left(1 - \frac{d'}{d}\right) \qquad (b)$$

From (*b*)

$$p_2 = \frac{M_2}{bd^2} \cdot \frac{1}{f_s\left(1 - \frac{d'}{d}\right)} \qquad (59)$$

From (*a*) and (*b*)

$$p' = p_2\frac{f_sk}{nf_c} \cdot \frac{1}{k - \frac{d'}{d}}$$

$$= p_2\frac{1 - k}{k - \frac{d'}{d}} \qquad (61)$$

(a) Check the shearing unit stress by Formula (18).
(b) Compute K by Formula (57) and K_2 by Formula (58) taking the value of K_1 from Table 1 or 2. While consulting this table, also record the values of k and p_1.
(c) Compute the value of p_2 by Formula (59) and of p by (60).
(d) Compute the value of p' by Formula (61).
(e) Compute A_s ($= pbd$) and A' ($= p'bd$) and determine the number and size of bars.
(f) Check the bond unit stress on the tensile reinforcement by Formula (17).
(g) Complete the design of the web reinforcement.

34. Designing Details.—If it is desired to cut off the compressive reinforcement short of the point of inflection, the compressive unit stress must be computed by Formula (53) or Diagram 3, p. 474. This diagram gives the embedment necessary for any compressive unit stress at any specified bond unit stress.

After the steel areas have been determined at the center and at each end of the member, considerable study is frequently necessary to determine the most effective way of arranging and bending the bars to provide these areas with the least waste. Unless the load is uniformly distributed the load-shear-moment graph will be found a most useful aid in detailing a beam with compressive reinforcement. From this graph the location of the section where the compression steel is no longer required is easily found, using Formula (57) rewritten as follows:

$$M_1 = K_1bd^2 \tag{62}$$

in which M_1 is the value of the ordinate to the moment curve at that section.

The tension and web reinforcement is detailed in the same manner already described for rectangular and T-beams. The compression steel is generally required to be tied in the same manner as the longitudinal steel in a tied column. For this purpose ¼-in. round ties at 8 in. on centers are not always sufficient. A safe rule is to use sufficient ties so that their cross-sectional area per foot of length of member shall not be less than 10 per cent of the area of the compressive steel. The spacing at the section of maximum stress should be not over 8 in. and the tie area may be decreased as the stress decreases, where such refinement is warranted. Figure 10 shows details of the proper bending of compression ties.

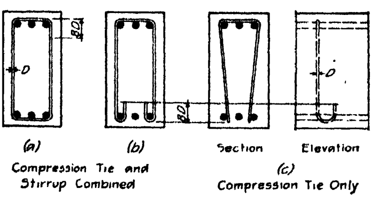


Fig. 10.—Ties for compression steel.

When compression reinforcement is used in an isolated beam (as between two elevator shafts), care must be taken that the beam is not made too slender for its span. In general the width of an isolated beam should be not less than $\frac{1}{30}$ of its clear span. If necessary to reduce the width below this figure, the compressive unit stress in the concrete should be reduced in accordance with the formula

$$\text{Per cent reduction in stress} = 3L/b - 90 \tag{63}$$

Illustrative Problem.—To design a beam whose web dimensions are limited to 36 in. wide by 14 in. deep, for a concentrated load at the center of 20,000 lb. and a uniformly distributed superimposed load of 850 lb. per ft. over the entire 20-ft. clear span. At the center, a 2½-in. slab permits of a total effective flange width of 64 in. This beam is the

middle span of a continuous beam of five opal spans, the adjoining span being loaded in an identical manner. Design the T-section at the center and the rectangular section at the support for $f_c = 700$ and $f_s = 18{,}000$.

For solution see Design Sheet 8. The following notes apply to that sheet: (*a*) Moment coefficients taken from three-moment diagrams; (*b*) moment coefficients taken from 1921 J. C. Specifications; (*c*) allows for ⅝-in. slab rods over 1¼-in. compression bars; (*d*) allows for ⅝-in. slab rods over 1-in. tension bars; (*e*) from Table 6, p. 472; (*f*) seven 1¼-in. square bars in compression, or equal area.

DESIGN SHEET 8

Moment due to concentrated load[a]

$$\text{At center} \quad +M = (0.13)\ (20{,}000)(20)(12) = 625{,}000''\#$$

$$\text{At support} \quad -M = (0.119)(20{,}000)(20)(12) = 572{,}000''\#$$

Moment due to uniform load[b]

$$\text{Weight beam} = 450\#/'$$

$$\text{Superimposed load} = 850\#/'$$

$$1{,}300\#/'$$

$$\text{At center} \quad +M = \left(\tfrac{12}{16}\right)(2{,}300)(20)^2 = 390{,}000''\#$$

$$\text{At support} \quad -M = \left(\tfrac{12}{12}\right)(2{,}300)(20)^2 = 520{,}000''\#$$

$$\text{Total moment at center} = +1{,}015{,}000''\#$$

$$\text{Total moment at support} = -1{,}092{,}000''\#$$

T-beam Design:

For $f_c = 700$, $f_s = 18{,}000$ and $n = 15$, from Table 1 or Table 2, $k = 0.368$, $d = 14'' - 2 - \frac{5}{8} = 11.37''$, $\frac{t}{d} = \frac{4}{11.37}$

From Table 2, $K_1 = 97.6$, $p_1 = 0.0060$, $j = 0.907$

$$d' = 1 + \tfrac{5}{8} + \tfrac{5}{8}^{(c)} = 2\tfrac{1}{4}'', \quad \frac{d'}{d} = \frac{2.25}{11.37} = 0.20$$

$$K \text{ at center } (b = 64'') = \frac{1{,}015{,}000}{(64)(11.37)^2} = 123$$

By (58) $\quad K_2 = K - K_1 = 123 - 97.6 = 25.4$

By (59) $\quad p_2 = \dfrac{25.4}{18{,}000\ (1 - 0.20)} = 0.00176$

By (60) $\quad p = 0.00176 + 0.006 = 0.00776$

$$A_s = (0.00776)(11.37)(64) = 5.65\square''$$
$$= 7\text{–}1''\phi \text{ bars, tension}$$

By (61) $\quad p' = 0.00176\left(\dfrac{1 - 0.368}{0.368 - 0.20}\right) = 0.00661$

$$A' = (0.00661)(11.37)(64) = 4.81\square'' = 6\text{–}1''\phi \text{ bars, compression}$$

Rectangular Beam Design:

$$d \text{ at support} = 14 - 1 - \tfrac{1}{2} - \tfrac{5}{8}^{(d)} = 11.87''$$

$$K \text{ at support} = \frac{1{,}092{,}000}{(36)(11.87)^2} = 216$$

$$d' = 2 + \tfrac{5}{8} = 2.62''$$

$$\frac{d'}{d} = \frac{2.62}{11.87} = 0.22$$

By (58) $\quad K_2 = 216 - 113 = 113$

By (59) $\quad p_2 = \dfrac{113}{18{,}000(1 - 0.22)} = 0.00805$

By (60) $\quad p = 0.0072 + 0.00805 = 0.01525$

$$A_s = (0.01525)(11.87)(24) = 4.34\square'' = 6\text{–}1''\phi \text{ bars, tension}$$

By (61) $\quad p' = 0.00805\left(\dfrac{1 - 0.368}{0.368 - 0.22}\right) = 0.0343$

$$A' = (0.0343)(11.87)(24) = 9.80\square'' = 7\text{–}1\tfrac{1}{4}''\square \text{ bars, compression}$$

Width required at support $= (6)(2.66)^{(e)} + (7)(1.25) + 3 = 27.2''$ O.K.

By (17), at support $\quad u = \dfrac{(10)(1{,}300) + 10{,}000}{(6)(314)(\frac{7}{8})(11.87)} = 118\#/\square''$. *Hook ends.*

By (17), at $\frac{1}{10}$ point[f] $\quad u = \dfrac{(8)(1{,}300)(10{,}000)}{(7)(5.00)(\frac{7}{8})(11.37)} = 59\#/\square''$

TABLE 1.—RECTANGULAR BEAMS

n	f_s	f_c	k*	j	kj	p	K	$\frac{f_s}{nf_c}$	$\frac{k^*}{n(1-k)}$
15	16,000	650	0.379	0.874	0.330	0.0077	107.5	1.64	0.0405
		700	0.396	0.868	0.344	0.0087	120.4	1.52	0.0437
		750	0.413	0.862	0.356	0.0097	133.5	1.42	0.0468
		800	0.429	0.857	0.367	0.0107	146.9	1.34	0.0499
		850	0.444	0.852	0.377	0.0118	160.6	1.26	0.0531
		900	0.458	0.848	0.387	0.0129	174.5	1.19	0.0561
	18,000	650	0.351	0.883	0.310	0.0063	100.8	1.85	0.0361
		700	0.368	0.877	0.323	0.0072	113.1	1.72	0.0388
		750	0.385	0.872	0.335	0.0080	125.7	1.60	0.0416
		800	0.400	0.867	0.346	0.0089	138.7	1.50	0.0444
		850	0.415	0.862	0.357	0.0098	151.9	1.41	0.0471
		900	0.429	0.857	0.367	0.0107	165.4	1.34	0.0500
12	16,000	850	0.389	0.870	0.338	0.0103	143.2	1.57	0.0531
		900	0.402	0.866	0.348	0.0113	156.5	1.48	0.0560
		950	0.415	0.862	0.357	0.0123	170.0	1.41	0.0591
		1,000	0.428	0.857	0.366	0.0134	183.5	1.33	0.0623
		1,050	0.441	0.853	0.375	0.0145	197.3	1.27	0.0658
		1,100	0.453	0.849	0.384	0.0156	211.5	1.21	0.0690
		1,150	0.464	0.845	0.392	0.0167	225.8	1.16	0.0721
		1,200	0.474	0.842	0.398	0.0178	239.8	1.11	0.0750
	18,000	850	0.361	0.880	0.318	0.0085	134.3	1.77	0.0471
		900	0.375	0.875	0.328	0.0093	146.8	1.67	0.0500
		950	0.388	0.871	0.337	0.0102	159.5	1.58	0.0529
		1,000	0.400	0.867	0.346	0.0111	172.3	1.50	0.0556
		1,050	0.412	0.863	0.355	0.0120	186.3	1.43	0.0583
		1,100	0.423	0.859	0.363	0.0129	199.4	1.37	0.0610
		1,150	0.434	0.855	0.371	0.0138	212.7	1.31	0.0639
		1,200	0.444	0.852	0.378	0.0148	226.3	1.25	0.0665
10	16,000	1,050	0.396	0.868	0.343	0.0130	180.3	1.53	0.0656
		1,100	0.407	0.864	0.351	0.0140	193.5	1.46	0.0687
		1,150	0.418	0.861	0.359	0.0150	216.9	1.39	0.0718
		1,200	0.428	0.857	0.366	0.0161	220.4	1.34	0.0749
		1,250	0.438	0.854	0.373	0.0171	234.0	1.28	0.0780
		1,300	0.448	0.851	0.380	0.0182	247.8	1.23	0.0811
		1,350	0.457	0.848	0.387	0.0193	261.8	1.19	0.0843
		1,400	0.466	0.845	0.393	0.0204	276.0	1.11	0.0875
	18,000	1,050	0.368	0.877	0.323	0.0108	170.4	1.72	0.0582
		1,100	0.379	0.874	0.331	0.0116	182.9	1.64	0.0610
		1,150	0.390	0.870	0.339	0.0125	195.5	1.57	0.0638
		1,200	0.400	0.867	0.346	0.0133	208.3	1.50	0.0666
		1,250	0.410	0.863	0.353	0.0142	221.3	1.44	0.0694
		1,300	0.419	0.860	0.360	0.0151	234.5	1.39	0.0722
		1,350	0.428	0.857	0.367	0.0161	247.8	1.34	0.0750
		1,400	0.437	0.854	0.373	0.0170	261.4	1.29	0.0778

* Applies to any type of beam.

DIAGRAM 1.

VALUES OF f_c, f_s, p, K, k AND j FOR RECTANGULAR BEAMS WITH TENSION STEEL ONLY.

$n = 15$

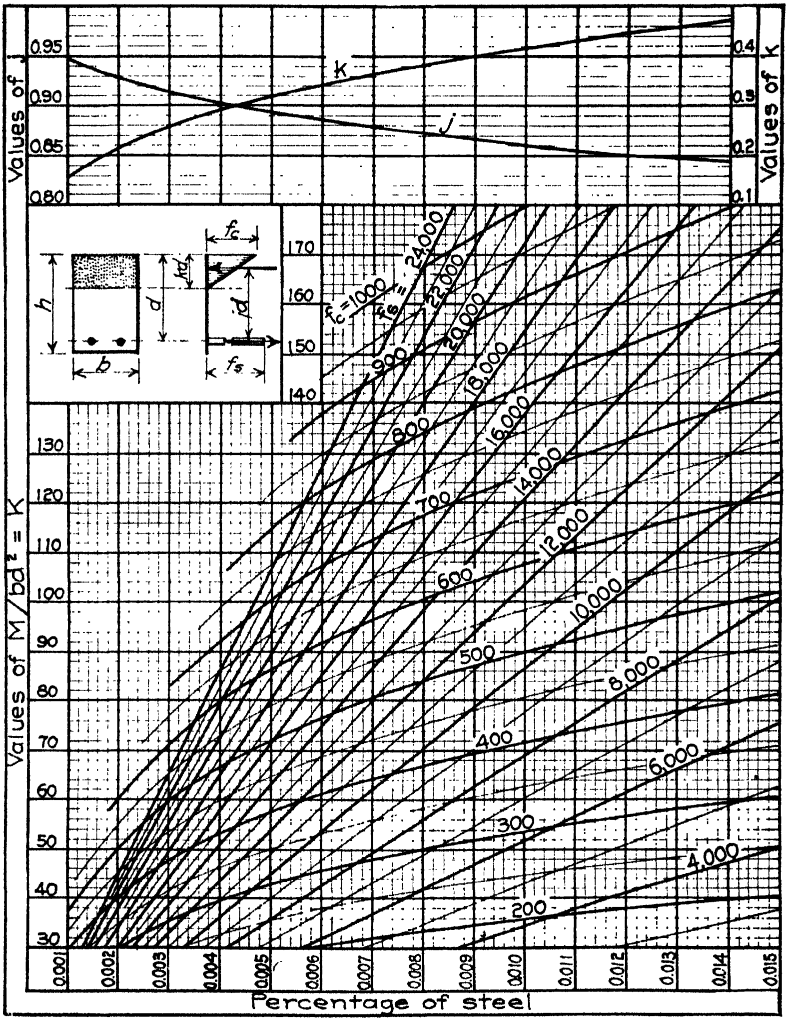


Table 2.—T-beams

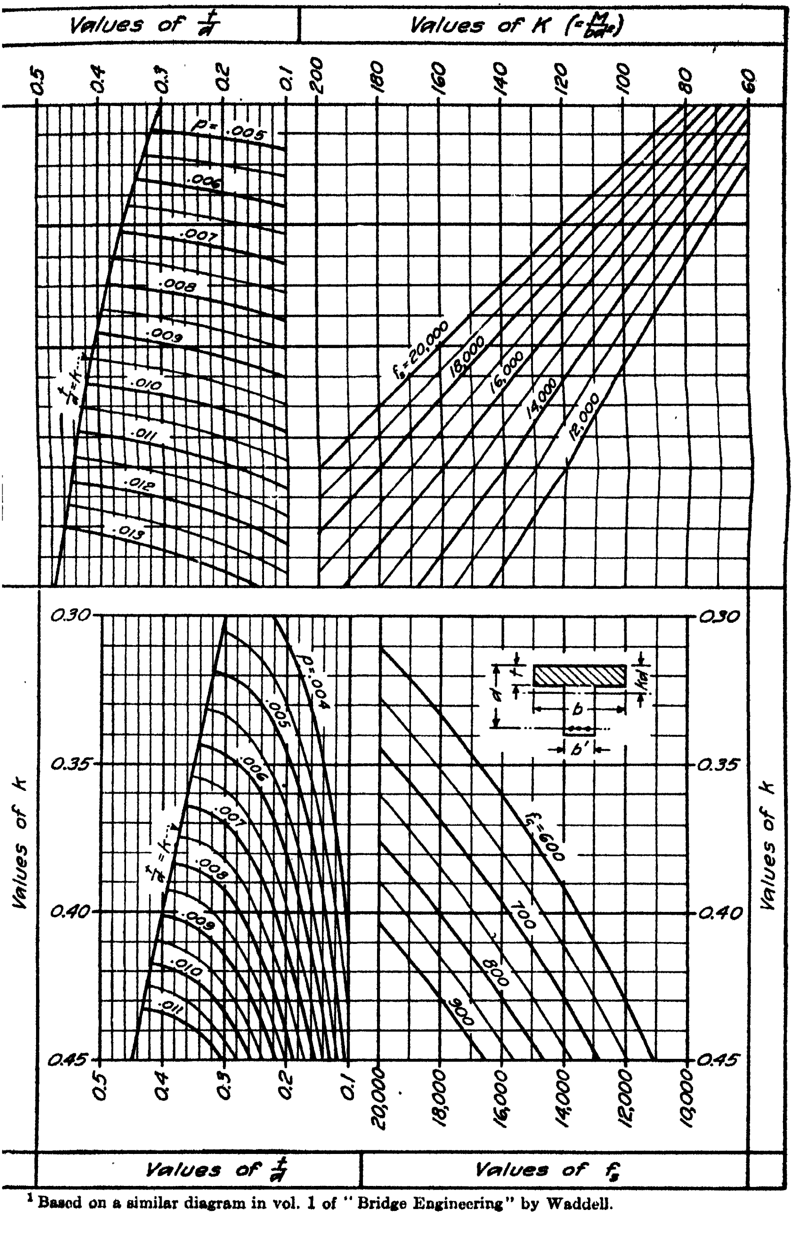
$n = 15$

Values of $\frac{t}{d}$		Values of K, j, and p							
		$f_s = 16{,}000$				$f_s = 18{,}000$			
		$f_c = 650$	700	750	800	650	700	750	800
0.16	K	76.4	83.0	89.7	96.6	74.5	81.3	87.7	94.7
	j	0.927	0.927	0.926	0.0926	0.928	0.928	0.927	0.926
	p	0.00513	0.00558	0.00606	0.00650	0.00445	0.00486	0.00527	0.00570
0.18	K	82.1	89.5	97.0	104.6	79.9	87.4	95.1	102.7
	j	0.919	0.919	0.918	0.917	0.921	0.920	0.919	0.918
	p	0.00557	0.00608	0.00659	0.00710	0.00483	0.00527	0.00574	0.00620
0.20	K	87.0	95.4	103.6	111.4	84.8	92.9	101.1	109.3
	j	0.912	0.911	0.910	0.909	0.914	0.913	0.912	0.911
	p	0.00597	0.00653	0.00709	0.00765	0.00516	0.00566	0.00616	0.00666
0.22	K	91.7	100.7	109.2	118.0	89.0	97.6	106.4	115.3
	j	0.905	0.904	0.903	0.902	0.908	0.906	0.905	0.904
	p	0.00634	0.00696	0.00756	0.00818	0.00545	0.00600	0.00654	0.00709
0.24	K	95.7	105.1	114.3	123.8	92.4	101.9	111.2	120.5
	j	0.899	0.897	0.896	0 895	0.902	0.900	0.899	0.897
	p	0.00666	0.00733	0.00798	0.00863	0.00570	0.00630	0.00688	0.00746
0.26	K	98.9	108.8	118.9	128.7	95.2	105.2	115.1	125.0
	j	0.893	0.891	0.890	0.888	0.897	0 895	0 893	0.891
	p	0.00694	0.00765	0.00834	0.00905	0.00591	0.00655	0.00717	0.00780
0.28	K	101.7	112.1	122.7	133 2	97.4	107.9	118.5	128.7
	j	0.888	0.885	0.884	0.882	0 892	0.890	0.888	0.885
	p	0.00717	0.00792	0.00867	0.00941	0.00607	0.00675	0.00742	0.00809
0.30	K	103.8	114.9	125.9	136.8	99.1	110 0	121.1	132.0
	j	0 883	0.880	0.879	0.876	0.889	0.886	0.883	0 880
	p	0.00736	0.00815	0.00895	0.00975	0.00621	0.00692	0.00762	0 00833
0.32	K	105.5	117.0	128.4	139.8	100 2	111.5	123.0	134.5
	j	0.879	0.876	0.874	0.871	0.887	0 882	0.879	0.876
	p	0.00751	0.00834	0.00919	0.01002	0.00629	0.00705	0 00778	0.00853
0.34	K	106.6	118.6	130.4	142 3	100.7	112.5	124 4	136 4
	j	0.876	0.872	0.870	0.866	0.885	0.880	0.876	0.872
	p	0.00761	0.00849	0.00937	0.01025	0.00633	0.00715	0.00790	0.00869
0.36	K	107.3	119.7	131.8	144.2	....	113.0	125.2	137.6
	j	0.874	0.870	0.867	0.862	....	0 879	0 875	0.870
	p	0.00767	0.00859	0.00952	0.01043	....	0.00716	0.00798	0.00880
0.38	K	....	120.2	132.9	145.6	....	....	125.7	138.3
	j	....	0.868	0.865	0 859	....	....	0.874	0.868
	p	....	0.00867	0.00961	0.01058	....	....	0.00801	0.00886
0.40	K	....	....	133 4	146.5	....	....	....	138.7
	j	....	....	0.863	0.857	....	....	....	0.867
	p	....	....	0.00966	0.01067	....	....	....	0.00889
0.42	K	....	....	....	146.9				
	j	....	....	....	0.855				
	p	....	....	....	0.01071				
k	K	107.52	120.36	133.51	146 94	100.81	113.11	125.74	138.66
	j	0.874	0.868	0.862	0.857	0.883	0.877	0.872	0.867
	p	0.00769	0.00867	0.00968	0.01071	0.00634	0.00716	0.00801	0.00889
All values	k	0.3786	0.3962	0.4128	0.4287	0.3513	0.3684	0.3846	0.4000

DIAGRAM 2.

DESIGN DIAGRAM FOR T-BEAMS.

(Case II)[1] $n = 15$



[1] Based on a similar diagram in vol. 1 of "Bridge Engineering" by Waddell.

TABLE 3.—BEAMS REINFORCED FOR COMPRESSION AND TENSION

Values of p and p' for Various Values of K/f_s and K/f_c

Values of K/f_c		Values of $K \div f_s$: 0.007	0.008	0.009	0.010	0.011	0.012	0.013	0.014	0.015	0.016	0.017	0.018	0.019	0.020	0.021	0.022
0.18	p'	0.0018	0.0008														
	p	0.0080	0.0092														
0.20	p	0.0049	0.0036	0.0026	0.0016												
	p	0.0079	0.0091	0.0104	0.0117												
0.22	p'	0.0082	0.0067	0.0054	0.0044	0.0035											
	p	0.0078	0.0090	0.0103	0.0115	0.0128											
0.24	p'	0.0114	0.0096	0.0084	0.0072	0.0063	0.0055										
	p	0.0078	0.0089	0.0102	0.0114	0.0127	0.0138										
0.26	p'	......	0.0130	0.0115	0.0100	0.0090	0.0081	0.0073									
	p	......	0.0089	0.0101	0.0113	0.0126	0.0137	0.0149									
0.28	p'	......	......	0.0145	0.0131	0.0119	0.0110	0.0100	0.0093								
	p	......	......	0.0101	0.0113	0.0125	0.0136	0.0148	0.0160								
0.30	p'	......	......	......	0.0160	0.0149	0.0148	0.0128	0.0120	0.0113							
	p	......	......	......	0.0112	0.0124	0.0135	0.0147	0.0159	0.0172							
0.32	p'	......	......	......	......	0.0178	0.0166	0.0156	0.0147	0.0138	0.0133						
	p	......	......	......	......	0.0123	0.0134	0.0147	0.0158	0.0171	0.0184						
0.34	p'	......	......	......	......	......	0.0195	0.0184	0.0175	0.0165	0.0158	0.0151					
	p	......	......	......	......	......	0.0134	0.0146	0.0157	0.0170	0.0183	0.0194					
0.36	p'	......	......	......	......	......	0.0225	0.0213	0.0202	0.0193	0.0185	0.0178	0.0171				
	p	......	......	......	......	......	0.0133	0.0146	0.0157	0.0169	0.0182	0.0193	0.0205				
0.38	p'	......	......	......	......	......	......	0.0242	0.0232	0.0220	0.0212	0.0203	0.0196	0.0189			
	p	......	......	......	......	......	......	0.0146	0.0157	0.0168	0.0181	0.0192	0.0204	0.0217			
0.40	p'	......	......	......	......	......	......	......	0.0258	0.0247	0.0238	0.0230	0.0223	0.0215	0.0208		
	p	......	......	......	......	......	......	......	0.0157	0.0168	0.0180	0.0192	0.0203	0.0216	0.0228		
0.42	p'	......	......	......	......	......	......	......	0.0288	0.0277	0.0266	0.0257	0.0248	0.0241	0.0235	0.0228	
	p	......	......	......	......	......	......	......	0.0156	0.0167	0.0179	0.0191	0.0203	0.0215	0.0227	0.0238	
0.44	p'	......	......	......	......	......	......	......	......	0.0305	0.0294	0.0283	0.0275	0.0268	0.0262	0.0263	0.024
	p	......	......	......	......	......	......	......	......	0.0167	0.0179	0.0191	0.0202	0.0214	0.0226	0.0237	0.025
0.46	p'	......	......	......	......	......	......	......	......	......	0.0322	0.0310	0.0305	0.0296	0.0290	0.0281	0.027
	p	......	......	......	......	......	......	......	......	......	0.0178	0.0190	0.0202	0.0213	0.0225	0.0237	0.024

$n = 15$

$\dfrac{d'}{d} = 1.0$

TABLE 4.—BEAMS REINFORCED FOR COMPRESSION AND TENSION

Values of p and p' for Various Values of K/f_s and K/f_c

Values of K/f_c		Values of $K \div f_s$: 0.006	0.007	0.008	0.009	0.010	0.011	0.012	0.013	0.014	0.015	0.016	0.017	0.018
0.16	p'	0.0003	0.0000	0.0000										
	p	0.0070	0.0081	0.0093										
0.18	p'	0.0072	0.0039	0.0012	0.0000									
	p	0.0070	0.0081	0.0093	0.0105									
0.20	p'	0.0148	0.0100	0.0068	0.0043	0.0025								
	p	0.0070	0.0082	0.0094	0.0105	0.0118								
0.22	p'	0.0247	0.0169	0.0128	0.0096	0.0074	0.0055							
	p	0.0071	0.0083	0.0094	0.0106	0.0118	0.0130							
0.24	p'	...	0.0260	0.0194	0.0153	0.0127	0.0103	0.0085						
	p	...	0.0083	0.0094	0.0106	0.0118	0.0130	0.0142						
0.26	p'	...	...	0.0276	0.0220	0.0184	0.0155	0.0133	0.0115					
	p	...	...	0.0095	0.0107	0.0118	0.0130	0.0143	0.0154					
0.28	p'	...	...	...	0.0298	0.0248	0.0212	0.0187	0.0166	0.0147				
	p	...	...	...	0.0107	0.0119	0.0131	0.0143	0.0155	0.0167				
0.30	p'	...	...	...	...	0.0320	0.0279	0.0243	0.0217	0.0197	0.0178			
	p	...	...	...	...	0.0119	0.0131	0.0143	0.0156	0.0188	0.0180			
0.32	p'	...	...	...	...	...	0.0354	0.0306	0.0275	0.0249	0.0228	0.0210		
	p	...	...	...	...	...	0.0132	0.0144	0.0157	0.0168	0.0180	0.0193		
0.34	p'	...	...	...	...	...	...	0.0376	0.0342	0.0306	0.0280	0.0260	0.0241	
	p	...	...	...	...	...	...	0.0144	0.0157	0.0169	0.0181	0.0193	0.0205	
0.36	p'	...	...	...	...	...	...	...	...	0.0369	0.0332	0.0313	0.0291	0.0272
	p	...	...	...	...	...	...	...	...	0.0169	0.0181	0.0194	0.0205	0.0218
0.38	p'	...	...	...	...	...	...	...	...	...	...	0.0370	0.0345	0.0327
	p	...	...	...	...	...	...	...	...	...	...	0.0194	0.0206	0.0218

$\frac{d'}{d} = 0.2$

$n = 15$

TABLE 5.—AREAS AND PERIMETERS OF 1, 2, 3 OR 12 BARS

A_s (sq. in.)				Size of bar	Σ_0 (sq. in. per in.)		
1	2	3	12	No. of bars	1	2	3
0.1104	0.22	0.33	1.32	⅜ φ	1.178	2.36	3.53
0.1963	0.39	0.59	2.36	½ φ	1.571	3.14	4.71
0.2500	0.50	0.75	3.00	½ □	2.000	4.00	6.00
0.3068	0.61	0.92	3.68	⅝ φ	1.964	3.93	5.89
0.4418	0.88	1.33	5.30	¾ φ	2.356	4.71	7.07
0.6013	1.20	1.80	7.22	⅞ φ	2.749	5.50	8.25
0.7845	1.57	2.36	9.41	1 φ	3.142	6.28	9.42
1.0000	2.00	3.00	12.00	1 □	4.000	8.00	12.00
1.2656	2.53	3.80	15.18	1⅛ □	4.500	9.00	13.50
1.5625	3.12	4.69	18.75	1¼ □	5.000	10.00	15.00

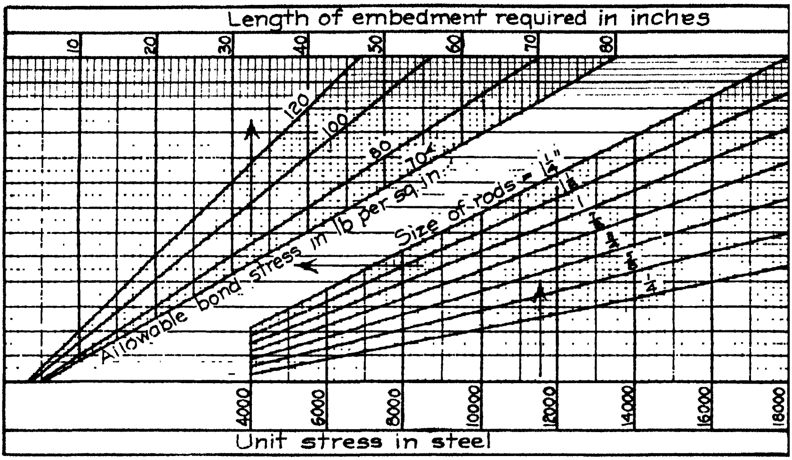
TABLE 6.—AREAS, PERIMETERS AND WEIGHTS, COMBINATION OF FOUR BARS

Bar sizes									A_s (sq. in.)	Σ_0 (sq. in.) per in.	Weight (lb.) per ft.
½ φ	½ □	⅝ φ	¾ φ	⅞ φ	1 φ	1 □	1⅛ □	1¼ □			
4	..	..	..	..	..	..	..	..	0.78	6.28	2.67
2	2	..	..	..	..	..	..	..	0.89	7.14	3.03
2	..	2	..	..	..	..	..	..	1.00	7.07	3.42
	4	..	..	..	..	..	..	..	1.00	8.00	3.40
	2	2	..	..	..	..	..	..	1.11	7.93	3.79
	..	4	..	..	..	..	..	..	1.23	7.86	4.17
2	..	..	2	..	..	..	..	..	1.28	7.85	4.34
	2	..	2	..	..	..	..	..	1.38	8.71	4.70
		2	2	..	..	..	..	..	1.50	8.61	5.09
	2	..	..	2	..	..	..	..	1.70	9.50	5.79
			4	..	..	..	..	..	1.77	9.42	6.01
		2	..	2	..	..	..	..	1.82	9.43	6.17
			2	2	..	..	..	..	2.09	10.21	7.09
		2	..	..	2	..	..	..	2.18	10.21	7.40
				4	..	..	..	..	2.40	11.00	8.18
			2	..	2	..	..	..	2.45	11.00	8.34
				2	2	..	..	..	2.77	11.78	9.42
			2	..	..	2	..	..	2.88	12.71	9.80
					4	..	..	..	3.14	12.56	10.67
				2	..	2	..	..	3.20	13.49	10.89
					2	2	..	..	3.57	14.28	12.13
				2	..	..	2	..	3.74	14.49	12.69
						4	..	..	4.00	16.00	13.60
					2	..	2	..	4.10	15.28	13.94
						2	2	..	4.53	17.00	15.41
					2	..	..	2	4.70	16.28	15.96
							4	..	5.06	18.00	17.21
						2	..	2	5.13	18.00	17.42
							2	2	5.66	19.00	19.23
								4	6.25	20.00	21.25

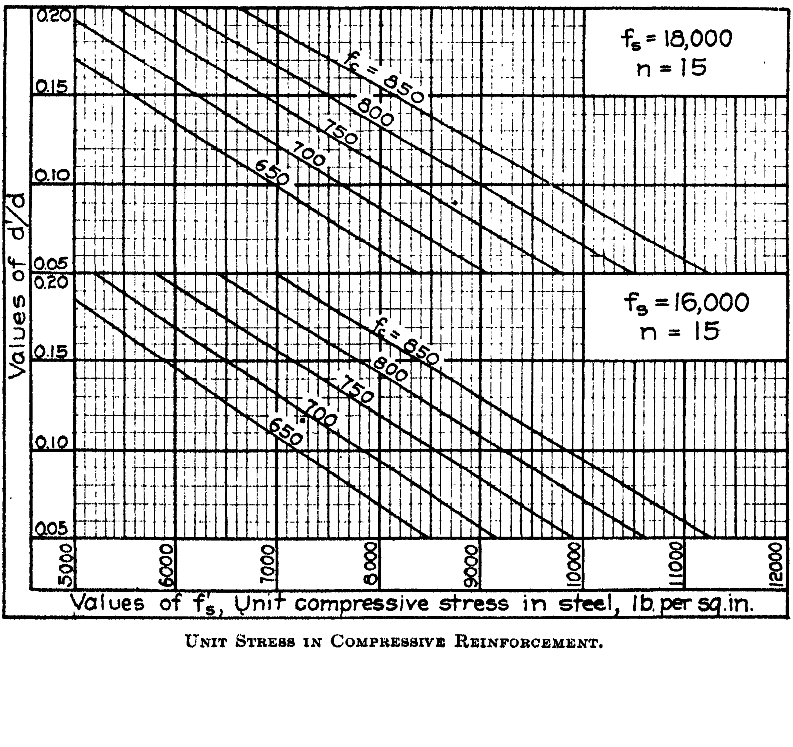
TABLE 7.—AREAS, PERIMETERS AND WEIGHTS, COMBINATION OF SIX BARS

Bar sizes						A_s (sq. in.)	Σ_0 (sq. in. per in.)	Weight (lb. per ft.)
¾ φ	⅞ φ	1 φ	1 □	1⅛ □	1¼ □			
6	..	..	..	..	..	2.65	14.14	9.01
3	3	..	..	..	..	3.13	15.31	10.64
	6	..	..	..	..	3.61	16.49	12.26
3	..	3	..	..	..	3.68	16.49	12.50
2	..	4	..	..	..	4.02	17.27	13.67
	3	3	..	..	..	4.16	17.67	14.13
3	..	..	3	..	..	4.32	19.07	14.71
		6	..	..	..	4.71	18.84	16.00
	3	..	3	..	..	4.80	20.24	16.33
2	..	..	4	..	..	4.88	20.71	16.60
		4	2	..	..	5.14	20.56	17.47
	2	..	4	..	..	5.20	21.50	17.77
		3	3	..	..	5.36	21.42	18.20
	3	..	..	3	..	5.60	21.74	19.04
		4	..	2	..	5.67	21.56	19.27
			6	..	..	6.00	24.00	20.40
		3	..	3	..	6.15	22.92	20.91
	2	..	..	4	..	6.26	23.50	21.30
			4	2	..	6.53	25.00	22.21
			3	3	..	6.80	25.50	23.11
		3	..	..	3	7.04	24.42	23.94
			4	..	2	7.13	26.00	24.22
				6	..	7.60	27.00	25.82
			3	..	3	7.69	27.00	26.14
		2	..	..	4	7.82	26.28	26.58
				4	2	8.19	28.00	27.84
			2	..	4	8.25	28.00	28.05
				3	3	8.48	28.50	28.84
				2	4	8.78	29.00	29.85
					6	9.38	30.00	31.87

DIAGRAM 3.



LENGTH OF EMBEDMENT TO TAKE OFF STRESS IN BOND.



UNIT STRESS IN COMPRESSIVE REINFORCEMENT.

DIAGRAM 4.

SHEARING RESISTANCE OF BENT-UP BARS AT VARIOUS SLOPES.

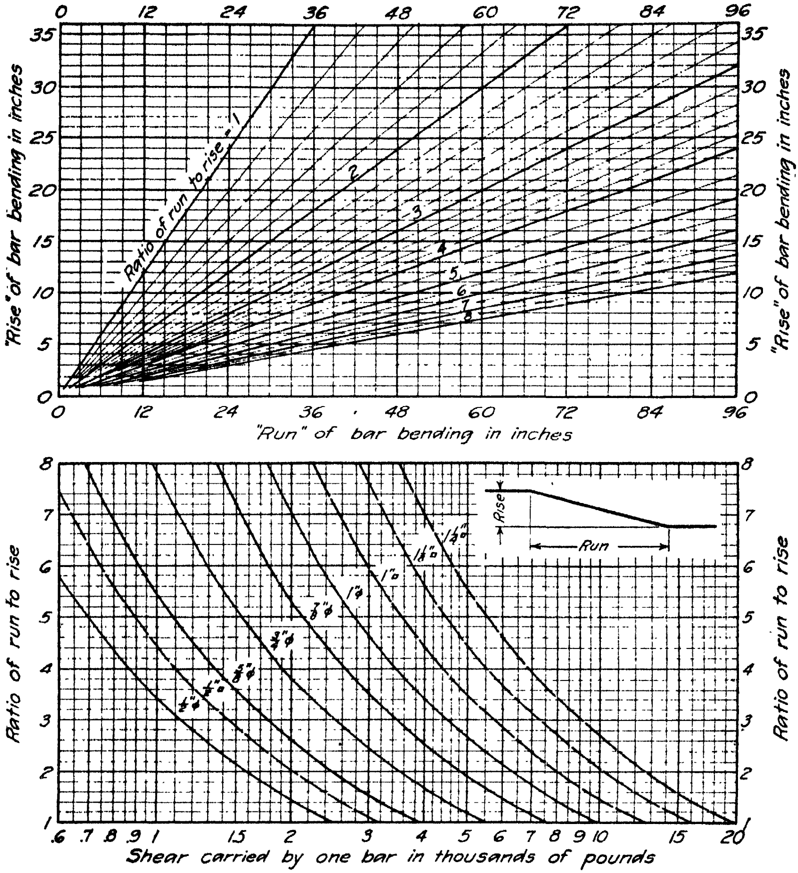


DIAGRAM 5.

NUMBER AND SPACING OF VERTICAL STIRRUPS UNDER UNIFORM LOAD.

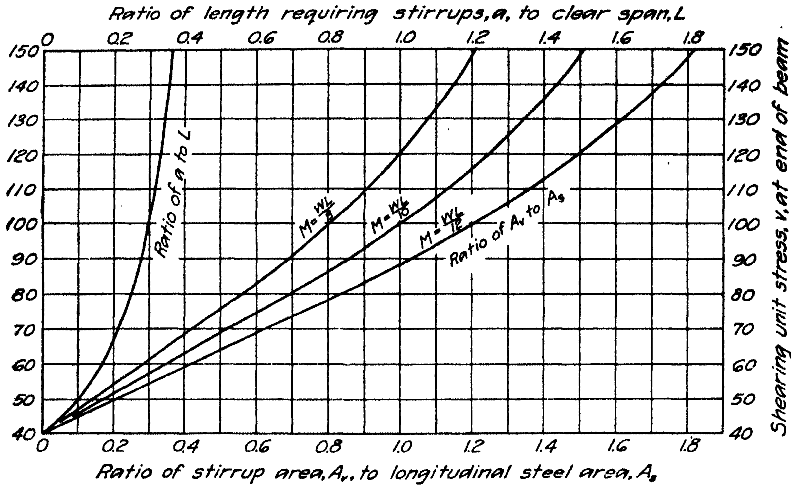


DIAGRAM 5.—*Continued*

Number of stirrups at each end	Distance of 1st stirrup from edge of support	Spacing c. to c. stirrups in terms of "a": Group 1 No.	Group 1 @ Spacing	Group 2 No.	Group 2 @ Spacing	Group 3 No.	Group 3 @ Spacing	Group 4 No.	Group 4 @ Spacing	Group 5 No.	Group 5 @ Spacing	Group 6 No.	Group 6 @ Spacing	Total No. of stirrups	Area A_v of all stirrups in beam—U shaped: ¼" round	⁵⁄₁₆" round	⅜" round	½" round	½" square	⅝" round
20	0.013a	7	0.028a	5	0.035a	3	0.046a	2	0.062a	1	0.081a	1	0.114a	40	3.93	6.14	8.84	15.71	20.00	24.55
19	0.014a	7	0.03 a	5	0.038	3	0.052	2	0.075	1	0.117	..	.....	38	3.73	5.83	8.40	14.92	19.00	23.32
18	0.015a	7	0.031a	5	0.043	3	0.06	1	0.085	1	0.122	..	.....	36	3.54	5.52	7.95	14.14	18.00	22.10
17	0.015a	6	0.033a	5	0.044	3	0.061	1	0.086	1	0.127	..	.....	34	3.34	5.21	7.51	13.36	17.00	20.87
16	0.016a	6	0.035a	4	0.047	3	0.062	1	0.087	1	0.13	..	.....	32	3.14	4.91	7.07	12.58	16.00	19.64
15	0.017a	6	0.038a	4	0.052	2	0.07	1	0.091	1	0.133	..	.....	30	2.95	4.60	6.63	11.79	15.00	18.41
14	0.018a	5	0.04 a	4	0.054	2	0.072	1	0.094	1	0.138	..	.....	28	2.75	4.29	6.19	11.00	14.00	17.19
13	0.020a	5	0.044a	3	0.057	2	0.076	1	0.099	1	0.141	..	.....	26	2.55	3.99	5.75	10.21	13.00	15.96
12	0.022a	5	0.048a	3	0.066	2	0.092	1	0.151	..	......	..	.....	24	2.36	3.68	5.30	9.43	12.00	14.73
11	0.024a	5	0.053a	3	0.077	1	0.11	1	0.156	..	......	..	.....	22	2.16	3.37	4.86	8.64	11.00	13.50
10	0.026a	4	•0.058a	3	0.08	1	0.114	1	0.164	..	......	..	.....	20	1.96	3.07	4.42	7.86	10.00	12.28
9	0.029a	4	0.066a	2	0.09	1	0.117	1	0.174	..	......	..	.....	18	1.77	2.76	3.98	7.07	9.00	11.05
8	0.032a	3	0.073a	2	0.095	1	0.126	1	0.183	..	......	..	.....	16	1.57	2.45	3.53	6.29	8.00	9.83
7	0.037a	3	0.085a	2	0.122	1	0.197	..	......	..	......	..	.....	14	1.37	2.14	3.09	5.50	7.00	8.60
6	0.043a	3	0.104a	1	0.145	1	0.212	..	......	..	......	..	.....	12	1.18	1.84	2.65	4.71	6.00	7.37
5	0.052a	2	0.12 a	1	0.161	1	0.23	..	......	..	......	..	.....	10	0.98	1.53	2.21	3.93	5.00	6.14
4	0.065a	1	0.145a	1	0.178	1	0.258	..	......	..	......	..	.....	8	0.78	1.22	1.77	3.14	4.00	4.91
3	0.095a	1	0.198a	1	0.297	..	......	..	......	..	......	..	.....	6	0.59	0.92	1.33	2.36	3.00	3.68
2	0.147a	1	0.353a	..	......	..	......	..	......	..	......	..	.....	4	0.39	0.61	0.88	1.57	2.00	2.46
1	0.333a	..	......	..	......	..	......	..	......	..	......	..	.....	2	0.19	0.30	0.44	0.78	1.00	1.23
			For plain stirrups d must equal or exceed →												12.5"	15.5"	18.8"	25.0"	25.0"	31.1"
			For deformed stirrups d must equal or exceed →												10.0"	12.5"	15.6"	20.0"	20.0"	26.0"

DIAGRAM 6.

PROPORTIONAL PART OF LONGITUDINAL STEEL PERMITTED BY MOMENT TO BE BENT DOWN NEAR SUPPORT.

NOTE.—By reversing terms *bent* and *straight* in left-hand legend, the proportional part of longitudinal steel permitted by moment to be bent up at sections away from the center may be determined from this diagram.

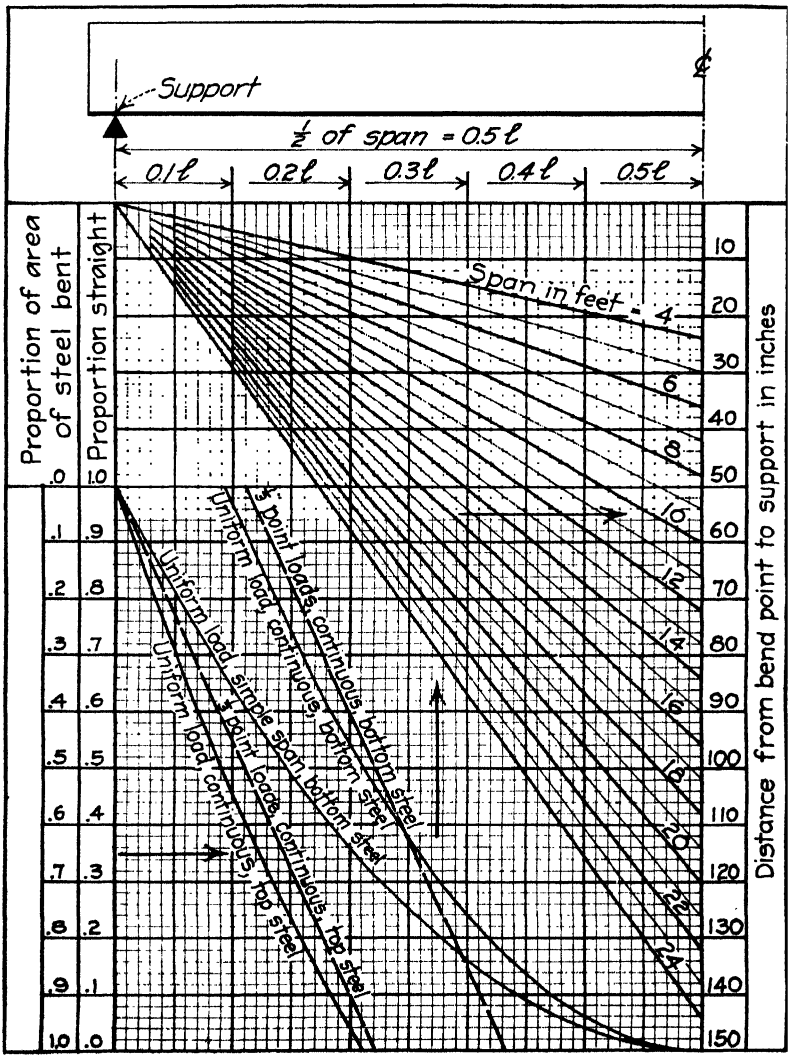
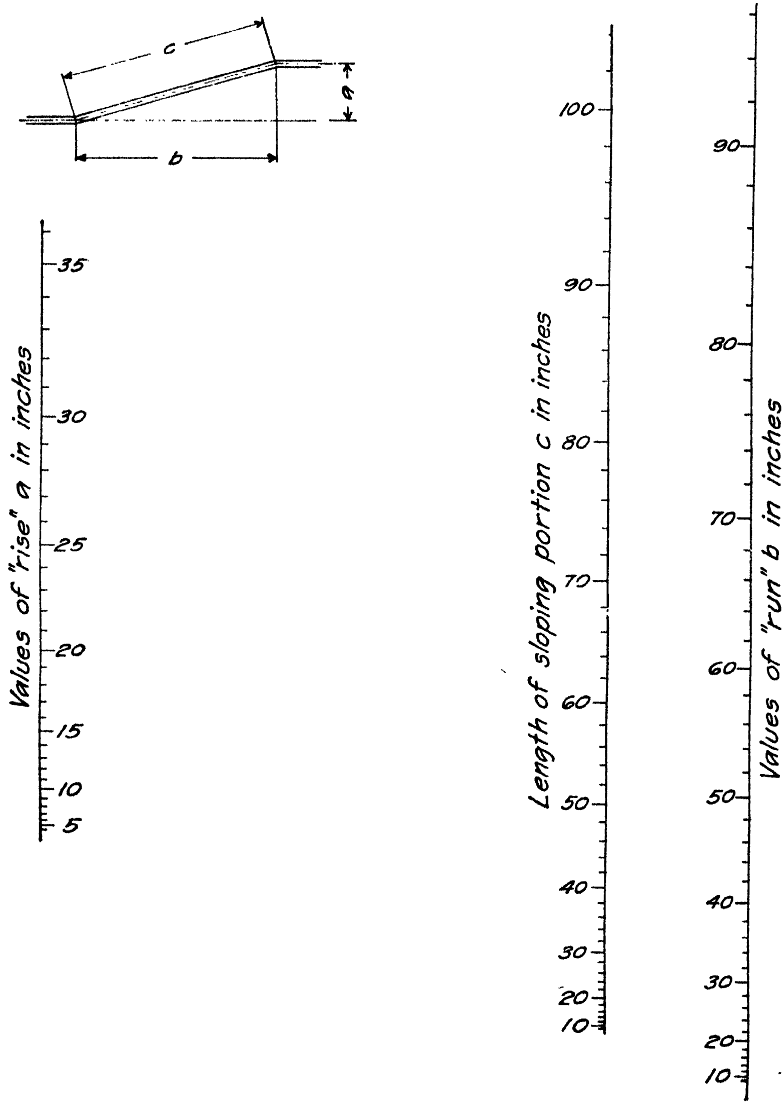


Diagram 7.

Length of Sloping Portion of Longitudinal Beam Bars.

Note.—Set straight edge on any two known quantities and read concurrent value of third quantity.



For small values use some multiplying factor for increased accuracy of reading.

DIAGRAM 8.

APPROXIMATE DIMENSIONS OF RIBBED SLABS AS GOVERNED BY SHEARING STRESS OF 40 LB. PER SQ. IN.

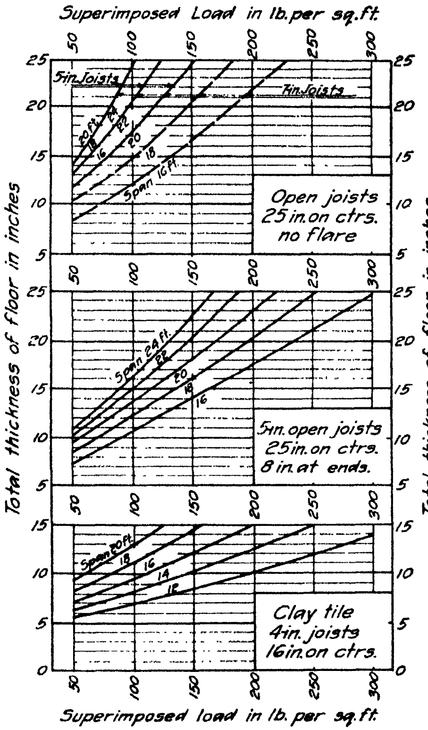
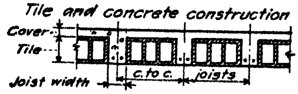
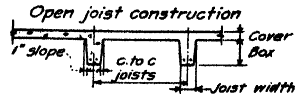


TABLE 8.—WEIGHTS OF RIBBED SLAB CONSTRUCTION





Depth		Width of joist[1] (in.)	Spacing c. to c. joists (in.)	Weight		Depth		Width of joist[2] (in.)	Spacing c. to c. joists (in.)	Weight	
Tile (in.)	Cover (in.)			Per foot of joist (lb.)	Per sq. ft. of floor (lb.)	Box (in.)	Cover (in.)			Per foot of joist (lb.)	Per sq. ft. of floor (lb.)
4 +	2	4	16	66	50	8 +	2	4	24	80	40
4 +	3	4	16	82	62			5	25	90	43
5 +	2	4	16	72	54	8 +	3	4	24	104	52
5 +	3	4	16	88	66			5	25	115	55
6 +	2	4	16	78	58	10 +	2	4	24	88	44
		5	17	86	61			5	25	100	48
6 +	3	4	16	90	68	10 +	3	4	24	112	56
		5	17	103	73			5	25	125	60
8 +	0	4	16	59	44			6	26	138	64
		6	18	75	50	12 +	2	4	24	96	48
8 +	2	4	16	91	68			5	25	110	53
		6	18	111	71			6	26	124	57
8 +	3	4	16	107	80	12 +	3	4	24	120	60
		6	18	129	86			5	25	135	65
10 +	0	4	16	72	54			6	26	150	69
		6	18	92	61	15 +	2	5	25	125	60
10 +	2	4	16	101	78			6	26	142	66
		6	18	128	85			7	27	159	71
10 +	3	4	16	120	90	15 +	3	5	25	150	72
		6	18	146	97			6	26	168	78
12 +	0	4	16	84	63			7	27	186	83
		6	18	108	72	18 +	3	6	26	186	86
		8	20	132	80			7	27	207	92
12 +	2	4	16	116	87			8	28	228	98
		6	18	144	96	21 +	3	6	26	204	91
12 +	3	4	16	132	104			8	28	252	108
		6	18	162	108			10	30	300	120

[1] Width of joist in shear may be taken as 1 in. larger than joist width.

[2] Width of joist in shear may be taken as 1 in. larger than joist bottom width if joists are flared at top as shown above.

Three-inch cover is required to embed conduits where no separate floor finish is used with open joist construction.

SLABS SUPPORTED UPON FOUR EDGES

When designed in accordance with the provisions of most city codes, the two way slab supported at its four edges, is uneconomical and its use is therefore limited. It has long been known, however, that such slabs possess a surprisingly high load-carrying capacity and it is the conviction of engineers familiar with tests that such members should be figured as slabs and not as beam strips. The design of such members is therefore in a period of transition, the old formulas being in disrepute while no agreement has been reached among engineers generally as to the proper moment coefficients to use in the light of recent test data.

35. Beam-strip Method of Design.—Considered purely as a member, using the beam-strip method of design, with the bending moment diagram fully known, there are no new design problems involved. The formulas are the same as those stated for one-way slabs in Art. 1, p. 432, and the design procedure is the same, except that the weight of the slab is considered to be distributed between the two sets of reinforcing bars in the same proportion as the superimposed load. For square slabs the load is considered as equally divided between the two directions. For rectangular slabs, the Joint Committee report of 1916 recommends the following division of the load between the long and short directions:

$$r = \frac{a}{b} - 0.5$$

in which r = proportion of total load carried by shorter span. a = length of longer span. b = length of shorter span.

Figure 11 gives the distribution of the load between the long and short directions in accordance with the formula above.

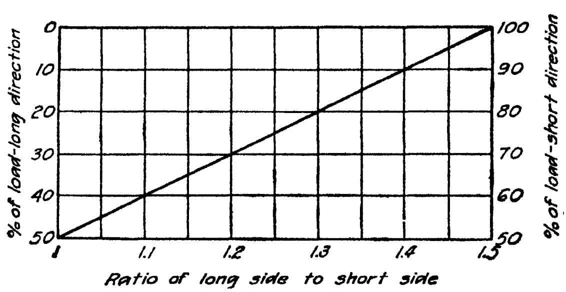


Fig. 11.—Load distribution in rectangular slabs.

The formula $r = \frac{a}{2b}$ is an average of a number of formulas in general use and is considered by many to represent the actual distribution of load more closely than the Joint Committee formula.

When designing by present beam-strip methods, some allowance is made for the decrease in moment near the supported edges. A common allowance is to permit of a 20 per cent reduction in total steel area, effected by an increased spacing of the bars in the outer quarters of the panel.

36. Slab-analysis Methods of Design.—The proper design of such slabs by true slab-analysis methods has not yet been standardized and various proposals are continually being made. The designer is advised to consult an analysis by Prof. Westergaard of square and rectangular slabs.[1]

37. Detailing the Reinforcing Steel.—In detailing the reinforcing steel for two-way slabs, considerable confusion in the field may be avoided by using straight bars in the outer quarters of the panel and only bending up the bars near the center. Bent-up bars at the corner run afoul of each other. The steel for negative moment over the column may well be bent into the shape of squares (or rectangles), the bending being all in a horizontal plane. These squares may then be laid directly upon the beam steel, thus saving one layer and increasing the value of d for both beam and slab.

[1] Proceedings A.C.I., vol. 17, 1921, pp. 430–439.

Illustrative Problem.—Design a two-way interior floor panel 16 ft. 0 in. × 20 ft. 0 in., supported on each edge on steel I-beams and carrying a superimposed load of 200 lb. per sq. ft. Use Chicago stresses[1] and take the positive and negative moment on each beam strip as $\frac{wl^2}{12}$. The total steel may be reduced 20 per cent. Use 1 in. for protective cover.

For solution see Design Sheet 9. The following notes apply to that sheet: (*a*) Since the supports are I-beams and not monolithic the span is the full panel dimension; (*b*) the steel in the long direction will be placed above that in the short direction at the center and below it at the panel edges.

DESIGN SHEET 9

$a = 20' - 0''\ \ b = 16' - 0''\ \ a/b = 1.25\ \ r = 1.25 - 0.5 = 0.75$

Superimposed load $= 200$

Assume 8″ slab $= \underline{100}$ $\quad 300^{\#/\square'}$

Load in short direction $= (0.75)(300) = 225^{\#/\square'}$

Load in long direction $= (0.25)(300) = 75^{\#/\square'}$

$f_c = 700\ \ f_s = 18{,}000\ \ n = 15\ \ K = 113.1\ \ p = 0.0072$ (Table 1, p. 466)

Short Way:

$$M = (225)^{(a)}\,(16)^2 \left(\frac{12}{12}\right) = 57{,}600^{\prime\prime\#/\prime}$$

$$b = 12''.\quad d^2 = \frac{57{,}600}{(12)\,(113.1)} = 42.5.\quad d = 6.6'' + 1.4 = 8''\text{ slab.}\quad \text{O.K.}$$

$$A_s = (0.0072)(12)(6.6) = 0.57^{\square''/'} = \tfrac{1}{2}''^{\square}\text{ at } 5\tfrac{1}{4}''\text{ on centers.}$$

Outer quarters $= (0.6)(0.57) = 0.34^{\square''/'} = \frac{1}{2}''^{\square}$ at 9″ on centers.

Long Way:

$$M = (75)(20)^2 \left(\frac{12}{12}\right) = 30{,}000^{\prime\prime\#/\prime}$$

$$d = 8 - 1 - 0.75 = 6.25''^{(b)}$$

$$A_s = \frac{30{,}000}{(18{,}000)(\tfrac{7}{8})(6.25)} = 0.305^{\square''/'} = \tfrac{1}{2}''\phi\text{ at } 7\tfrac{3}{4}''\text{ on centers.}$$

Outer quarters $= (0.6)(0.305) = 0.183^{\square''} = \frac{1}{2}''\phi$ at 13″ on centers.

FLAT SLAB FLOOR PANELS

38. General Description.—A flat slab floor consists of a slab of concrete resting directly upon regularly spaced columns without supporting beams or girders except at its edges, or to carry heavy concentrations of load. The further limitation that the percentage of reinforcing steel shall not exceed $p = .01$ in any direction at any point is commonly applied. The more common types are distinguished by:

1. The upper ends of the columns are flared out into column capitals.
2. The slab surrounding the column is thickened by means of a square or rectangular *drop*.
3. The slab at the center may be made thinner by a panelled ceiling effect (this is used only under very heavy loadings).
4. The slab may be lightened by means of hollow tile or other fillers, in a manner generally similar to ribbed slab construction.

[1] See table on p. 436.

As flat slabs have or do not have the various features listed above they are distinguished by various names, as follows:

1. *Drop construction* in which column capitals and drops are both present. This is the most common type.

2. *Cap construction* has the column capital but no drop.

3. *Column construction* without either column capitals or drops.

4. *Panel construction*, same as drop construction but with slab reduced in thickness at center by means of panelled ceiling.

5. *Tile construction*, in which tile fillers are used to lighten the weight and which may conform otherwise to any of the above types.

The differences in design are not great and only the drop construction will receive full treatment here. Figure 12 gives sections through four of the above types and will visualize the essential differences in construction.

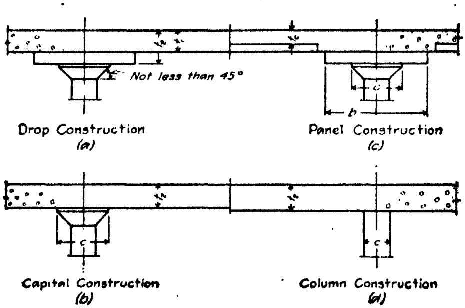


FIG. 12.—Types of flat slabs.

Flat slabs are classified in still a third way into "systems" depending upon the arrangement of their reinforcement, as follows:

1. *Four-way system*, having bands of rods running directly from column to column in both directions and also bands of rods running from column to column in both diagonal directions.

2. *Three-way system*, having bands of rods running from column to column only, but with the columns arranged at the corners of triangles.

3. *Two-way system*, having bands running directly from column to column in two directions, and having the intervening space reinforced by other sets of rods parallel to these bands.

4. *Ring system*, in which at least the portions of the slab surrounding the columns and surrounding the panel center are reinforced by series of rings or by continuous flat spirals. In some cases the remaining portions of the slab are reinforced by groups of straight rods, in others additional rings are forced to do an unnatural duty in these parts.

The four-way system has been longest in use and the patents have largely expired for this type. The two-way and three-way systems are also standardized through long and successful use. The ring system is the latest to arrive and considerable change from present practice would not be surprising, ere this type becomes standardized.

Flat slabs are used commonly for structures carrying floor loads of 100 to 150 lb. per sq. ft. or more. For the types of occupancy requiring lighter floor loads than these, the column capitals and drops are generally objectionable, while column construction which eliminates the capitals and drops is very much more expensive.

39. Comparison of Various Specifications.—In discussing the design of reinforced concrete beams and slabs as structural members, we have assumed that

the external loads, shears, and moments were fully known in all cases. In the case of flat slabs, in like manner, some specification must be selected as a guide in determining the external conditions. Of all such specifications, the Chicago flat slab ordinance comes the nearest to being in general use and has been so long in use as to demonstrate its conservative character. The American Concrete Institute Specifications are in most respects equivalent to the Chicago code and have the advantage of being so drawn as to apply to varying sizes of column capitals where the Chicago code is drawn for one fixed size, $c = 0.225L$. The New York code is much more recently adopted and its slightly lower moment requirements are offset by the lower unit stresses permitted in design. The 1921 J. C. report, while not fully considered at this writing, recognizes much the same moment coefficients for the reinforcing steel but increases them for the concrete, thus requiring thicker slabs.[1] Tables 9 and 10 give a resumé of the principal provisions of these specifications.

TABLE 9.—DESIGN PROVISIONS FOR FOUR-WAY SQUARE INTERIOR FLAT SLAB PANELS WITH DROPS AND CAPITALS. BASED UPON 2,000-LB. CONCRETE AND $n = 15$

	A.C.I.	Chicago	New York	1921 Joint Com.
Concrete stress, f_c	750	700	650	800[1]
Steel stress, f_s	16–18,000	16–18,000	16,000	16–18,000
Fixed col. capital		$0.225L$[2]	$0.225L$	
Formula No.	(64)	(65)	(64)	(64)
Slab thickness	$t_1 = 0.2L\sqrt{w} + 1''$	$t_1 = 0.0227\sqrt{W}$	$t_1 = 0.2L\sqrt{w} + 1''$	$t_1 = 0.2L\sqrt{w} + 1''$
Slab + drop, t_2	$< 1.67t_1$	$< 1.67t_1$	$> 1.33t_1$	
Min. drop width	$0.3L$	$0.33L$	$0.33L$	$0.33L$
Shearing stress[7] (edge of capital)	$100 \#/\square''$	$120 \#/\square''$		
Diagonal tension[8] (edge of drop)	$60 \#/\square''$	$60 \#/\square''$		
Formula No.	(66)	(67)	(68)	(69)
Total moment coeff.[3]	$M_0 = 0.09wl_1 (l_2 - qc)^2$	$M_0 = WL/16$	$M_0 = WL/17$	$M_0 = 0.09wL \left(l - \frac{2c}{3L}\right)^2$
$-M_c$ Column strip[4]	50–55 % M_0[6]	$WL/30$	$WL/32$	51–57 % M_0
$+M_c$ Column strip	18–20 % M_0[6]	$WL/80$	$WL/100$	18–20 % M_0
$-M_m$ Middle strip[5]	18–20 % M_0[6]	$WL/120$	$WL/133$	18–20 % M_0
$+M_m$ Middle strip	10–12 % M_0[6]	$WL/120$	$WL/100$	7–9 % M_0

1. Maximum fiber stress computed by special formula taking into account variation in intensity of stress across the column strip.

2. See Chicago code for special formula for flat slab on columns without capitals or drops.

3. The moment coefficients in table are for four-way type only (see Table 10 and various specifications for coefficients for other types).

4. Effective steel area, Chicago = one direct + one diagonal band. Other specifications = one direct + component of two diagonal bands.

5. Effective steel area, Chicago = one diagonal band. Other specifications = component of two diagonal bands.

6. The A. C. I. specificational lows 10 per cent of total moment to be assigned to sections by designer. The percentages tabulated are proper for four-way type with drop. One hundred per cent of M_0 must be used on the 4 sections.

7. Figured on vertical section through edge of column capital.

8. Figured on vertical section of depth jd on edge of drop. Figures in parentheses are formula numbers for text reference.

[1] Changed to conform with practice, Jan., 1923.

Table 10.—Design Provisions for Two-way Square Interior Flat Slab Panels with Drops and Capitals. Based on 2,000-lb. Concrete and $n = 15$

	A. C. I.	Chicago	New York	1921 Joint Com.
Total moment coeff	(66) $M_0 = 0.09wl_1 (l_2 - gc)^2$	(70) $M_0 = WL/15$	(68) $M_0 = WL/17$	(69) $M_0 = 0.09\ wL \left(1 - \frac{2}{3}\frac{c}{L}\right)^2$
$-M_c$ Column strip	50–55%	$WL/30$	$WL/32$	47–53%
$+M_c$ Column strip	18–20%	$WL/60$	$WL/80$	19–21%
$-M_m$ Middle strip	14–16%	$WL/120$	$WL/133$	14–16%
$+M_m$ Middle strip	14–16%	$WL/120$	$WL/133$	14–16%

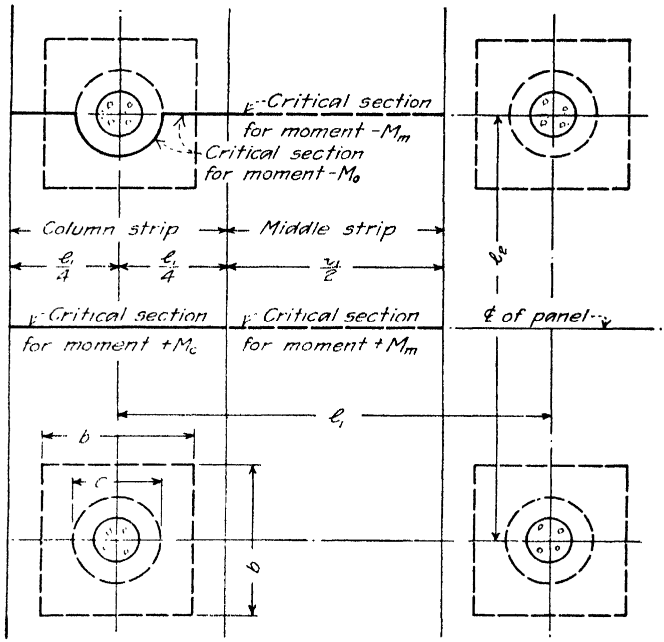
The allowable concrete and steel fiber stresses, the size of column capital, the slab and drop thicknesses, the width of the drop and the allowable shearing stresses are the same for two-way as given for four-way in Table 9.

The various specifications should be studied in all their details. Note that the A. C. I. specification applies to all shapes of column capitals and to all proportions of panel dimensions (within the permitted variation) without modification. It is recommended for use wherever some other specification is not compulsory.

The symbols used in flat slab formulas, and not heretofore listed, are given in Appendix A. Figure 13 shows the usual division of a flat slab panel into design strips. In the designation of the strips the 1921 J. C. report has been followed, while in the notations the more flexible A. C. I. specifications are used.

40. Assumption as to Tensile Stress in Concrete.—One of the assumptions commonly underlying the design of reinforced concrete is that the tensile stress which admittedly exists in the concrete shall be neglected. This assumption is reasonable in the case of beams and girders in which large percentages of steel are crowded into small stems, and its use greatly simplifies the design formulas. When, however, the percentage of reinforcing steel is low and distributed widely, as is practically always the case in flat slabs, the tensile resistance becomes important and dependable. To take account of this tensile resistance and still use the simple formulas, the moment coefficients in flat slab design have been reduced. Where the theoretical total moment coefficient is $M = 0.125wl_1(l_2 - qc)^2$ the A. C. I. specification (which agrees with the others cited in the table within very narrow limits) calls for a design total moment of $M_o = 0.09wl_1(l_2 - qc)^2$. Thus the concrete has been credited with approximately 28 per cent of the entire tensile resistance. To be consistent, this moment coefficient reduced on account of tension in the concrete, should not be applied to the concrete in compression. The 1921 J. C. report proposes for the first time to design the concrete for the full statical moment, and for good measure, takes account of a variation, alleged to be very large, in the compressive stress across the column strip. Since it does not recognize any increased compressive unit stress at the support (as is done in beams), the net result is to require abnormally heavy slabs. In view of the very satisfactory experience with present Chicago slab thicknesses under severe and long continued test loadings, in many of which tests the stresses have been measured, it is doubtful if a designer is warranted in following the new J. C. to its extreme position. The presence of the drop undoubtedly largely reduces the

variation in compressive stress across the column strip. There would seem to be an ample basis of experience beneath the provisions of the various specifications in Tables 9 and 10. The fiber stress in compression at the support should not be increased as it is in beams.



Design strips at right angles are similarly located.

FIG. 13.—Principal design sections in accordance with 1921 J. C. and A. C. I. specifications.

41. Steps to be Taken in Design.—For design purposes a flat slab floor panel is divided into two strips (in each direction) called the *column strip* and the *middle strip*. The column strip is bounded by two lines parallel to the panel edge and distant $\frac{1}{4}l_1$ on either side of it, while the middle strip comprises the slab between the column strips on either side. Figure 13 illustrates these design strips and, together with Fig. 12, much of the notation of the design formulas. On each strip the maximum positive moment at the center and the maximum negative moment at the panel edge are assigned definite values by Table 9. The first step in design of a four-way flat slab panel in accordance with the Chicago code is to compute the values of w, W, and WL, and to do this the weight of the slab must be assumed. Table 11 may be used as a guide and will enable this assumption to be made with great accuracy. Check the slab thickness by Formula (65)[1] and revise if necessary. Two-way, three-way and ring types take the same slab thickness as four-way.

[1] See Table 9.

SQUARE INTERIOR FLAT SLAB PANEL
American Concrete Institute Specifications

$f_s = 18{,}000$ $f_c = 750$ $n = 15$ $c = 0.225L$

TABLE 11

Side of panel	Side of drop		Diam. of cap		Slab thickness	Slab & drop	Approximate weight of slab steel	Four-way type: Steel area column strip	Four-way type: Rods required, Column strip	Four-way type: Rods required, Middle strip		Two-way type: Steel area column strip	Two-way type: Rods required, Column strip	Two-way type: Rods required, Middle strip	
								$-M_c$	$+M_c$	$+M_m$	$-M_m$	$-M_c$	$+M_c$	$+M_m$	$-M_m$
Ft.	Ft.	In.	Ft.	In.	In.	In.	#/□'	□"				□"			
50 #/□' superimposed roof load															
16	4	10	3	6	6	8½	1.10	1.86	10-3/8 φ	10-5/16 φ	11-1/4 φ	1.70	10-3/8 φ	11-5/16 φ	11-5/16 φ
17	5	2	4	0	6	8½	1.20	2.25	11-3/8 φ	12-5/16 φ	10-1/4 □	2.17	11-3/8 φ	13-5/16 φ	13-5/16 φ
18	5	6	4	0	6	8½	1.35	2.67	14-3/8 φ	14-5/16 φ	10-5/16 φ	2.48	14-3/8 φ	11-3/8 φ	11-3/8 φ
19	5	8	4	6	6	8½	1.50	3.13	16-3/8 φ	12-3/8 φ	11-5/16 φ	2.89	12-3/8 □	12-3/8 φ	12-3/8 φ
20	6	0	4	6	6	8½	1.65	3.70	18-3/8 φ	14-3/8 φ	13-5/16 φ	3.40	14-3/8 □	14-3/8 φ	14-3/8 φ
21	6	4	4	6	6½	9	1.80	4.10	16-3/8 □	15-3/8 φ	14-5/16 φ	3.85	12-1/2 φ	12-3/8 □	12-3/8 □
22	6	8	5	0	6¾	9¼	1.90	4.75	13-1/2 φ	13-3/8 □	12-3/8 φ	4.45	13-1/2 φ	14-3/8 □	14-3/8 □
23	7	0	5	0	7	9¾	2.05	5.20	15-1/2 φ	15-3/8 □	13-3/8 φ	4.90	15-1/2 φ	15-3/8 □	15-3/8 □
24	7	4	5	6	7¼	10¼	2.20	5.68	16-1/2 φ	17-3/8 □	14-3/8 φ	5.37	16-1/2 φ	17-3/8 □	17-3/8 □
25	7	6	5	6	7½	10¾	2.30	6.20	18-1/2 φ	18-3/8 □	16-3/8 φ	5.90	18-1/2 φ	14-1/2 φ	14-1/2 φ
26	7	10	6	0	8	11¼	2.40	6.90	19-1/2 φ	15-1/2 φ	17-3/8 φ	6.50	19-1/2 φ	15-1/2 φ	15-1/2 φ
100 #/□' superimposed floor load															
16	4	10	3	6	6	8	1.45	2.59	10-3/8 □	10-3/8 φ	10-5/16 φ	2.40	10-3/8 □	10-3/8 φ	10-3/8 φ
17	5	2	4	0	6½	9	1.60	3.06	12-3/8 □	11-3/8 φ	11-5/16 φ	2.84	12-3/8 □	12-3/8 φ	12-3/8 φ
18	5	6	4	0	6¾	9¼	1.70	3.54	13-3/8 □	13-3/8 φ	12-5/16 φ	3.32	13-3/8 □	11-3/8 □	11-3/8 □
19	5	8	4	6	7¼	9¾	1.85	4.00	15-3/8 □	14-3/8 φ	14-5/16 φ	3.78	11-1/2 φ	12-3/8 □	12-3/8 □
20	6	0	4	6	7½	10	2.00	4.65	12-1/2 φ	13-3/8 □	11-3/8 φ	4.40	12-1/2 φ	13-3/8 □	13-3/8 □
21	6	4	4	6	8	10½	2.15	5.20	14-1/2 φ	14-3/8 □	12-3/8 φ	4.90	14-1/2 φ	15-3/8 □	15-3/8 □
22	6	8	5	0	8¼	11	2.25	5.80	16-1/2 φ	16-3/8 □	14-3/8 φ	5.65	16-1/2 φ	17-3/8 □	17-3/8 □
23	7	0	5	0	8¾	11½	2.40	6.50	17-1/2 φ	17-3/8 □	15-3/8 φ	6.20	17-1/2 φ	13-1/2 φ	13-1/2 φ
24	7	4	5	6	9	12	2.55	7.10	19-1/2 φ	14-1/2 φ	17-3/8 φ	6.80	15-1/2 □	15-1/2 φ	15-1/2 φ
25	7	6	5	6	9½	12¾	2.65	7.63	16-1/2 □	15-1/2 φ	15-3/8 □	7.36	16-1/2 □	16-1/2 φ	16-1/2 φ
26	7	10	6	0	9¾	13½	2.80	8.40	18-1/2 □	17-1/2 φ	16-3/8 □	8.00	18-1/2 □	18-1/2 φ	18-1/2 φ

150#/□′ superimposed floor load

16	5 0	3 6	6	8¾	1.90	3.24	13-⅜□	13-⅜ϕ	12-5/16ϕ	3.03	13-⅜□	10-⅜□	10-⅜□
17	5 2	4 0	6½	9½	2.10	3.65	15-⅜□	11-⅜□	10-⅜ϕ	3.43	11-½ϕ	12-⅜□	12-⅜□
18	5 6	4 0	6¾	10	2.20	4.20	12-½ϕ	13-⅜□	11-⅜ϕ	3.98	12-½ϕ	13-⅜□	13-⅜□
19	5 10	4 6	7¼	10½	2.35	4.65	14-½ϕ	14-⅜□	12-⅜ϕ	4.40	11-½□	11-½ϕ	11-½ϕ
20	6 2	4 6	7½	11	2.50	5.20	16-½ϕ	12-½ϕ	11-⅜□	5.00	12-½□	12-½ϕ	12-½ϕ
21	6 6	4 6	8	11½	2.65	5.84	17-½ϕ	13-½ϕ	12-⅜□	5.55	13-½□	13-½ϕ	13-½ϕ
22	6 10	5 0	8¼	12	2.80	6.55	15-½□	14-½ϕ	14-⅜□	6.25	15-½□	15-½ϕ	15-½ϕ
23	7 2	5 0	8¾	12½	3.00	7.30	17-½□	16-½ϕ	15-⅜□	7.00	14-⅝ϕ	16-½ϕ	16-½ϕ
24	7 6	5 6	9	13¼	3.15	8.00	18-½□	17-½ϕ	16-⅜□	7.65	15-⅝ϕ	14-½□	14-½□
25	7 10	5 6	9½	14	3.30	8.65	17-⅝ϕ	15-½□	18-⅜□	8.30	17-⅝ϕ	16-½□	16-½□
26	8 4	6 0	9¾	14½	3.45	9.70	18-⅝ϕ	17-½□	14-½ϕ	9.30	18-⅝ϕ	17-½□	17-½□

200#/□′ superimposed floor load

16	5 4	3 6	6½	9¼	2.20	3.76	11-½ϕ	11-⅜□	10-⅜ϕ	3.54	11-½ϕ	12-⅜□	12-⅜□
17	5 10	4 0	6¾	9¾	2.40	4.35	12-½ϕ	13-⅜□	11-⅜ϕ	4.10	12-½ϕ	14-⅜□	14-⅜□
18	6 2	4 0	7¼	10¼	2.60	4.92	11-½ϕ	11-½ϕ	13-⅜ϕ	4.66	11-½□	11-½ϕ	11-½ϕ
19	6 6	4 6	7½	10¾	2.80	5.53	16-½ϕ	12-½ϕ	11-⅜□	5.25	13-½□	13-½ϕ	13-½ϕ
20	6 10	4 6	8	11½	2.95	6.10	18-½ϕ	13-½ϕ	12-⅜□	5.80	14-½□	14-½ϕ	14-½ϕ
21	7 2	4 6	8¼	12	3.10	6.70	20-½ϕ	15-½ϕ	14-⅜□	6.40	13-⅝ϕ	15-½ϕ	15-½ϕ
22	7 6	5 0	8¾	12¾	3.25	7.50	17-½□	16-½ϕ	15-⅜□	7.20	15-⅝ϕ	13-½□	13-½□
23	7 10	5 0	9¼	13¼	3.40	8.30	16-⅝ϕ	14-½□	17-⅜□	8.00	16-⅝ϕ	14-½□	14-½□
24	8 4	5 6	9½	13¾	3.55	9.30	17-⅝ϕ	15-½□	14-½ϕ	8.90	17-⅝ϕ	16-½□	16-½□
25	8 8	5 6	10	14½	3.70	10.00	19-⅝ϕ	17-½□	15-½ϕ	9.65	19-⅝ϕ	18-½□	18-½□
26	9 2	6 0	10½	15	3.90	11.00	21-⅝ϕ	18-½□	16-½ϕ	10.70	21-⅝ϕ	16-⅝ϕ	16-⅝ϕ

250#/□′ superimposed floor load

16	5 8	3 6	7	9¾	2.40	4.25	12-½ϕ	12-⅜□	11-⅜ϕ	4.00	12-½ϕ	13-⅜□	13-⅜□
17	6 2	4 0	7½	10½	2.60	4.80	13-½ϕ	14-⅜□	12-⅜ϕ	4.50	13-½ϕ	11-½ϕ	11-½ϕ
18	6 6	4 0	7¾	10¾	2.80	5.55	16-½ϕ	12-½ϕ	11-⅜□	5.30	12-½□	12-½ϕ	12-½ϕ
19	6 10	4 6	8¼	11½	3.00	6.10	14-½□	13-½ϕ	12-⅜□	5.80	14-½□	13-½ϕ	13-½ϕ
20	7 2	4 6	8½	12	3.20	6.90	15-½□	14-½ϕ	13-⅜□	6.55	13-⅝ϕ	12-½□	12-½□
21	7 6	4 6	9	12¾	3.40	7.70	17-½□	16-½ϕ	11-½ϕ	7.30	14-⅝ϕ	13-½□	13-½□
22	8 0	5 0	9½	13½	3.55	8.50	19-½□	14-½□	12-½ϕ	8.20	16-⅝ϕ	15-½□	15-½□
23	8 4	5 0	10	14	3.70	9.40	17-⅝ϕ	15-½□	13-½ϕ	9.10	17-⅝ϕ	16-½□	16-½□
24	8 8	5 6	10½	14¾	3.90	10.20	19-⅝ϕ	16-½□	14-½ϕ	9.80	19-⅝ϕ	14-⅝ϕ	14-⅝ϕ
25	9 2	5 6	11	15¼	4.05	11.30	20-⅝ϕ	18-½□	16-½ϕ	11.00	20-⅝ϕ	16-⅝ϕ	16-⅝ϕ
26	9 8	6 0	11¼	15¾	4.20	12.30	22-⅝ϕ	17-⅝ϕ	17-½ϕ	12.00	22-⅝ϕ	17-⅝ϕ	17-⅝ϕ

300#/□′ superimposed floor load

16	6 0	3 6	7½	10¼	2.70	4.65	13-½ϕ	10-½ϕ	11-⅜ϕ	4.45	13-½ϕ	10-½ϕ	10-½ϕ
17	6 6	4 0	7¾	10¾	2.90	5.40	15-½ϕ	11-½ϕ	10-⅜□	5.10	15-½ϕ	12-½ϕ	12-½ϕ
18	6 10	4 0	8¼	11¼	3.05	6.13	17-½ϕ	12-½ϕ	12-⅜□	5.86	13-½□	13-½ϕ	13-½ϕ
19	7 2	4 6	8¾	12	3.25	6.75	15-½□	14-½ϕ	13-⅜□	6.43	15-½□	11-½□	11-½□
20	7 6	4 6	9¼	12¾	3.40	7.65	16-½□	15-½ϕ	11-½ϕ	7.30	14-⅝ϕ	13-½□	13-½□
21	7 10	4 6	9¾	13½	3.60	8.40	18-½□	17-½ϕ	12-½ϕ	8.00	15-⅝ϕ	14-½□	14-½□
22	8 6	5 0	10	14	3.80	9.40	20-½□	15-½□	13-½ϕ	9.10	17-⅝ϕ	16-½□	16-½□
23	9 0	5 0	10½	14½	4.00	10.40	19-⅝ϕ	16-½□	14-½ϕ	10.00	19-⅝ϕ	14-⅝ϕ	14-⅝ϕ
24	9 4	5 6	11	15¼	4.20	11.30	20-⅝ϕ	18-½□	16-½ϕ	10.90	20-⅝ϕ	16-⅝ϕ	16-⅝ϕ
25	9 8	5 6	11½	15¾	4.40	12.50	22-⅝ϕ	19-½□	17-½ϕ	12.00	15-¾ϕ	17-⅝ϕ	17-⅝ϕ
26	10 0	6 0	12	16½	4.60	13.50	24-⅝ϕ	18-⅝ϕ	15-½□	13.00	17-¾ϕ	18-⅝ϕ	18-⅝ϕ

Next determine the bending moment $+M_c$ at the center of the column strip. Determine d from the slab thickness and compute the steel area for the direct band of rods. The two formulas used are

$$+M_c = WL/80 \quad \text{and} \quad A_s = \frac{+M_c}{f_s jd}$$

Determine the number and size of rods to give a spacing in a band $0.4L$ wide of between 4 and 9 in.

Next determine the design for the center of the middle strip, using the formulas

$$+M_m = WL/120 \quad \text{and} \quad A_s = \frac{+M_m}{f_s jd}$$

d for this section will be less because there are two layers of rods. Determine the number and size of rods to give a spacing in the diagonal band ($0.4L$ wide) of 4 to 9 in. also. For the two positive moment sections and for the negative moment section of the middle strip, the compression in the concrete practically never controls, as the formula for slab thickness is determined by deflection considerations.

Next determine the negative moment on the column strip from the formula $-M_c = WL/30$. The compressive stress governs at this section. Compute the required value of bd^2 and select values of b and d to agree. The value of b will be found in Table 11 or any other value between the limits of $0.3L$ and $0.5L$ may be substituted. The value of K will depend upon the stresses f_c and f_s as in beam design. The value of d may be solved directly from the formula $d = \sqrt{\frac{-M_c}{bK}}$. Compute A_s for this value of d and determine whether or not one direct band plus one diagonal band will provide sufficient steel area. By bending up some fraction of the rods in each band the required area is generally made up without difficulty, but in some cases it is advantageous to increase either b or d to facilitate a convenient steel arrangement. The total thickness through the slab and drop will be d plus two rod diameters plus the necessary protective cover for a slab. Check to see that this total thickness does not exceed $1\frac{2}{3}$ times the slab thickness alone.

Next design the slab for negative moment on the middle strip using the two formulas

$$-M_m = WL/120 \quad \text{and} \quad A_s = \frac{-M_m}{f_s jd}$$

and determine the number and size of rods.

Check the design for shear and diagonal tension. Along the periphery of the column capital the load causing shear will be $V = w\left(L^2 - \frac{\pi c^2}{4}\right)$ and the shearing unit stress will be $v = \frac{V}{\pi cd}$ in which d must be taken for the combined slab and drop. This stress must not exceed 120 lb. per sq. in. for 2,000-lb. concrete. The A. C. I. limits this stress to 100 lb. per sq. in., which is a better stress limit to use.

At the periphery of the drop the load causing shear (as a measure of the diagonal tension) will be $V = w(L^2 - b^2)$, and the unit stress will be $v = \frac{V}{4bjd}$. This stress must not exceed 60 lb. per sq. in. Figure 14 illustrates shear design assumptions by various specifications.

If panel construction is used, the compressive stress at both sections of positive moment must be checked.

The bond stress, so troublesome in beam design, will rarely require investigation in flat slab design, as small rods, shallow depths and long spans make an ideal combination for low bond stresses.

The protective cover on flat slabs should be the same as on other slabs. For ordinary protection only, ½ in. clear of concrete above and below the steel is adequate with proper spacing and supporting devices. One inch will be sufficient cover for highly fire-resistive construction.

In the steps given above a design in accordance with the Chicago flat slab code has been outlined. If any of the other three specifications given in Table 9 had been used, the steps would have been the same, using specified moments, except that the area of steel in the diagonal band resisting positive moment on the middle strip must be taken as the right sectional area of two bands multiplied by the sine of the angle between the bands and the section. In like manner for the negative moment in the column strip, the *component* of the two diagonal bands will be used.

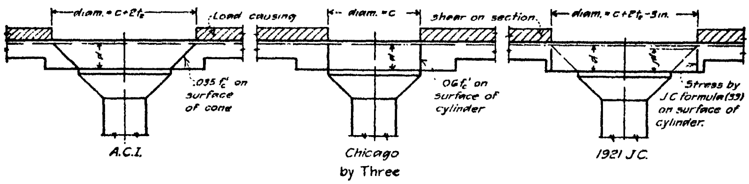


FIG. 14.—Shear at column capital by three specifications.

For the two-way and three-way type of flat slab the various specifications should be consulted. The number of layers of steel at various sections may also be different, which will affect the value of d. Very commonly also the size of bars used in these types will be somewhat larger than in the four-way type and this again will affect the value of d. In Table 10 the design moments at the principal sections are listed in the two-way type.

42. Floor Loads Coming on Lintel Beams.—Where walls of sill height occur at the boundaries of flat slab floors, it is generally possible to design a beam of no greater depth than the combined drop and slab thickness and, if this is done, no floor load need be figured as coming upon this lintel beam (some city ordinances do not agree with this). If, however, the wall is a solid brick wall for the story height, the beam must be deeper and will be so stiff as to take a portion of the load and weight of the floor. With the four-way or three-way types the area of load considered as coming upon the lintel beam should be taken as a triangle equal to one-fifth of the wall panel area. With the two-way type this should be increased to one-fourth.

43. Interior Beams.—Where fire walls occur on interior column center lines, the beams must carry the wall plus the above portion of the floor load on *each* adjacent panel. For other beams the designer must make similar allowances for floor load unless their depth can be kept within the combined thickness of slab and drop.

44. Diameter of Column Capital.—The column capital diameter, c, may not be less than $0.225L$ in Chicago. If it be made greater, no reductions in the moments are permitted. Under the A. C. I. specifications the column capital diameter should preferably be not less than $0.2L$ but may be any diameter and the moment depends upon the size. Very large column capitals should not be used upon small columns. The diameter of the column capital should not in any caes exceed three times the column diameter. Bending in the columns due to unbalanced floor loading must always be taken into account in design.

45. Size of Drop.—The drop should be not less than $0.3L$ in width and not over 0,5L in any case. The combined thickness of slab and drop must not exceed one and two-thirds times the slab thickness alone. The diagonal tension along the edge of the drop will commonly determine the minimum drop size. Very large drops are objectionable since they cut up the forms badly. In rectangular panels the drop is made rectangular also and the above limits apply to parallel dimensions of drop and panel.

46. Slab Thickness in Panel Construction.—The slab thickness, t_c, at the center when panel construction is used, should not be less than two-thirds of the nominal slab thickness, t_1, and the width of the ceiling panel should not exceed two-thirds of the parallel dimension center to center of column. When a drop is used with panel construction the panel dimension should not exceed $(L - b)$.

47. Slab Thickness without Drop.—When no drop is used, the slab thickness is determined by the compressive unit stress at the column capital (the width of beam being taken as the full width of the column strip) or by the shearing unit stress along the periphery of the column capital. This shearing stress should be determined as in footings for a section concentric with the column capital and removed a distance d from it. The total load causing shear will be $V = w\left(L^2 - \frac{\pi}{4}(c + 2d)^2\right)$ and the shearing unit stress will be $v = \frac{V}{\pi jc +)p\,2d)}$ This stress should not exceed 60 lb. per sq. in.

48. Tile Fillers.—Tile fillers should not be permitted within the area commonly covered by the drop and where the shearing stresses are increasing rapidly. In general the ends of rows of tile should be not less than $0.2L$ from the column center lines to which the rows are perpendicular.

49. Details of Design.—Steel detailing of flat slab design has been badly abused in many cases in the past and unsatisfactory results can be traced to this fault in some cases. Flat slab bars should not be lap-spliced at sections of maximum moment. When any splicing at such sections is necessary, each bar end should be carried to the quarter point of the adjoining span and the lap counted as two full bars in tension. A portion (not less than one-quarter) of the bars in the direct bands should be carried through in the bottom of the slab to an embedment in the drop, for the same reason as in beam design. At the outer walls this portion may well be increased to one-third. Of the bars which bend down from the upper level over the column to the lower level at the center, a portion should be bent just outside of the drop (there is generally a supporting bar at this point) and the rest at a greater distance, where they can be spared in tension, thus reinforcing a zone just outside the drop against diagonal tension In considering the point at which this second group of bars may be bent down, the designer must remember that while the moment reduces much the same as in

a beam, the tensile stress in the steel is sharply increased at the edge of the drop. This section will commonly require the same steel area as the section at the column capital. Hence, the bending down of bars depends upon the decrease in the moment between the edge of the drop and the point of inflection. The point of inflection will be farther from the column center and is generally considered to fall at the quarter point of the span center to center of columns.

The rods in the upper part of the slab near the column are commonly supported by two bars placed just outside the edge of the drop and themselves resting upon precast concrete blocks of the proper height. The rods in the top of the slab for negative moment in the middle strip are supported in like manner upon two bars upon slightly higher precast blocks. These supporting bars, also the number and height of concrete blocks, should be shown and specified upon the design drawings. A small allowance for *crowning* or lifting of the rods at the column is generally necessary.

Radial head bars or column bars bent out into the slab were common features of certain designs but have been shown by test to be inefficient reinforcement and are no longer used. Circumferential bars (rings or spiral) were also formerly used under the slab steel at the column. They are now placed on top if used at all, but their effectiveness as tension reinforcement is largely offset by the weakness they cause. Such bars tend to form continuous circular cracks following the bar extending deep into the slab and tending to precipitate a shear failure. Further, the concrete under the exceedingly high local compression against the bar yields plastically, with the result that these circular cracks tenb constantly to increase in size, and deflections are increased. The recovery from deflection with circular reinforcement is much less than with the usual bands of small rods.

50. Compression Reinforcement in Place of Drop.—Compression reinforcement in the bottom of a flat slab across the column has been used in some cases in place of a drop. It is far from economical. Further, the shearing stresses are greatly increased and it is very difficult to reinforce a flat slab against high diagonal tension stresses. Such web reinforcement must be very small to be effective on so small a depth and the labor of placing it is excessive.

51. Use of Wire Mesh.—Wire mesh has been frequently proposed and occasionally used for flat slab reinforcement. If provided with closely-spaced welded intersections, it is especially adapted to this use and somewhat higher tensile unit stresses may properly be permitted. New York city code permits 20,000 lb. per sq. in. on such material and some cities even more.

52. Rectangular Panels.—The A. C. I. moment formula is so expressed as to apply directly to rectangular panels and is recommended for use wherever possible. The Chicago code is somewhat irrational in its provisions for rectangular panels and gives designs that do not agree with test indications. Using the A. C. I. specifications, the procedure is exactly the same as with square panels, the values of l_1 and l_2 interchanging in the two directions. The computations for sections parallel to the shorter dimensions of the panel should be made first since they determine the drop thickness for both directions. The slab thickness may be based upon the average panel dimension, except that the minimum value of L in Formula 64 should be 0.9 of the longer panel dimension. Almost all specifications agree that when the ratio of longer side of panel to shorter side exceeds four

to three the floor should be designed in some beam type of construction and not as a flat slab.

53. Flat Slabs Supported on Walls.—Flat slabs should not be designed for *support on walls at one or more edges. Such support requires special consideration as it materially affects the curvature of the slab and the moments developed. The safe rule is to provide column support at the edges of all flat slabs and carry the walls as curtain walls on these columns.*

Illustrative Problem.—To design a four-way flat slab interior panel, 20 ft. square, for a superimposed load of 200 lb. per sq. ft., including floor finish. Allow $\frac{3}{4}$-in. protective cover and use Chicago stresses for 2,000-lb. concrete and hard grade steel.

For solution see Design Sheet 10. The following notes apply to that sheet: (*a*) Deductions are: 1 in. for two layers of rods, $\frac{3}{4}$ in. for cover and $\frac{1}{4}$ in. for *crowning* of the slab steel, (*b*) the higher fiber stresses permitted at the support of beams and girders are not available for flat slab design.

Illustrative Problem.—To design a four-way flat slab interior panel, 18 ft. by 22 ft., for a superimposed live load of 250 lb. per sq. ft. and a wood finish floor weighing 22 lb. per sq. ft., allowing $\frac{3}{4}$-in. cover and using A.C.I. specifications.

For solution see Design Sheet 11. The following notes apply to that sheet: (*a*) This trial is made to determine whether the average span or 0.9 times the longer span is to be used in thickness formula; (*b*) there are two bands, the effective area of each being its right area times sin α; (*c*) two bands, with seven lapped rods each, provide twenty-eight $\frac{1}{2}$-in. round rods whose effective area in this direction is 0.775 times their right area; (*d*) same steel, but only 0.635/0.775 as effective in this direction as they were in the 22-ft. direction; (*e*) would not appear well to have drop wider in 18-ft. direction than in 22-ft. direction; (*f*) see wording of the A. C. I. specification.

Illustrative Problem.—To design a two-way flat slab exterior panel for a superimposed load, including floor finish, of 125 lb. per sq. ft. The panel dimensions are 18 ft. c. to c. columns perpendicular to the wall and 20 ft. c. to c. columns parallel to the wall. The superimposed lintel load is 250 lb. per ft. Use A. C. I. specifications for 2,000-lb. concrete and hard grade steel.

For solution see Design Sheet 12. The following notes apply to that sheet: (*a*) $qc/2$ is the distance from the column center to the center of gravity of the semi-perimeter of the column capital, and is the reduction in span at either end due to the presence of the capital; (*b*) column strip on first interior column center line ($\frac{1}{2}$ in wall panel); (*c*) half column strip along wall, not including lintel; (*d*) moment over first interior column; (*e*) 80 per cent of moment of interior strip $= \frac{2}{3}$ of moment of wall strip; (*f*) considering the lintel beam only.

DESIGN SHEET 10

Superimposed load $= 200^{\#/\square'}$ $\qquad f_c = 700$

From Table 11 dead weight $= \underline{100}$ (8″ slab) $\qquad f_s = 18{,}000$

Total dead and live load $= 300^{\#/\square'}$ $\qquad K = 113.1^{(a)}$

$L = 20'-0''$ $\qquad c = (0.225)(20) = 4'-6''$ diam.

$W = (300)(20)^2 = 120{,}000'^{\#}$ $\qquad WL = 2{,}400{,}000 = 28{,}800{,}000''^{\#}$

By (65) $t_1 = (0.0227)(120{,}000)^{\frac{1}{2}} = 7.86''$ $\quad$ 8″ slab O.K.

Column strip, $+M_o = \dfrac{28{,}800{,}000}{80} = 360{,}000''^{\#}$

$d = 8 - \frac{3}{4} - \frac{1}{4} = 7''^{(a)}$ $\qquad A_s = \dfrac{360{,}000}{(18{,}000)(\frac{7}{8})(7)} = 3.26\square'' = 17 - \frac{1}{2}''\phi$ rods

Band width $= (0.4)(20) = 8'-0''$ $\qquad$ Spacing rods $= \dfrac{96}{16} = 6''$ O.K.

Middle strip, $+M_m = \dfrac{28{,}800{,}000}{120} = 240{,}000''^{\#}$

$d = 8 - \frac{3}{4} - \frac{1}{2} = 6.75''$ $A_s = \dfrac{240{,}000}{(18{,}000)(\frac{7}{8})(6.75)} = 2.26\,\square'' = 12 - \frac{1}{2}''\ \phi$ rods

Spacing $= \dfrac{96}{11} = 8\frac{3}{4}''$ O.K.

Column strip, $-M_c = \dfrac{28{,}800{,}000}{30} = 960{,}000''^{\#}$

From Table 11, $b = 6'-8'' = 80''$ $d^2 = \dfrac{960{,}000}{(113.1)(80)} = 106$ $d = 10.3''$

Total slab and drop thickness $= 10.3 + 1 + 1 = 12.3''^{(b)}$

Use $4\frac{1}{2}''$ drop, $d = 10.5''$ $A_s = \dfrac{960{,}000}{(18{,}000)(\frac{7}{8})(10.5)} = 5.81\,\square'' = 30 - \frac{1}{2}''\ \phi$ rods

Bend up $8 - \frac{1}{2}''\phi$ from direct band and lap $0.25L$ past column center

Bend up $7 - \frac{1}{2}''\phi$ from diagonal band and lap $0.35L$ past column center

Middle strip $-M_m = +M_m = 240{,}000''^{\#}$

$d = 8 - \frac{3}{4} - \frac{1}{4} = 7''$ $A_s = \dfrac{240{,}000}{(18{,}000)(\frac{7}{8})(7)} = 2.18\,\square'' = 11 - \frac{1}{2}''\ \phi$ bars 10'-0" long

$9 - \frac{1}{2}''\phi$-15'-0" in direct band, straight in bottom

$5 - \frac{1}{2}''\phi$-15'-0" in diagonal band, straight in bottom

Edge of column capital $V = 300(20)^2 - \frac{\pi}{4}(4.5)^2 = 115{,}000^{\#}$

$v = \dfrac{115{,}000}{(3.14)(54)(10.5)} = 65^{\#}/\square''$ O.K.

Edge of drop $V = 300(20)^2 - (6.67)^2 = 106{,}600^{\#}$

$v = \dfrac{106{,}600}{(4)(80)(\frac{7}{8})(6)} = 63^{\#}/\square''$

Too high. Make drop 7'-0" square $v = \dfrac{105{,}300}{(4)(84)(\frac{7}{8})(6)} = 59.7^{\#}/\square''$ O.K.

DESIGN SHEET 11

A.C.I. Specifications, $f_c = 750$ $f_s = 18{,}000$ $K = 125.7$

For Moments in 22' Direction

$l_1 = 18'-0''$ $l_2 = 22'-0''$ $L = 20'-0''$ $(0.9)(22) = 19.8'$ $L^{(a)}$ O.K.

Diagonal span $= l_1^2 + l_2^2 = 28.4$ $\sin\alpha = \dfrac{22}{28.4} = 0.775$

Since no limitations are stated, assume $c = 5'-0''$ circular capital. $q = \frac{3}{4}$

Superimposed $LL = 250^{\#}/\square'$

Wood floor finish $= 22$

Assume 9" slab $= 113$ $385^{\#}/\square'$

Slab thickness, by (64), $t_1 = (0.2)(20)\sqrt{385} + 1 = 8.87''$ 9" slab O.K.

Total moment (in 22' direction) $M_o = (0.09)(385)(18)\left(22 - \dfrac{(2)(5)}{3}\right)^2(12)$

$= 2{,}610{,}000''^{\#}$

Column strip, $+M_c = (0.20)(2{,}610{,}000) = 522{,}000''^{\#}$

$d = 9 - \frac{3}{4} - \frac{1}{4} = 8''$ $A_s = \dfrac{522{,}000}{(18{,}000)(\frac{7}{8})(8)} = 4.15\,\square'' = 21 - \frac{1}{2}''\ \phi$

Middle strip, $+M_m = (0.20)(2{,}610{,}000) = 522{,}000''^{\#}$

$d = 9 - \frac{3}{4} - \frac{1}{2} = 7.75''$ $A_s = \dfrac{522{,}000}{(18{,}000)(\frac{7}{8})(7.75)} = 4.28^{\square''} \div (2)(0.775)^{(b)}$

$= 2.76^{\square''} = 14 - \frac{1}{2}''\phi$

Column strip, $-M_c = (0.50)(2{,}610{,}000) = 1{,}305{,}000''^{\#}$

Assume 4″ drop, $d = 13 - 1 - 1 = 11''$

$b = \dfrac{1{,}305{,}000}{(125.7)(11)^2} = 86''$. Drop 7′−2″wide (⊥ to 22′ direction)

$A_s = \dfrac{1{,}305{,}000}{(18{,}000)(\frac{7}{8})(11)} = 7.53^{\square''}$ $\begin{cases} 9 - \frac{1}{2}''\phi \text{ from direct band, lapped} & = 3.54 \\ 7 - \frac{1}{2}''\phi \text{ from } each \text{ diag. band, lapped} & = 4.26^{(c)} \end{cases}$

$7.80^{\square''}$
O.K.

Middle strip, $-M_m = (0.10)(261{,}000) = 261{,}000$

$d = 8''$ $A_s = \dfrac{261{,}000}{(18{,}000)(\frac{7}{8})(8)} = 2.05^{\square''} = 11 - \frac{1}{2}''\phi - 9'-0''$ long

$12 - \frac{1}{2}''\phi - 17'-0''$ in long direct band straight in bottom
$7 - \frac{1}{2}''\phi - 15'-0''$ in each diag. band straight in bottom

For moments in 18′ direction

$l_1 = 22'\text{-}0''$ $l_2 = 18'\text{-}0''$ $\sin\alpha = \dfrac{18}{28.4} = 0.635$

Total moment $M_o = (0.09)(385)(22)\left(18 - \dfrac{(2)(5)}{3}\right)^2(12) = 1{,}970{,}000''^{\#}$

Column strip, $+M_c = (0.20)(1{,}970{,}000) = 394{,}000''^{\#}$

$d = 8''$ $A_s = \dfrac{394{,}000}{(18{,}000)(\frac{7}{8})(8)} = 3.13^{\square''} = 16 - \frac{1}{2}''\phi$

Middle strip, $+M_m = +M_c = 394{,}000''^{\#}$

$d = 7.75''$ $A_s = 3.23^{\square''} \div (2)(0.635) = 2.54^{\square''} = 13 - \frac{1}{2}''\phi$ bars

Use $14 - \frac{1}{2}\phi$ rods in each diagonal band

Column strip, $-M_c = (0.50)(1{,}970{,}000) = 985{,}000''^{\#}$

$d = 11''$ $A_s = 5.65^{\square''}$ $\begin{cases} 7 - \frac{1}{2}''\phi \text{ from each diag. band, lapped} & = 3.49^{\square''(d)} \\ 6 - \frac{1}{2}''\phi \text{ from direct band, lapped} & = 2.35 \end{cases}$

$5.84^{\square''}$ O.K.

Width in compression, $b = \dfrac{985{,}000}{(125.7)(11)^2} = 66''$

Make drop 7′−2″ square[e]

Middle strip, $-M_m = (0.10)(1{,}970{,}000) = 197{,}000''^{\#}$

$d = 8''$ $A_s = 1.56^{\square''} = 8 - \frac{1}{2}''\phi$ rods 11′−0″ long

Edge of capital $V = (385)(22 \times 18) - \dfrac{\pi}{4}\left(5 + \dfrac{22}{12}\right)^{2(f)} = 138{,}500^{\#}$

$$v = \frac{138{,}500}{(3.14)(70)(1.414)(\frac{7}{8})(11)} = 47^{\#/\square''} \text{ (70 allowed)}$$

Edge of drop $V = (385)(22)(18) - (7.16)^2 = 132{,}500^{\#}$

$$v = \frac{132{,}500}{(4)(86)(\frac{7}{8})(7)} = 63$$

Too large, make drop 7′−6″ square $v = \dfrac{130{,}900}{(4)(90)(\frac{7}{8})(7)}$ $59.5^{\#/\square''}$ O.K.

DESIGN SHEET 12

Use A. C. I. specification $f_c = 750$ $f_s = 18{,}000$ $K = 125.7$.
Assume octagonal interior column capitals 4′−6″ diam. $q = \frac{2}{3}$
Assume wall column brackets 2′−6″ wide × 1′−3″ deep. Col. 2′−0″ × 2′−6″

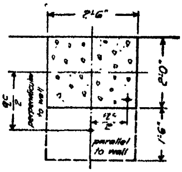
$$\perp \text{ to wall } \frac{(5)(1.25) + (2.5)(2.5)}{7.5} = 1.67' = q\frac{c}{2}\ (a)$$

$$\parallel \text{ to wall } \frac{(2.5)(0.625) + (3.5)(1.25)}{6.0} = 1.00 = q\frac{c}{2}$$

Reduction in span

$\perp$ to wall $= 1.67' + (\frac{1}{3})(4.5) = 3.17'$
$\parallel$ to wall, int. col. $= (\frac{2}{3})(4.5) = 3.00'$
$\parallel$ to wall, wall col. $= (2)(1.00) = 2.00'$

L.L = 125 By (64), $t_1 = (0.2)(19)\sqrt{213} + 1 = 6.85$ 7″ slab O.K.



7″ slab = 88
$w = 213$

For moments $\parallel$ to wall

$l_1 = 18'-0''$ $l_2 = 20'-0''$ $(l_2 - qc) = \begin{cases} 20 - 3 = 17'-0'' \text{ for int. col.} & \text{℄} \\ 20 - 2 = 18'-0'' \text{ for ext. col.} & \text{℄} \end{cases}$

Column strip, $^{(b)}$ $+M_c = (0.20)(0.09)(213)(18)(17)^2(12) = 240{,}000''^{\#}$

$d = 7 - \frac{3}{4} - \frac{1}{4} = 6''$ $A_s = \dfrac{240{,}000}{(18{,}000)(\frac{7}{8})(6)} = 2.54\ \square'' = 13 - \frac{1}{2}''\phi$ rods

Half column strip, $^{(c)}$ $+M_c = (\frac{1}{2})(0.20)(0.09)(213)(18)(18)^2(12) = 134{,}000''^{\#}$

$d = 6''$ $A_s = \dfrac{134{,}000}{(18{,}000)(\frac{7}{8})(6)} = 1.42\ \square'' = 7 - \frac{1}{2}''\phi$ rods

Middle strip, $+M_m = (0.16)(0.09)(213)(18)(17.5)^2(12) = 203{,}000''^{\#}$

$d = 7 - \frac{3}{4} - \frac{1}{2} = 5.75''$ $A_s = \dfrac{203{,}000}{(18{,}000)(\frac{7}{8})(5.75)} = 2.25\ \square'' = 12 - \frac{1}{2}''\phi$ rods

Column strip, $^{(b)}$ $-M_c = \dfrac{0.50}{0.20}(+M_c) = 600{,}000''^{\#}$

Assume 6′−8″ drop. $b = 80''$ $d^2 = \dfrac{600{,}000}{(125.7)(80)^2} = 74$ $d = 8.6''$

Total thickness slab and drop $= 8.6 + 1 + 1 = 10.6$

Make drop $3\frac{3}{4}''$ thick $d = 8.75''$ $A_s = \dfrac{600{,}000}{(18{,}000)(\frac{7}{8})(8.75)} = 4.35\ \square'' = 22 - \frac{1}{2}''\phi$

Half column strip, $^{(c)}$ $-M_c = \dfrac{0.50}{0.20}(134{,}000) = 336{,}000''^{\#}$

$d = 8.75''$ $A_s = 2.44\ \square'' = 13 - \frac{1}{2}''\phi$

Middle strip, $-M_m = (0.14)(0.09)(213)(18)(17.5)^2(12) = 284{,}000''^{\#}$

$d = 6''$ $A_s = 3.0\ \square'' = 16 - \frac{1}{2}''\phi$ rods

For moments $\perp$ to wall. (Moments increased 20 per cent.)

$l_1 = 20'-0''$ $l_2 = 18'-0''$ $(l_2 - qc) = 14.83'$

Column strip, $+M_c = (1.2)(0.20)(0.09)(213)(20)(14.83)^2(12) = 243{,}000''^{\#}$

Same design as in other direction, $13 - \frac{1}{2}\phi$ rods

Middle strip, $+M_m = \dfrac{0.16}{0.20}(+M_c) = 194{,}000''^{\#}$

$d = 5.75''$ $A_s = 2.15\ \square'' = 11 - \frac{1}{2}''\phi$ rods

Column strip, $^{(d)}$ $-M_c = \dfrac{0.20}{0.20}(+M_c) = 608{,}000''^{\#}$

$d = 8.75$ $A_s = 4.41\ \square'' = 23 - \frac{1}{2}''\phi$ rods

Middle strip, $-M_m = \frac{0.14}{0.16}(+M_m) = 170{,}000''^{\#}$

$d = 6''.\ A_s = 1.8^{\square''} = 9 - \frac{1}{2}''\phi$ rods $\left\{\text{Provide this no. of rods at wall and at int. ends. Hook rods into lintel.}\right.$

At wall end of column strip provide $\frac{2}{3}^{(e)}$ as many rods as at interior end or $15 - \frac{1}{2}''\phi$ hooked into column and lintel.

Lintel Beam:

Superimposed load = 250		Span = 20 − 2 = 18′−0″
Weight of beam = 150	400	

$$\text{Moment at center} = (400)(18)^2(\tfrac{12}{12}) = 130{,}000''^{\#}$$

$$d = 8.75\ b = \frac{130{,}000}{(125.7)(8.75)^2} = 13.5 \text{ O.K.}$$

$$A_s = \frac{130{,}000}{(18{,}000)(\frac{7}{8})(8.75)} = 0.95^{\square''} = 4\text{–}\tfrac{1}{2}''^{\square} \text{ bars}$$

Moment at support same as at center

$$\text{Shear at edge of } 6'-8'' \text{ drop}(f) = \frac{(400)(10 - 3.33)}{(13)(\frac{7}{8})(8.75)} = 27^{\#/\square''} \text{ for beam}$$

$$\text{Shear at edge of drop}\begin{cases}\text{Beam} = (2)(400)(6.67) & = 5{,}330\\ \text{Slab} = (0.5)(213)(18)(20) & = 38{,}370\end{cases}$$

$$43{,}700$$

$$\text{Resisting section} = (2)(39) + 40 = (118)(\tfrac{7}{8})(5) = 515$$

$$+ 2\ (13)(\tfrac{7}{8})(8.75) = 200$$

$$715$$

$$v = \frac{43{,}700}{715} = 61^{\#/\square''}$$

Make lintel beam 18″ wide.

$$\text{Shear at edge of cap}\begin{cases}\text{Beam } (2)(400)(8.75) & = 7{,}000\\ \text{Slab } (0.5)(213)(18)(20) - (3.96)(1.23) & = 37{,}800\end{cases}$$

$$44{,}800$$

$$v = \frac{44{,}800}{(46 + 46 + 38)(\frac{7}{8})(1.414)(8.75)} = 64 \text{ O.K.}$$

MEMBERS SUBJECT TO DIRECT AXIAL COMPRESSION

There are cases in design where concrete and reinforced concrete members receive direct axial load only. There are many more cases where they are wrongly considered to receive such load by neglecting the eccentricity. Most building columns come in this latter class, the justification being, presumably, that present factors of safety are based on this practice. While this matter has no bearing on the design of structural members, it is perhaps worth while to remember that in actual buildings a column under true axial compression is more rare even than a simple beam.

54. Formulas.—The symbols used in the formulas that follow are as given in Appendix A with the following additions:

A' = loaded portion of area of member (considered as surrounded by unloaded portions).

p' = percentage of spiral reinforcement.

P' = total safe load on long column or strut.

P_c = load on concrete of composite column.

P_s = load on steel or cast iron of composite column.

r_a = permissible compressive stress on area A'.

R = least radius of gyration of net section of member or of structural steel or cast iron core section in composite members.

The commonly used formulas for designing compression members are: For plain concrete piers, caissons, or walls

$$P/A = f_c = 0.25f_c' \tag{71}$$

For columns or struts reinforced with ½ to 4 per cent vertical rods tied with not less than ¼-in. round rods at 8-in. centers

$$P/A = f_c(1 - p + np) \tag{72}$$

For columns or struts reinforced with closely spaced spirals[1] enclosing vertical rods[1]

Chicago Ordinance, $P/A = f_c(1 + 2.5np')(1 - p + np)$ (73)
New York Ordinance, $P/A = f_c(1 - p + np) + 2f_s p'$ (74)
1921 J. C. Report, $P/A = [300 + (0.1 + 4p)f_c'](1 - p + np)$ (75)
A. C. I. Specifications, $P/A = f_c[(1 + 4np') - p + np]$ (76)

For composite columns or struts of structural steel[1] and spiral-encased concrete core, the J. C. recommends

$$P = 0.25f_c'A + (18{,}000 - \frac{70h}{R} - 0.25f_c')A_s \tag{77}$$

(The quantity in the parentheses must not exceed 16,000.)

For composite columns or struts of cast iron[1] and spiral-encased concrete core

$$P = 0.25f_c'A + (12{,}000 - \frac{60h}{R} - 0.25f_c')A_s \tag{78}$$

(The quantity in parentheses must not exceed 10,000.)

For long columns and struts (h/R exceeds 40) the J. C. recommends

$$P'/P = 1.33 - \frac{h}{120R} \tag{79}$$

For loading over a portion only of the area of a column the compressive unit stress may be increased to the value given by the J. C. formula

$$r_a = P/A' = 0.25f_c'\sqrt[3]{A/A'} \tag{80}$$

The radius of gyration of reinforced concrete sections is obtained by considering the reinforcement to be of n times its actual area and to act at the same centroid.

In the design of structural members subject to axial compression it is assumed that no external bending moment is applied to the member at any point. The only flexural stress taken into account is that due to the length of the column and its deflection under axial load. The ends of these members are presumed to be substantially fixed inasmuch as pin connections are practically unknown in reinforced concrete.

55. Height or Length of Member.—The height or length of a member, h, is taken as the distance between those points at either end where lateral support is present in at least two directions making an angle of not less than 75 deg. and not more than 105 deg. with each other. For columns in flat slab construction, h is the distance from the floor to the under side of the column capital. Where beams frame into the column the shallower set of beams governs the value of h. If beams occur along one center line only, h must be taken as the clear distance between floor slabs.

[1] For limiting conditions see the various specifications.

56. General Considerations.—It is always cheapest to use a minimum amount of reinforcing, say ½ of 1 per cent, in carrying direct compression. Tied columns with low percentages of vertical steel are generally less expensive than spiral columns. The reason for this is that concrete itself is the cheapest column reinforcement. There is another designers' axiom that reads, "Cement is the cheapest concrete reinforcement," which means that a richer mix is generally cheaper than a larger column of the usual mix, or a column of the usual mix more heavily reinforced. As against the saving in the first cost of construction the tied column occupies more floor space than the spiral column and that is a continuing disadvantage that remains "long after the price is forgotten."

The relative strength and advantages of tied versus spiral columns has long been a matter of rather heated discussion among engineers and equally a matter of serious investigation among technical committees. Tied columns have been charged with many weaknesses of which they are not guilty and spiral columns credited with a reserve strength which is probably not fully available without very serious damage to the supported construction. Most well-informed engineers would certainly prefer a spiral column to a tied column at the same cost per ton of load carried. But the tied column designed in accordance with Formula (72) is frequently much cheaper while it unquestionably has an adequate factor of safety as compared with the rest of the structural members. In many codes and ordinances an insufficient number of ties is permitted. Very few concrete buildings have been constructed without tied columns at least in the upper stories. A designer should adopt a spiral column type, however, wherever the cost comparison is not too heavily against it.

The Emperger column, combining a cast-iron core with a spirally reinforced concrete shell, is of particular advantage where ventilating ducts are placed at the center of columns. In this combination the cast iron is always under compression and its great compressive strength is fully utilized. Where very heavy loads must be carried on very small columns, a structural steel core is commonly used.

Where concrete columns are supported laterally for their entire height, as is the case with caissons, very long members may be designed without reinforcement. When without lateral support, plain concrete piers are commonly limited to a height of three to four times their least breadth. Plain concrete walls are built to much greater heights when under very light stress and when protected from large temperature changes.

Where load is applied to a small portion only of the area of a member, the stress on the loaded area may be larger than when the entire area is loaded. The unloaded concrete acts to restrain the portion under load. Where a column rests on a much larger pedestal or on top of a caisson, such a condition exists. The capacity of the loaded area depends upon the extent of the unloaded area that surrounds it and, with a very small proportionate part loaded, compressive strengths several times the ultimate strength of the same concrete in fully-loaded test cylinders may be realized. The stress, r_a, under such load concentrations may be computed by Formula (80).

57. Design of Plain Concrete Piers.—The simplest compression member is the plain concrete pier or caisson. The allowable working compressive stress is generally taken as $0.25f_c'$, although Chicago limits it to $0.2f_c'$. To find the diam-

eter of a circular pier or caisson, divide the load (including the estimated weight of the pier but not including the weight of the caisson) by the product of the area of the pier or caisson times $0.25f_c'$. For this case Formula (71) reduces to

$$\text{Diameter} = 2.25\sqrt{\frac{P}{f_c'}} \qquad (81)$$

For a square pier of side b, Formula (71) reduces to

$$b = 2\sqrt{\frac{P}{f_c'}} \qquad (82)$$

and, for a rectangular pier of dimensions b and d, to

$$bd = A = 4\frac{P}{f_c'} \qquad (83)$$

Areas and perimeters of caissons and column are given in Table 15.

TABLE 12.—VALUES OF n FOR VARIOUS STRENGTHS OF CONCRETE

Value of n	Values of f_c' for which n applies			
	A. C. I.	Chicago	New York	1921 J. C.
15	1,200–2,200	2,000	1:2:4 mix	1,200–2,200
12	2,201–3,300	2,500	1:1.5:3 mix	2,201–2,900
10	3,301 up	2,900	....	2,901 up

58. Design of Tied Columns.—The tied column is next in order of ease in design. A design graph from which the safe value of P/A for any percentage of reinforcement may be read directly is commonly used. Diagram 9 is a *general* graph on this order in which the steel has been *transformed* into equivalent area and an average safe load, P/A, plotted for the core area of the column. Table 12 gives the proper value of n to use for various values of f_c' for tied columns. At the right hand margin of Diagram 9, values of P/A in terms of f_c' are given in accordance with the J. C. and A. C. I. specifications. The value of f_c for tied columns is given as $0.2f_c'$ by the Chicago code and the 1921 J. C. report, as $0.25f_c'$ by the A. C. I., and as 500 lb. per sq. in. by the New York code. For any particular value of f'_c, a design graph is readily prepared in which the safe load P/A in pounds per square inch is plotted against the ratio of vertical steel. Diagram 10, p. 509, is such a graph for the Chicago ordinance and the 1921 J. C. report for 2,000, 2,500, and 2,900-lb. concrete, and for the A. C. I. and New York codes for 2,000 and 2,400-lb. concrete.

To design a tied column, the first step is to assume a core area and find the value of P/A for the superimposed load plus the weight of the assumed column. Enter the graph (for the ordinance or specification to be used) with this value of P/A and determine the steel percentage p. If no size limit exists on the column, the cheapest columns will be found when $p = 0.005$ (= 0.5 per cent). Table 13 gives the core areas, perimeters, and weights per foot of concrete columns, round, square, or octagonal in shape. The volumes of column capitals in flat slab construction must be included in the column weight computation and may be taken direct from Table 14 for the range of sizes found in practice.

If the concrete to be used on the work is a 2,800-lb. concrete the general graph may be used. The value of n from Table 12 will be 12 and the ratio of P/A to f'_c is readily determined from Diagram 9 in accordance with the A. C. I. or 1921 J. C. specifications given by the scales at the right.

To complete the design after sufficient trials to eliminate any error in the column weight assumption, compute the number of ¼-in. round ties required. These should be spaced 8 in. on centers and should be detailed as explained in Art. 64 in accordance with the number of vertical bars. When the percentage of vertical steel is considerable a check should be made to see that the area of ties per foot length of column, as cut by a section through the column center, is not less than 5 per cent of the area of the vertical steel.

59. Design of Spiral Columns.—The spiral column is designed by more complex formulas, which differ widely in different specifications. The 1921 J. C. formula should be used wherever possible, as tests show conclusively that some of the column formulas in common use result in dangerously high stresses in the vertical steel. The J. C. formula is also the simplest to use and is covered completely by the single design Diagram 11, p. 510. It is as economical as the Chicago code for example.

To design a column by the J. C. formula, the weight of the column must be assumed and the P/A determined for the assumed column. With this value of P/A, enter Diagram 11, follow up vertically to the strength of concrete proposed to be used, and from this intersection proceed horizontally to the percentage, p, of vertical steel. Now lay a straight edge on Diagram 17 so that it passes through this value of p and also through the assumed diameter of column core (used to determine A), and, from the intersections of this straight edge with the vertical steel scale, the number and size of vertical bars may be read directly.

The size of wire and pitch of spiral are also determined from Diagram 17, using p' equal to one-fourth of p (as found from Diagram 11), laying the straight edge so as to pass through p' and the assumed diameter and reading on the spiral scale. The weight of the spiral so found may be obtained from Tables 17 to 22 and the sectional area of an equivalent cylindrical shell is also given in the table. The weight of vertical column steel is found from Table 16.

To design a column by the Chicago code determine P/A as before and enter Diagram 12, 13, or 14 (according as 1:6, 1:4½ or 1:3 concrete is used). Place a straight edge through this value of P/A on the middle scale and the corresponding values of p and p' are read on the two outer scales. Economy of design tells us to use as small a percentage of vertical steel as possible, the code forbids using p less than p', and the straight edge should be swung on the fixed P/A value to secure proper relations of p and p'. The designer, however, should not aim at minimum reinforcement ($p = p' = 0.005$) but must remember that this ordinance will result in exceedingly high stresses if low values of p are used. The high stress in small amounts of vertical bars is due to the shrinkage and yield of the concrete which throws an initial compression into the steel, and the smaller the percentage, the higher this initial compression will be.[1]

For this reason a designer should use not less than 1½ per cent of vertical steel when designing in accordance with the Chicago, New York or A. C. I. column specifications. Additional factor of safety will be obtained if hard grade

[1] *A. C. I. Proceedings*, vol. 17, 1921, p. 150.

steel is used and in this case as low as 1 per cent of vertical steel is justified. The maximum percentage of vertical steel should be determined by the periphery of the column core and the arrangement should be such as to give not more than one ring of vertical bars, with the spacing of rods in this ring large enough to permit the concrete to flow through freely to fill up the shell. In general, this requires somewhat greater spacing of bars than in a beam with only one layer of steel, as the closely-spaced spiral forms a network with the verticals that tends to hold back the concrete and form pockets in the shell. A clear spacing of 1½ in. is recommended as a minimum.

Having by a sufficient number of trials arrived at a proper relation of p, p' and column size, the size and spacing of the column spiral and the number and size of column vertical bars may be read by means of a straight edge direct from Diagram 17.

Spiral column design in accordance with the New York code is made in exactly the same manner as with the Chicago code, and the same limitations as to vertical steel percentages apply. This code results in the same excessive compressive stresses in small amounts of vertical steel due to the shrinkage of the concrete. Determine p, p' and column size from Diagrams 15 and 16, and the spiral size and pitch and the number and size of vertical rods from Diagram 17 as before.

60. Design of Columns with Structural Steel Core.—To design a concrete column reinforced by a structural steel core the total load P including an assumed column weight must be determined. In such design the maximum dimension of the column is commonly fixed by architectural limitations so that A can be assumed very closely. The total load carried by the concrete is the first term of Formula (77), $P_c = 0.25f_c'A$. This really includes the area of the structural steel core also, but A is used instead of A_c for the reason that A_s and hence A_c is an unknown to be determined and it is simpler to use A and deduct the concrete stress from the permissible steel stress than to guess at the steel area at the start. The load left to be carried by the structural steel core is $P_s = P - P_c$. Having computed this numeral value of P_s, design a steel column in the usual way for a stress of $\left(18{,}000 - \frac{70h}{R} - 0.25f_c'\right)$ but not to exceed 16,000 lb. per sq. in. If the detail of the steel column is not such as to enclose and restrain the core concrete in a degree equal to a ½ per cent spiral, then a ½ per cent spiral must be provided.

61. Design of Columns with Cast-iron Core.—A composite column of concrete reinforced by a cast-iron core is designed in the same manner as above except that the cast-iron column is designed for a stress of $\left(12{,}000 - \frac{60h}{R} - 0.25f_c'\right)$, but not to exceed 10,000 lb. per sq. in. The diameter of the cast-iron core must not exceed one-half of the core diameter of the concrete column. Also a ½ per cent spiral must be provided.

62. Load Applied on Part of Area Only.—Where one compression member rests upon another, as a column upon a caisson or pier, the top of the lower member must frequently be designed for a heavy load concentration over a small area. In such design, the concentrated load is known and no dead weight need be added. The allowable bearing stress may be computed by Formula (80). If

P/A' exceeds $0.25f_c' \sqrt[3]{A/A'}$ the caisson top must be enlarged or reinforced. For a reinforced top, Formula (80) becomes

$$P/A' = x \sqrt[3]{A/A'} \tag{84}$$

in which x = the safe load upon the whole area A as reinforced (but considered to be without the benefit of restraint of any surrounding unloaded concrete). Either ties and longitudinal rods or spiral and longitudinal rods may be used. Top reinforcement for a caisson is best provided by carrying the column vertical and lateral reinforcement down into the caisson. The verticals should extend a sufficient distance to deliver their compression to the concrete at safe bond values. The spiral or ties should extend to a distance, a (see Fig. 15), such that the load, x, computed for an extended base of diameter $(c + a)$, by Formula (84), shall not exceed $0.25f_c'$. The top of the caisson may be made of richer concrete to increase the value of f'_c. If this is done, the richer concrete should extend for a depth not less than the difference in diameter of the caisson and the supported column, and the vertical bars should extend into the caisson a proper length to transmit their load by bond. The extension of the verticals is commonly effected by using separate dowels extending up into the column and down into the caisson, while the spiral in the caisson is generally separate from the column spiral.

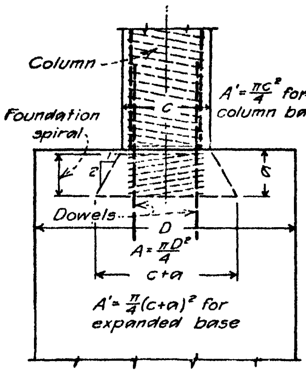


Fig. 15.—Design of caisson top.

63. Design of Plain Concrete Walls.—Plain concrete walls are designed the same as plain concrete piers using the formula

$$P/A = 0.2f_c' \tag{85}$$

A section of wall 1 ft. long is commonly taken as the designing unit. The height of such walls is limited by the same rules as apply to brick walls or other masonry construction, having due regard for the unit stresses permitted. Such walls have not been used as compression members to a large extent on account of the limited number of cases in which they are so situated as to be free of destructive temperature changes and movements. In basements and similar places of low temperature range plain concrete walls of heights up to sixteen times their thickness are used, and when the stress is kept down to $0.1f_c'$ this ratio may be increased to 24. If such walls are braced laterally by cross walls so that the length between cross walls does not exceed 30 times the wall thickness, a height ratio of 24 with a stress of $0.15f_c'$ may be permitted. Where such walls form a closed shaft, such as a vent, stair, or elevator shaft, whose greatest clear dimension in plan does not exceed 24 times the wall thickness, a stress of $0.15f_c'$ may be permitted without other height limitation.

Plain concrete walls subjected even to normal outdoor ranges of temperature develop unsightly cracks which may affect their strength. Exposed walls are therefore required to have temperature reinforcement both vertically and horizontally, even though no steel is required to take the direct compression. A

common specification is ¼ of 1 per cent in each direction. The percentage is properly a function of the size of rods used, however, as their effectiveness depends upon their size. The writers have proposed the following formula for which no sanction of established general usage can be claimed.

$$\frac{1}{p^\circ} = \frac{0.19u^\circ}{(0.0000055k^\circ t)^2 E_c o} + 1 \tag{86}$$

in which

p° = ratio of temperature reinforcement.
E_c = modulus of elasticity of the concrete.
k° = a coefficient depending upon the rigidity of the supports against movement due to temperature changes.[1]
n = ratio of moduli of elasticity of steel to concrete.
o = perimeter of one reinforcing bar in inches.
t = range of temperature *in the concrete* in degrees.
u° = safe working stress in sliding bond.[1]

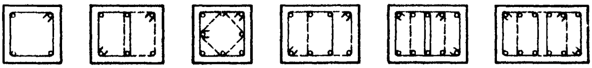
If we consider the probable temperature range in the concrete itself of an exposed wall as 70 deg., the percentage of deformed ½-in. round bars would be approximately 0.5 per cent horizontally and ⅙ per cent vertically. In the parapet wall the temperature range would be greater, say 90 deg. and the percentages would be approximately ⅝ per cent and ¼ per cent respectively. With larger bars, or with plain bars, larger percentages are necessary to have the same beneficial effect. The *minimum* percentage of temperature reinforcement that will do any good is generally considered to be approximately 0.1 per cent in each direction. Many engineers use double this percentage as a minimum. A superficial consideration of the problem will show that very small steel areas will have no value under loads sufficient to crack the concrete, even considering the relief afforded by the formation of the crack.

When the load on walls including their own weight is more than $0.2f_c'$, vertical reinforcement is required and may be computed by Formula (72) as for a tied column. Since the minimum percentage for such members is ½ per cent, no temperature steel need be added in a vertical direction but Formula (86) must still be used to determine the horizontal temperature reinforcement. This horizontal steel if properly arranged will replace the ties otherwise required longitudinally of the wall, so that only ties through the wall need be added.

64. Reinforcement Details.—Column vertical bars must possess some stiffness, as a bent or bowed bar under compression would throw tension into the concrete. Experience indicates that a ½-in. round bar is the smallest that should ever be used for a column vertical bar and that this size is permissible only in spiral columns where such tension is provided for. For tied columns the minimum size is ⅝-in. and the minimum reinforcement in any single member is four ⅝-in. round bars. These bars are held in alignment during construction and the effect

[1] Values of k° are recommended as follows:
¾ for bridge slabs built into heavy abutments.
⅝ for outside walls of building in horizontal direction.
0.5 for floor slabs on column support.
0.4 for roof slabs on column support.
0.3 for outside walls of buildings in a vertical direction.
Values of u° are recommended as follows:
$0.06f_c'$ for plain bars.
$0.10f_c'$ for deformed bars of proper design.

of any unavoidable slight bowing neutralized by ties placed at 8-in. centers. Ties are always ¼-in. round or larger (see Art. **58**). City ordinances permit wider spacing of ties, especially with larger bars, but the 1921 J. C. report recommends this fixed spacing for all bar sizes. Ties should be so detailed as to afford support against outward bending for every vertical bar at intervals of 8 in. Figure 16 shows proper details for (*a*) a 4-bar column, (*b*) a 6-bar column, (*c*) an 8-bar column, (*d*) a 10-bar column and (*e*) a 12-bar column. In a spiral column not less than six vertical bars should be used. The weight per foot of groups of column bars may be taken direct from Table 16, which also gives the sectional area of the group.



1 tie (a) 2 ties (b) 2 ties (c) 3 ties (d) 3 ties (e)

Fig. 16.—Column tie bending details.

Spiral details have become standardized by long experience. The maximum pitch is almost universally specified as 3 in. but not to exceed one-sixth of the core diameter. The minimum is set at 1⅜ or 1½ in. by the manufacturing process and by the practical requirement that the concrete must flow from the core into the shell when the column is poured. Every spiral 16 in. or more in diameter should have at least three spacers to insure the even spacing of the coils at all points. Several extraordinarily severe fires have removed the shells of concrete columns to such an extent as to show clearly that two spacers do not insure an even spacing of the wire. For spirals 30 in. or over in diameter four spacers are recommended. Each end of each spiral should be provided with an extra turn thoroughly wired or otherwise held securely against slipping. Where the wire must be spliced in the length of a spiral, each end of the wire at the splice should be carried half around the spiral and bent with a 6-in. hook around the spacing bar. The spiral length should be approximately 2 in. less than the clear height between floor slabs (counting the drop as a part of the floor slab in flat slab construction). Sizes of spiral wire are commonly stated in fractions of an inch although the actual cold drawn wire largely used in spiral manufacture provides sectional areas slightly less than these nominal sizes. Diagram 17 gives the percentages, p', for various sizes and pitches on any core area in which they are used, while from Tables 17, 18, 19, 20, 21, or 22, the weight of the wire per foot length of spiral may be read for the same percentage and core diameter. To this weight must be added ¾ to 1 lb. per ft. to cover the weight of spacers and to the length of spiral 3 to 4 in. must be added to cover weight of extra turns, etc., at each end.

Composite columns with structural steel or cast-iron cores must in general be detailed so as to provide effective means of getting the load into the metal section. Bond stress, which is effective on ordinary column bars, becomes unreliable in character and insufficient in amount in the case of large areas of flat or rounding metal surfaces. Each story of metal core should be provided with brackets or lugs of sufficient bearing area to take up at least that proportion of the load increment in the story that the last term in Formula (77) or (78) bears to the total value of P/A. Column splices are made in the same way as in ordinary steel or cast-iron design. Cast-iron cores must be cast in a vertical position.

Illustrative Problem.—To design a tied column to support an axial load of 180,000 lb. using $f_c = 400$ lb. per sq. in. and $n = 15$. Allow 1½-in. cover.

For solution see Design Sheet 13.

Illustrative Problem.—To design a spiral column to support an axial load of 800,000 lb. Allow 2-in. cover. Use the 1921 J. C. specification and the Chicago code, 2,900-lb. concrete and $n = 10$. The column may not exceed 30 in. in outside diameter.

For solution see Design Sheet 14.

Illustrative Problem.—To design a composite column at the center of which is a C. I. core with 8-in. ventilating duct. The superimposed load is 550,000 lb. Make the design in accordance with the 1921 J.C. specifications, using 3,000-lb. concrete. The height of the column is 10 ft.

For solution see Design Sheet 15.

Illustrative Problem.—To design the top of a caisson, 5 ft. in diameter, to carry a column 38-in. in diameter and carrying a load of 1,000,000 lb. The column vertical reinforcement consists of twelve 1¼-in. square bars. The caisson is made of 2,000-lb. concrete.

For solution see Design Sheet 16.

Illustrative Problem.—To design the basement story (9 ft. high in the clear) of an elevator shaft, 12- × 18-ft. inside dimensions, carrying a load of 70,000 lb. per ft. on the 18-ft. walls just below the first floor slab. Use 1921 J.C. stresses and 2,000-lb. concrete. Assume that the temperature range in the basement is so small as to be negligible.

For solution see Design Sheet 17.

DESIGN SHEET 13

$P = 180{,}000^{\#}$

Assume 22″ square column $A = (19)(19) = 361^{\square''}$

$$\frac{P}{A} = \frac{180{,}000}{361} = 499^{\#/\square''}$$

From Diagram 10, with $f_c = 400$ and $P/A = 500$

$$p = 0.0178 \quad A_s = (0.0178)(361) = 6.43^{\square''} = 8 - 1^{''\square} \text{ bars}$$

Ties $(0.05)(6.4) = 0.32^{\square''}$ per foot

$6 - \frac{1}{4}''\phi$ ties $= 0.30^{\square''}$ per foot provided, O.K.

Arrange ties as shown in Fig. 16c for square column

DESIGN SHEET 14

Joint Committee Design:

Assume 30″ dia. column. Core, 26″ dia. $A = 530.9^{\square''}$

$$P = 800{,}000^{\#} \quad \frac{P}{A} = \frac{800{,}000}{530.9} = 1{,}505^{\#/\square''}$$

From Diagram 11 for $P/A = 1{,}505$ and $2{,}900^{\#}$ concrete, $p = 0.0395$ and for 26″ core read from diagram.

Vertical bars = 14 − 1¼ ″□

Spiral = $\frac{7}{16}''\phi$ at 2⅝″ pitch

Chicago Design:

Assume same size of column. $P/A = 1{,}505$

From Diagram 14 (1:1:2 concrete, $f_c' = 2{,}900$) for $P/A = 1{,}505$

$p' = 0.015$ and $p = 0.0564$

From Diagram 17 for 26″ core dia.

$p = 0.0564$ Vertical bars = 19 − 1¼ ″□

$p' = 0.015$ Spiral = $\frac{7}{16}''\phi$ at 1½″ pitch

DESIGN SHEET 15

Assume a 26″ dia. column. Core dia. = 22″

Core area = $\frac{\pi}{4}\left[(22)^2 - (8)^2\right] = 330^{\square''}$

$$P_c = \text{first term of Eq. (78)} = (750)(330) = 248{,}000^{\#}$$

$$P_s = P - P_c = 550{,}000 - 248{,}000 = 302{,}000^{\#}$$

Assume 1⅛″ shell. Inside dia. C. I. core = 8″
Outside dia. C. I. core = 10.25″

Area $= \frac{\pi}{4}\left[(10.25)^2 - (8)^2\right] = 41^{\square''}$

Rad. of gyration $= \frac{1}{4}\sqrt{(10.25)^2 + (8)^2} = \frac{13}{4} = 3.25$

Allowable stress $= 12{,}000 - \frac{(60)(120)}{3.25} - 750 = 9{,}030^{\#/\square''}$

Stress required $= \frac{302{,}000}{41} = 7{,}370^{\#/\square''}$ Too low.

Try 1″ shell. Area $= 36^{\square''}$ $r = 3.21$.

Allowable stress $= 90{,}100^{\#/\square''}$

Stress required $= \frac{302{,}000}{36} = 8{,}400^{\#/\square''}$ O.K.

From Diagram 17 for $p' = 0.005$ and 22″ dia. core

Spiral = ¼″ ϕ at 1¾″ pitch

DESIGN SHEET 16

Caisson dia. = 5′-0″ Area $= A = 2{,}827^{\square''}$ $\sqrt[3]{\frac{A}{A'}} = \sqrt[3]{2.49} = 1.35$

Column dia. = 3′-2″ Area $= A' = 1{,}134^{\square''}$

$$P = 1{,}000{,}000 \qquad \frac{P}{A'} = \frac{1{,}000{,}000}{1{,}134} = 881^{\#/\square''}$$

By Formula (84) $X = \frac{881}{1.35} = 664^{\#/\square''}$

$0.25f'_c = 500$. Stress is too high.

Try $a = 8''$ $c + a = 3' - 10''$ New $A' = 1{,}662^{\square}$ (see Fig. 15

$$\frac{A}{A'} = \frac{2{,}827}{1{,}662} = 1.70 \qquad \sqrt[3]{1.70} = 1.19$$

$$\frac{P}{A'} = \frac{1{,}000{,}000}{1{,}662} = 605 \qquad x = \frac{605}{1.19} = 509^{\#/\square''}$$

Provide caisson spiral (same wire, pitch and diameter as column spiral) extending from top of caisson not less than 9″ into caisson.

DESIGN SHEET 17

Assume 13″ wall. Effective thickness = 12″

$$P = 70{,}000 + (162)(9) = 71{,}460$$

$$A = (12)(12) = 144^{\square''} \qquad \frac{P}{A} = \frac{71{,}460}{144} = 495^{\#/\square''}$$

From Diagram 10, $p = 0.017$

$A_s = (0.017)(144) = 2.45^{\square''}$ = ¾″ ϕ bars at 4.3″ on ctrs. each face

Ties through wall $(0.05)(2.45) = 0.123^{\square''/'}$ = ⅜″ϕ at 10″ ctrs. along every third vertical bar.

Horizontal steel area $= 0.123^{\square''}$ = ⅜″ ϕ at 13″ ctrs.

DIAGRAM 9.
GENERAL TIED COLUMN DESIGN GRAPH.

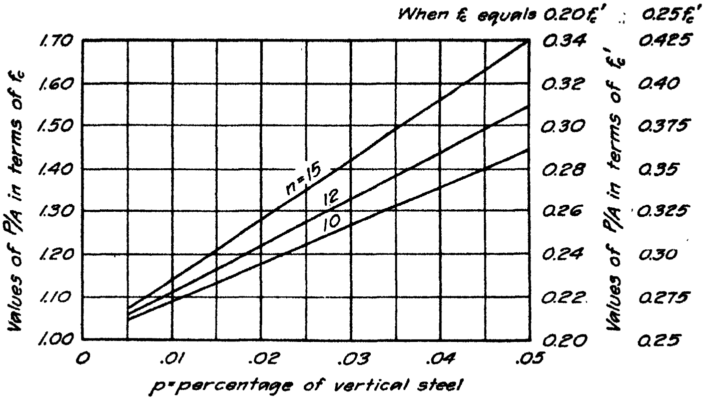


DIAGRAM 10.
TIED COLUMN DESIGN BY FOUR SPECIFICATIONS.

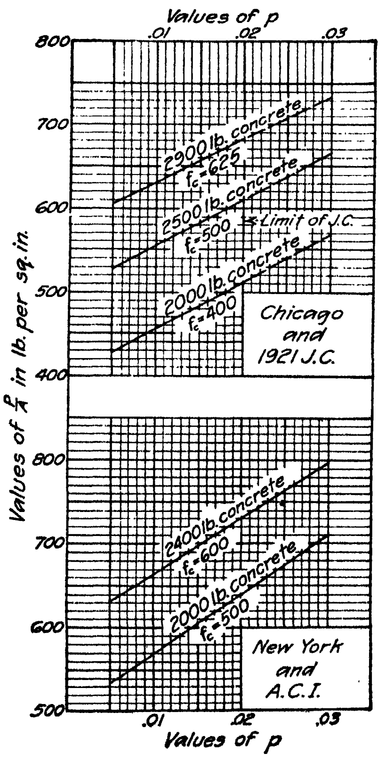


DIAGRAM 11.

PERCENTAGE OF REINFORCEMENT IN SPIRAL COLUMN BY 1921 JOINT COMMITTEE SPECIFICATIONS.

NOTE.—Percentage of Spiral Reinforcement equals one-fourth of percentage of vertical reinforcement in all cases.

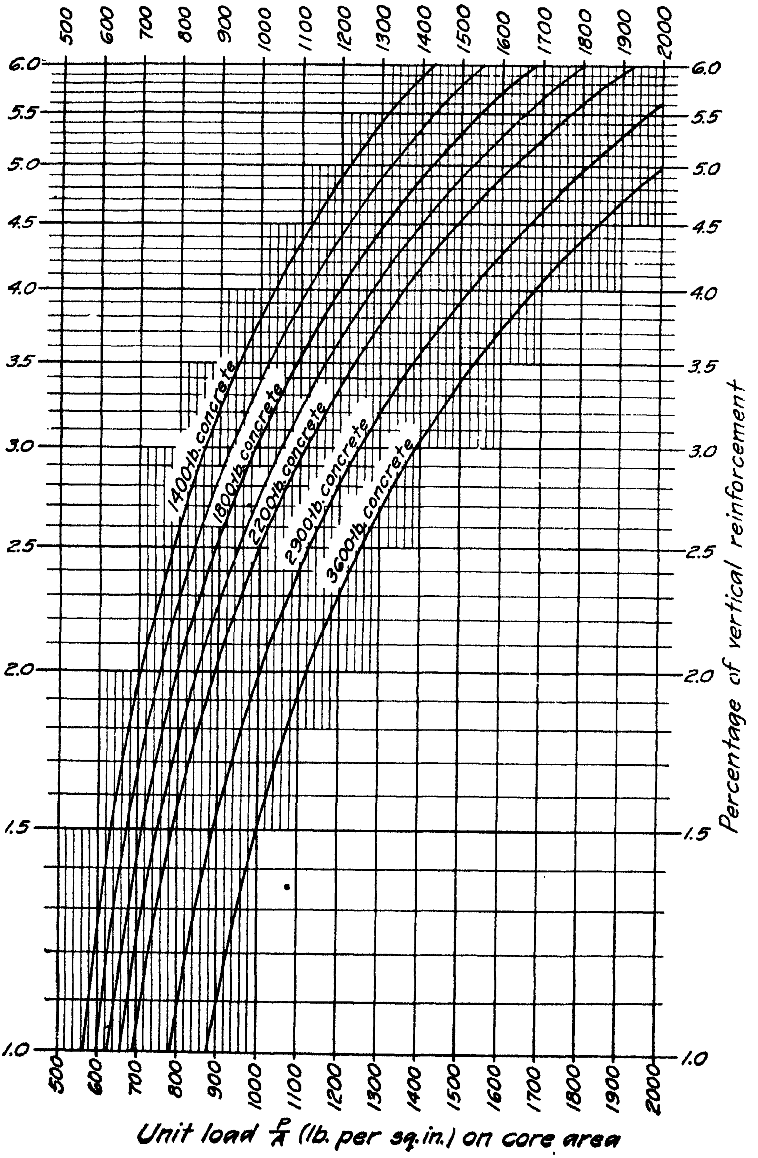
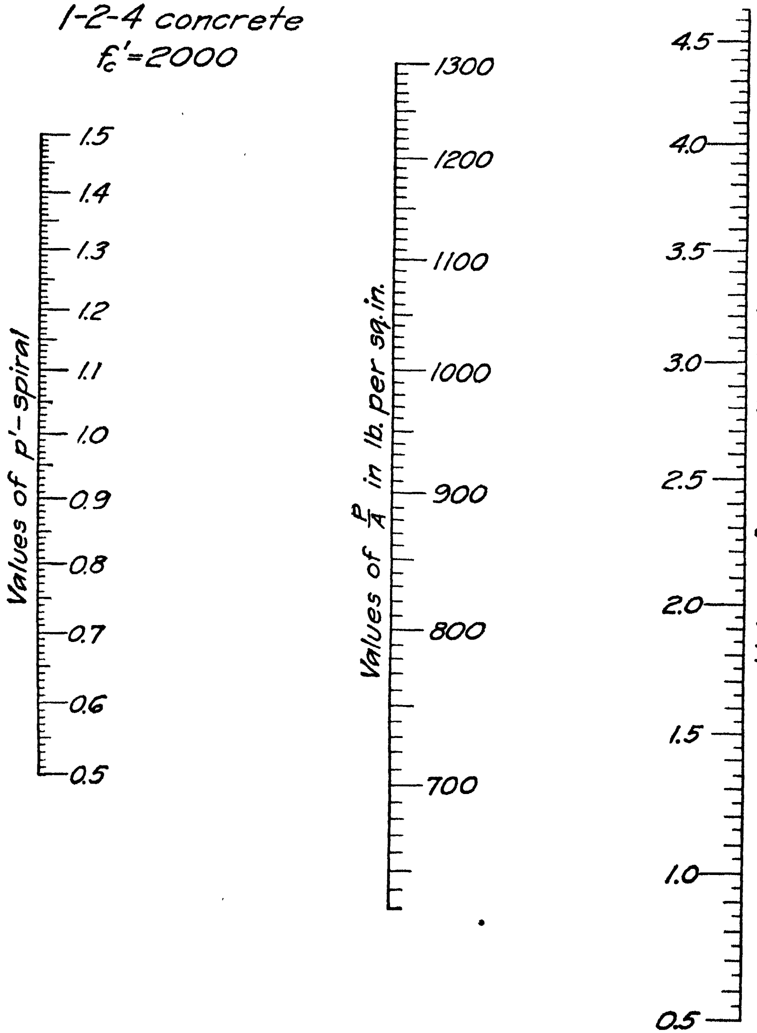


DIAGRAM 12.

CHICAGO SPIRAL COLUMN DESIGN[1]—1:6 CONCRETE.

NOTE.—Set straight edge on any two known quantities and read concurrent value of third quantity.

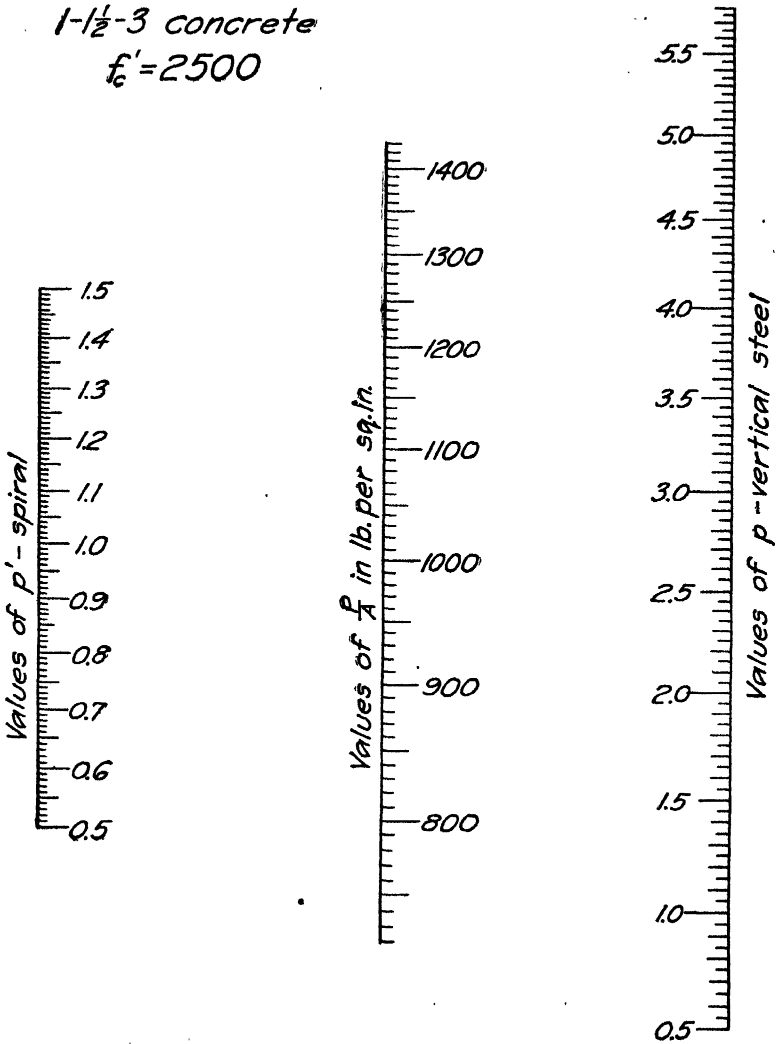


[1] Prepared by Gardner and Lindberg, Industrial Engineers, Chicago. Wallace Berger, Structural Engineer.

DIAGRAM 13.

CHICAGO SPIRAL COLUMN DESIGN[1]—1:4½ CONCRETE.

NOTE.—Set straight edge on any two known quantities and read concurrent value of third quantity.

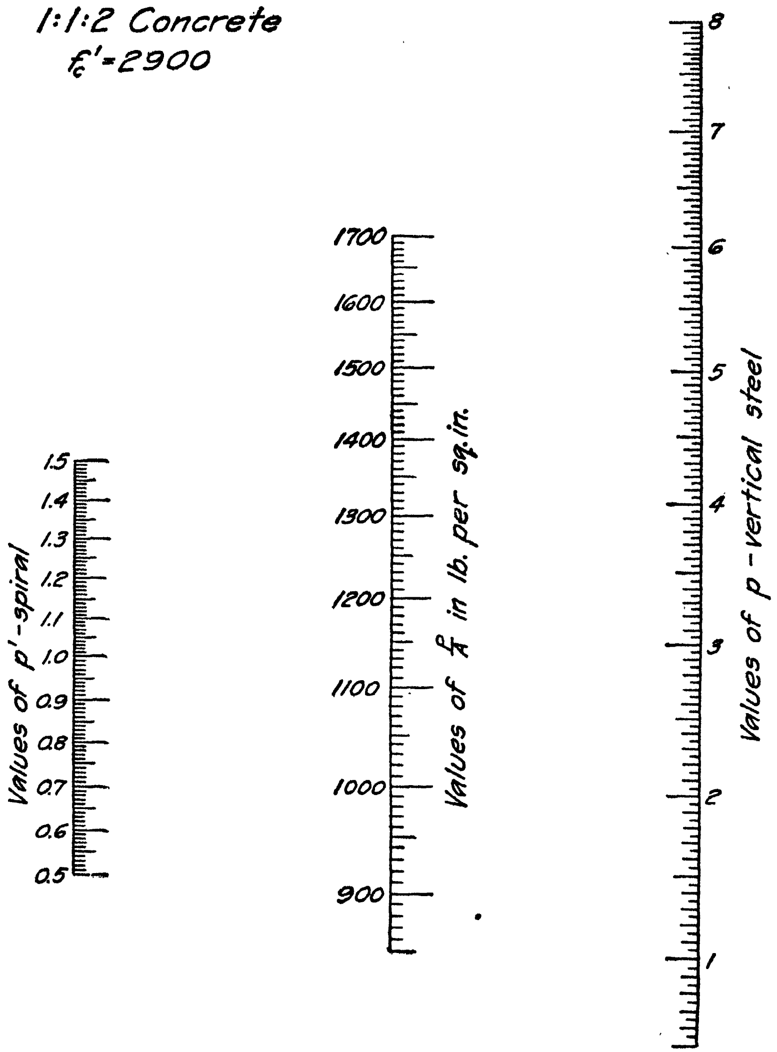


[1] Prepared by Gardner and Lindberg, Industrial Engineers, Chicago. Wallace Berger, Structural Engineer.

DIAGRAM 14.

CHICAGO SPIRAL COLUMN DESIGN[1]—1:3 CONCRETE.

NOTE.—Set straight edge on any two quantities and read concurrent value of third quantity.



[1] Prepared by Gardner and Lindberg, Industrial Engineers, Chicago. Wallace Berger, Structural Engineer.

DIAGRAM 15.

NEW YORK SPIRAL COLUMN DESIGN—$f_c = 500$.

NOTE.—Set straight edge on any two quantities and read concurrent value of third quantity.

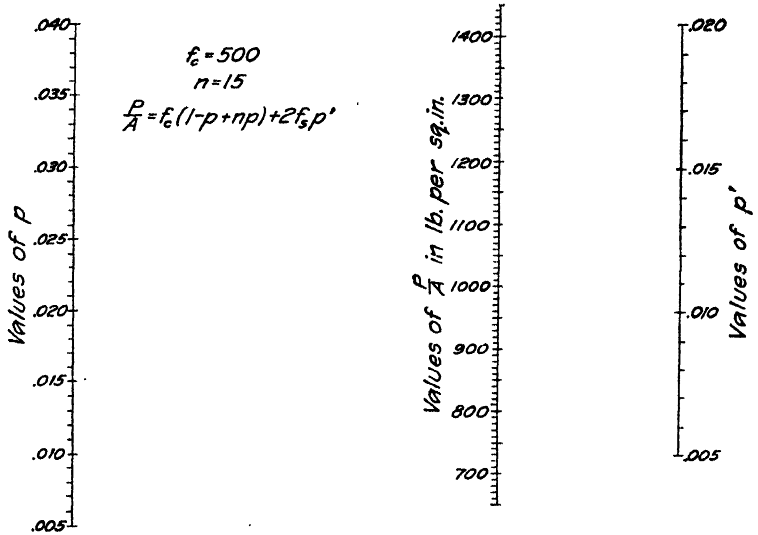


DIAGRAM 16.

NEW YORK SPIRAL COLUMN DESIGN—$f_c = 600$.

NOTE.—Set straight edge on any two quantities and read concurrent value of third quantity.

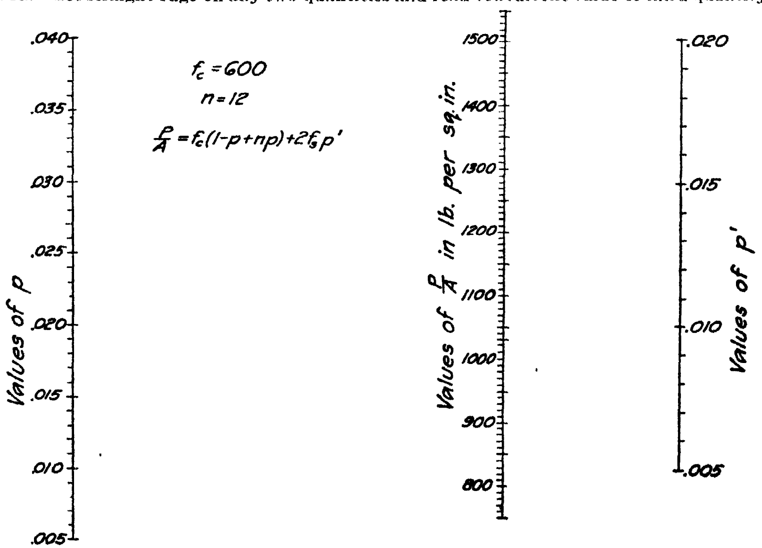
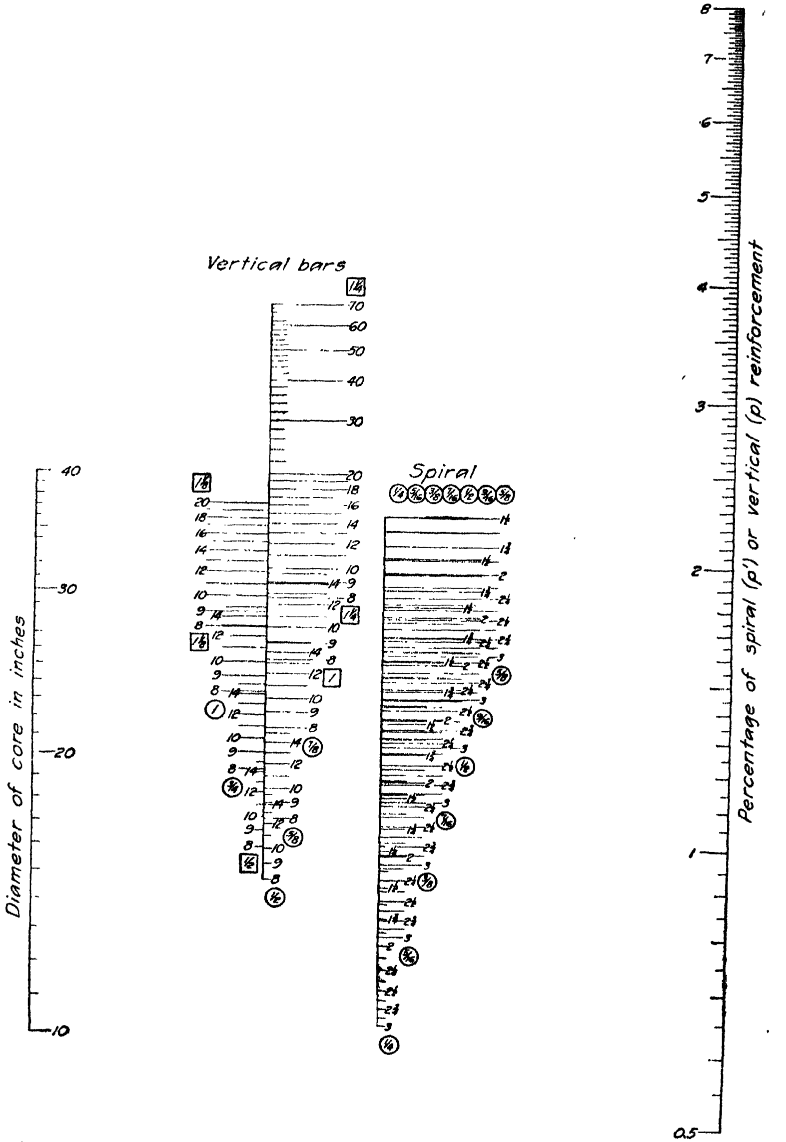


DIAGRAM 17.

PERCENTAGE OF SPIRAL REINFORCEMENT IN CIRCULAR CORES.[1]

NOTE.—Read with aid of straight edge.



[1] Based on similar diagram prepared by Gardner and Lindberg, Industrial Engineers, Chicago. Wallace Berger, Structural Engineer.

TABLE 13.—CORE AREAS, PERIMETERS, AND CONCRETE VOLUMES FOR COLUMNS

Diameter		Core area (sq. in.)	ϕ col. perimeter	Volume (cu. ft. per ft.)		
Col. (in.)	Core (in.)		Ft. In.	Round	Octagonal	Square
14	10	78.5	3—8	1.07	1.12	1.36
	11	95.0				
16	12	113.1	4—2	1.40	1.47	1.78
	13	132.7				
18	14	153.9	4—8	1.77	1.86	2.25
	15	176.7				
20	16	201.0	5—2	2.18	2.30	2.78
	17	226.9				
22	18	254.4	5—9	2.64	2.78	3.36
	19	283.5				
24	20	314.1	6—3	3.14	3.31	4.00
	21	346.3				
26	22	380.1	6—9	3.69	3.89	4.70
	23	415.4				
28	24	452.3	7—4	4.28	4.51	5.45
	25	490.8				
30	26	530.9	7—10	4.91	5.17	6.25
	27	572.5				
32	28	615.7	8—4	5.58	5.89	7.11
	29	660.5				
34	30	706.8	8—11	6.30	6.64	8.03
	31	754.7				
36	32	804.2	9—5	7.07	7.45	9.00
	33	855.3				
38	34	907.9	10—0	7.88	8.30	10.02
	35	962.1				
40	36	1,017.8	10—6	8.73	9.20	11.10
	37	1,075.2				
42	38	1,134.1	11—0	9.62	10.15	12.25

TABLE 14.—VOLUME OF CONCRETE IN COLUMN SHAFTS AND COLUMN CAPITALS

Round columns								Square columns							
Dia. of col. shaft	Vol. of col. shaft	Diameter of column capital						Side of col. shaft	Vol. of col. shaft	Side of column capital					
		3′ 6″	4′ 0″	4′ 6″	5′ 0″	5′ 6″	6′ 0″			3′ 6″	4′ 0″	4′ 6″	5′ 0″	5′ 6″	6′ 0″
14	1.07	4.87	....	.....	.....	.....	.....	14	1.36	6.13					
16	1.40	4.63	7.61	.....	.....	.....	.....	16	1.78	5.70	8.90				
18	1.77	4.38	7.05	10.60	.....	.....	.....	18	2.25	5.26	8.37	12.41			
20	2.18	3.94	6.49	9.92	14.46	.....	.....	20	2.78	4.76	7.79	11.70	16.68		
22	2.64	3.49	5.93	9.23	13.58	.....	.....	22	3.36	4.24	7.15	10.97	15.80	22.24	
24	3.14	3.04	5.37	8.54	12.69	17.80	24.15	24	4.00	3.80	6.50	10.23	15.02	21.10	28.00
26	3.69	2.60	4.80	7.81	11.81	16.78	23.00	26	4.70	3.18	5.83	9.43	14.07	19.95	26.80
28	4.28	2.15	4.22	7.08	10.91	15.76	21.80	28	5.45	2.65	5.13	8.61	13.10	18.78	25.60
30	4.91	1.70	3.64	6.35	10.00	14.74	20.55	30	6.25	2.14	4.46	7.78	12.10	17.58	24.40
32	5.58	....	3.10	5.65	9.11	13.68	19.35	32	7.11	....	3.78	6.73	11.09	16.38	23.10
34	6.30	....	2.55	4.95	8.22	12.59	18.10	34	8.03	....	3.12	5.68	10.05	15.16	21.75
36	7.07	....	2.00	4.24	7.32	11.47	16.80	36	9.00	....	2.44	4.63	9.00	13.92	20.25
38	7.88	....	....	3.59	6.49	10.44	15.52	38	10.02	....	....	4.03	8.00	12.70	18.83
40	8.73	....	....	2.94	5.66	9.40	14.22	40	11.10	....	....	3.45	6.99	11.46	17.35
42	9.62	....	....	2.29	4.83	8.34	12.92	42	12.25	....	....	2.88	5.96	10.22	15.86
44	10.56	....	....	....	4.08	7.37	11.70	44	13.44	....	....	....	4.88	9.02	14.40
46	11.54	....	....	....	3.33	6.40	10.49	46	14.69	....	....	....	3.78	7.84	12.95
48	12.57	....	....	....	2.59	5.43	9.28	48	16.00	....	....	....	2.69	6.68	11.52

For octagonal columns and capitals add 6 per cent to values given for round columns and capitals.

Volume of concrete in column shaft is given in cubic feet per foot of height. For column capitals it is given in cubic feet and includes only the concrete in the capital outside the surface of the column enclosed in the capital.

TABLE 15.—AREAS AND PERIMETERS OF COLUMNS AND CAISSONS

Diameter			Area		Circ.	Diameter			Area		Circ.
Ft.	In.	In.	Sq. in.	Sq. ft.	Ft.	Ft.	In.	In.	Sq. in.	Sq. ft.	Ft.
1	1	13	132.7	0.92	3.40	4	7	55	2,376	16.50	14.40
1	2	14	153.9	1.07	3.66	4	8	56	2,463	17.10	14.66
1	3	15	176.7	1.23	3.93	4	9	57	2,552	17.72	14.92
1	4	16	201.0	1.40	4.19	4	10	58	2,642	18.35	15.18
1	5	17	226.9	1.58	4.45	4	11	59	2,734	18.99	15.45
1	6	18	254.4	1.77	4.71	5	0	60	2,827	19.63	15.71
1	7	19	283.5	1.97	4.97	5	1	61	2,922	20.29	15.97
1	8	20	314.1	2.18	5.24	5	2	62	3,019	20.97	16.33
1	9	21	346.3	2.41	5.50	5	3	63	3,117	21.65	16.49
1	10	22	380.1	2.64	5.76	5	4	64	3,217	22.34	16.76
1	11	23	415.4	2.89	6.02	5	5	65	3,318	23.04	17.02
2	0	24	452.3	3.14	6.28	5	6	66	3,421	23.76	17.28
2	1	25	490.8	3.41	6.54	5	7	67	3,526	24.48	17.54
2	2	26	530.9	3.69	6.81	5	8	68	3,632	25.22	17.80
2	3	27	572.5	3.98	7.07	5	9	69	3,739	25.97	18.06
2	4	28	615.7	4.28	7.33	5	10	70	3,848	26.73	18.33
2	5	29	660.5	4.59	7.59	5	11	71	3,959	27.49	18.59
2	6	30	706.8	4.91	7.85	6	0	72	4,072	28.27	18.85
2	7	31	754.7	5.24	8.12	6	1	73	4,185	29.07	19.11
2	8	32	804.2	5.59	8.38	6	2	74	4,301	29.87	19.37
2	9	33	855.3	5.94	8.64	6	3	75	4,418	30.68	19.63
2	10	34	907.9	6.31	8.90	6	4	76	4,536	31.50	19.90
2	11	35	962.1	6.68	9.16	6	5	77	4,657	32.34	20.16
3	0	36	1,017.8	7.07	9.43	6	6	78	4,778	33.18	20.42
3	1	37	1,075.2	7.47	9.69	6	7	79	4,902	34.04	20.68
3	2	38	1,134.1	7.88	9.95	6	8	80	5,027	34.91	20.94
3	3	39	1,194.5	8.30	10.21	6	9	81	5,158	35.78	21.21
3	4	40	1,256.6	8.73	10.47	6	10	82	5,281	36.67	21.47
3	5	41	1,320.2	9.17	10.73	6	11	83	5,411	37.57	21.73
3	6	42	1,385.4	9.62	10.99	7	0	84	5,542	38.48	21.99
3	7	43	1,452.2	10.08	11.26	7	1	85	5,674	39.41	22.25
3	8	44	1,520.5	10.56	11.52	7	2	86	5,809	40.34	22.51
3	9	45	1,590.4	11.04	11.78	7	3	87	5,945	41.28	22.71
3	10	46	1,661.9	11.54	12.04	7	4	88	6,082	42.24	23.04
3	11	47	1,734.9	12.05	12.30	7	5	89	6,221	43.20	23.30
4	0	48	1,809.5	12.57	12.57	7	6	90	6,362	44.18	23.56
4	1	49	1,885.7	13.10	12.83	7	7	91	6,504	45.17	23.82
4	2	50	1,963.5	13.64	13.09	7	8	92	6,648	46.16	24.09
4	3	51	2,042.0	14.19	13.35	7	9	93	6,798	47.17	24.35
4	4	52	2,124.0	14.75	13.61	7	10	94	6,940	48.19	24.61
4	5	53	2,206.0	15.32	13.88	7	11	95	7,088	49.22	24.87
4	6	54	2,290.0	15.90	14.14	8	0	96	7,238	50.27	25.13

TABLE 16.—AREAS AND WEIGHTS OF COLUMN RODS

Heavy figures = area; Number of rods; Light figures = weight per foot

Size (in.)	1	4	6	8	10	11	12	13	14	15	16	17	18	19	20
½ φ	**0.1963** 0.668	**0.79** 2.67	**1.18** 4.01	**1.57** 5.34	**1.96** 6.68										
½ □	**0.2500** 0.850	**1.00** 3.40	**1.50** 5.10	**2.00** 6.80	**2.50** 8.50										
⅝ φ	**0.3068** 1.043	**1.23** 4.08	**1.84** 6.26	**2.46** 8.35	**3.07** 10.4	**3.38** 11.5	**3.68** 12.5								
¾ φ	**0.4418** 1.502	**1.77** 6.01	**2.65** 9.01	**3.54** 12.0	**4.42** 15.0	**4.86** 16.6	**5.30** 18.1	**5.75** 19.6	**6.19** 21.1						
⅞ φ	**0.6103** 2.044	**2.40** 8.18	**3.61** 12.3	**4.81** 16.4	**6.01** 20.4	**6.62** 22.5	**7.22** 24.6	**7.82** 26.6	**8.43** 23.7	**9.03** 30.7	**9.63** 32.8				
1 φ	**0.7854** 2.670	....	....	**6.29** 21.4	**7.85** 26.7	**8.65** 29.4	**9.43** 32.0	**10.2** 34.7	**11.0** 37.4	**11.8** 40.0	**12.6** 42.7	**13.4** 45.4	**14.1** 48.0	**14.9** 50.7	
1 □	**1.000** 3.400	....	....	**8.00** 27.2	**10.0** 34.0	**11.0** 37.4	**12.0** 40.8	**13.0** 44.2	**14.0** 47.6	**15.0** 51.0	**16.0** 54.4	**17.0** 57.8	**18.0** 61.2	**19.0** 64.6	
1⅛ □	**1.266** 4.303	....	....	....	**12.7** 43.0	....	**15.2** 51.6	**16.4** 56.0	**17.7** 60.3	**19.0** 64.5	**20.2** 68.9	**21.5** 73.1	**22.8** 77.5	**24.0** 81.8	**25.3** 86.1
1¼ □	**1.563** 5.312	....	....	....	**15.6** 53.1	....	....	....	....	....	**25.0** 85.0	**26.6** 90.3	**28.1** 95.6	**29.7** 101.0	**31.3** 106.2

TABLE 17.—AREAS AND WEIGHTS OF ¼-IN. WIRE SPIRALS

Core diam. (inches)	Areas of equivalent cylinders in square inches given in heavy figures Weights of wire in pounds per foot length of spiral in light figures Pitch of spiral in inches												
	3″	2⅞″	2¾″	2⅝″	2½″	2⅜″	2¼″	2⅛″	2″	1⅞″	1¾″	1⅝″	1½″
10	**0.515** 1.75	**0.537** 1.83	**0.562** 1.91	**0.588** 2.00	**0.618** 2.10	**0.651** 2.21	**0.686** 2.33	**0.726** 2.47	**0.771** 2.62	**0.824** 2.80	**0.882** 3.00	**0.950** 3.23	**1.03** 3.50
11	**0.566** 1.92	**0.591** 2.01	**0.618** 2.10	**0.646** 2.20	**0.679** 2.31	**0.715** 2.43	**0.754** 2.56	**0.798** 2.72	**0.848** 2.88	**0.905** 3.08	**0.971** 3.30	**1.04** 3.55	**1.13** 3.85
12	**0.617** 2.10	**0.645** 2.19	**0.674** 2.29	**0.705** 2.40	**0.741** 2.52	**0.781** 2.65	**0.823** 2.80	**0.871** 2.96	**0.925** 3.15	**0.987** 3.36	**1.06** 3.60	**1.14** 3.88	**1.23** 4.19
13	**0.668** 2.27	**0.699** 2.38	**0.730** 2.48	**0.764** 2.60	**0.803** 2.73	**0.846** 2.88	**0.892** 3.03	**0.944** 3.21	**1.00** 3.41	**1.07** 3.64	**1.15** 3.90	**1.23** 4.20	**1.33** 4.54
14	**0.720** 2.45	**0.752** 2.56	**0.786** 2.67	**0.823** 2.80	**0.865** 2.94	**0.911** 3.10	**0.961** 3.26	**1.02** 3.46	**1.08** 3.67	**1.15** 3.92	**1.23** 4.20	**1.33** 4.52	1% **1.44** 4.90
15	**0.771** 2.62	**0.807** 2.74	**0.843** 2.87	**0.882** 3.00	**0.926** 3.15	**0.976** 3.32	**1.03** 3.50	**1.09** 3.71	**1.16** 3.94	**1.23** 4.20	**1.32** 4.50	**1.42** 4.85	**1.54** 5.24
16	**0.823** 2.80	**0.860** 2.92	**0.898** 3.05	**0.940** 3.20	**0.987** 3.36	**1.04** 3.54	**1.10** 3.73	**1.16** 3.95	**1.23** 4.20	**1.31** 4.48	**1.41** 4.80	**1.52** 5.17	**1.64** 5.60
17	**0.874** 2.97	**0.913** 3.10	**0.954** 3.24	**1.00** 3.40	**1.05** 3.56	**1.11** 3.75	**1.17** 3.96	**1.23** 4.19	**1.31** 4.46	**1.40** 4.76	**1.50** 5.10	**1.61** 5.49	**1.74** 5.94
18	**0.926** 3.15	**0.966** 3.29	**1.01** 3.44	**1.06** 3.60	**1.11** 3.78	**1.17** 3.98	**1.24** 4.20	**1.31** 4.45	**1.39** 4.73	**1.48** 5.05	**1.59** 5.40	**1.71** 5.82	**1.85** 6.28
19	**0.978** 3.32	**1.02** 3.47	**1.07** 3.62	**1.12** 3.80	**1.17** 3.98	**1.24** 4.20	**1.30** 4.43	**1.38** 4.69	**1.46** 4.98	**1.56** 5.32	**1.67** 5.70	**1.80** 6.14	**1.95** 6.64
20	**1.03** 3.50	**1.07** 3.65	**1.12** 3.82	**1.17** 4.00	**1.23** 4.20	**1.30** 4.42	**1.37** 4.66	**1.45** 4.93	**1.54** 5.25	**1.64** 5.60	**1.76** 6.00	**1.90** 6.46	**2.06** 7.00
21	**1.08** 3.67	**1.13** 3.84	**1.18** 4.01	**1.23** 4.20	**1.30** 4.40	**1.37** 4.64	**1.44** 4.90	**1.53** 5.18	**1.62** 5.51	**1.73** 5.88	**1.85** 6.30	**1.99** 6.78	**2.16** 7.34
22	**1.13** 3.85	**1.18** 4.02	**1.23** 4.20	**1.29** 4.40	**1.36** 4.62	**1.43** 4.86	**1.51** 5.13	**1.59** 5.43	**1.70** 5.77	**1.81** 6.16	**1.94** 6.60	**2.09** 7.11	**2.27** 7.70
23	**1.18** 4.02	**1.23** 4.20	**1.29** 4.38	**1.35** 4.60	**1.42** 4.83	**1.49** 5.08	**1.58** 5.36	**1.67** 5.67	**1.77** 6.03	**1.89** 6.44	**2.03** 6.90	**2.18** 7.43	**2.37** 8.05
24	**1.24** 4.20	**1.29** 4.38	**1.35** 4.57	**1.41** 4.80	**1.48** 5.04	**1.56** 5.30	**1.65** 5.60	**1.74** 5.92	**1.85** 6.30	**1.97** 6.72	**2.12** 7.20	**2.28** 7.75	**2.47** 8.40
25	**1.29** 4.37	**1.34** 4.57	**1.40** 4.78	**1.47** 5.00	**1.54** 5.25	**1.63** 5.53	**1.71** 5.83	**1.82** 6.17	**1.93** 6.56	**2.06** 7.09	**2.20** 7.50	**2.37** 8.07	**2.57** 8.75
26	**1.34** 4.55	**1.40** 4.75	**1.46** 4.96	**1.53** 5.20	**1.61** 5.46	**1.69** 5.75	**1.78** 6.06	**1.89** 6.42	**2.00** 6.82	**2.14** 7.28	**2.29** 7.80	**2.47** 8.39	**2.67** 9.10
27	**1.39** 4.72	**1.45** 4.94	**1.52** 5.15	**1.59** 5.40	**1.67** 5.67	**1.76** 5.96	**1.85** 6.30	**1.96** 6.66	**2.08** 7.08	**2.22** 7.56	**2.38** 8.10	**2.56** 8.72	½% **2.78** 9.45
28	**1.44** 4.90	**1.50** 5.11	**1.57** 5.34	**1.64** 5.60	**1.73** 5.88	**1.82** 6.19	**1.92** 6.53	**2.03** 6.91	**2.16** 7.34	**2.30** 7.84	**2.47** 8.40	**2.66** 9.04	**2.88** 9.80
29	**1.49** 5.07	**1.56** 5.30	**1.63** 5.53	**1.70** 5.80	**1.79** 6.09	**1.89** 6.41	**1.99** 6.77	**2.11** 7.16	**2.24** 7.60	**2.38** 8.12	**2.56** 8.70	**2.75** 9.36	**2.98** 10.15

Spirals above and to right of zigzag lines are nearest commercial size fully equal to percentage of core area indicated at end of line.

Weights in above table include the wire only. To this must be added the weights of the spacing bars and the weight of extra turn at ends of spiral.

TABLE 18.—AREAS AND WEIGHTS OF 5/16-IN. WIRE SPIRALS

Core diam. (inches)	Areas of equivalent cylinders in square inches given in heavy figures Weights of wire in pounds per foot length of spiral in light figures													
	Pitch of spiral in inches													
	3″	2⅞″	2¾″	2⅝″	2½″	2⅜″	2¼″	2⅛″	2″	1⅞″	1¾″	1⅝″	1½″	
10	**0.803** 2.73	**0.838** 2.86	**0.876** 2.98	**0.918** 3.13	**0.964** 3.28	**1.01** 3.46	**1.07** 3.65	**1.13** 3.86	**1.20** 4.10	**1.28** 4.38	**1.38** 4.69	**1.48** 5.05	**1.61** 5.48	
11	**0.884** 3.01	**0.922** 3.14	**0.963** 3.28	**1.01** 3.44	**1.06** 3.61	**1.12** 3.80	**1.18** 4.01	**1.25** 4.25	**1.33** 4.52	**1.41** 4.82	**1.51** 5.16	**1.63** 5.56	**1.77** 6.02	
12	**0.965** 3.28	**1.01** 3.43	**1.05** 3.58	**1.10** 3.75	**1.16** 3.93	**1.22** 4.15	**1.28** 4.39	**1.36** 4.64	**1.44** 4.93	**1.54** 5.26	**1.65** 5.63	**1.78** 6.06	**1.93** 6.57	
13	**1.04** 3.55	**1.09** 3.72	**1.14** 3.88	**1.19** 4.06	**1.25** 4.27	**1.32** 4.50	**1.39** 4.75	**1.47** 5.02	**1.57** 5.34	**1.67** 5.70	**1.79** 6.10	**1.93** 6.57	**2.09** 7.11	1½ %
14	**1.12** 3.83	**1.17** 4.00	**1.23** 4.18	**1.28** 4.38	**1.35** 4.60	**1.42** 4.84	**1.50** 5.11	**1.59** 5.41	**1.69** 5.75	**1.80** 6.13	**1.93** 6.56	**2.08** 7.08	**2.25** 7.67	
15	**1.20** 4.10	**1.26** 4.29	**1.31** 4.48	**1.38** 4.69	**1.45** 4.93	**1.52** 5.19	**1.61** 5.48	**1.70** 5.79	**1.81** 6.16	**1.93** 6.57	**2.07** 7.04	**2.22** 7.59	**2.41** 8.21	
16	**1.29** 4.38	**1.34** 4.57	**1.40** 4.78	**1.47** 5.01	**1.54** 5.26	**1.62** 5.53	**1.71** 5.84	**1.81** 6.18	**1.93** 6.57	**2.06** 7.01	**2.20** 7.51	**2.38** 8.08	**2.57** 8.76	
17	**1.37** 4.65	**1.42** 4.86	**1.49** 5.08	**1.56** 5.32	**1.64** 5.58	**1.72** 5.88	**1.82** 6.21	**1.93** 6.56	**2.05** 6.98	**2.19** 7.45	**2.34** 7.98	**2.52** 8.59	**2.73** 9.31	
18	**1.45** 4.93	**1.51** 5.14	**1.58** 5.37	**1.65** 5.63	**1.73** 5.91	**1.83** 6.22	**1.93** 6.57	**2.04** 6.95	**2.17** 7.39	**2.31** 7.88	**2.48** 8.44	**2.67** 9.10	**2.89** 9.86	
19	**1.53** 5.20	**1.59** 5.43	**1.66** 5.68	**1.74** 5.94	**1.83** 6.24	**1.93** 6.57	**2.03** 6.94	**2.16** 7.34	**2.29** 7.80	**2.44** 8.32	**2.62** 8.92	**2.82** 9.60	**3.05** 10.4	
20	**1.61** 5.47	**1.68** 5.71	**1.75** 5.97	**1.83** 6.26	**1.93** 6.57	**2.03** 6.91	**2.14** 7.30	**2.27** 7.72	**2.41** 8.21	**2.57** 8.76	**2.75** 9.39	**2.97** 10.1	**3.21** 11.0	
21	**1.69** 5.74	**1.76** 6.00	**1.84** 6.27	**1.93** 6.56	**2.02** 6.89	**2.13** 7.26	**2.25** 7.67	**2.38** 8.11	**2.53** 8.62	**2.70** 9.20	**2.89** 9.85	**3.12** 10.6	**3.37** 11.5	1 %
22	**1.77** 6.02	**1.84** 6.29	**1.93** 6.57	**2.02** 6.88	**2.12** 7.22	**2.23** 7.61	**2.35** 8.03	**2.50** 8.49	**2.65** 9.04	**2.83** 9.65	**3.03** 10.3	**3.26** 11.1	**3.53** 12.0	
23	**1.85** 6.29	**1.93** 6.57	**2.02** 6.87	**2.11** 7.19	**2.22** 7.55	**2.33** 7.95	**2.46** 8.40	**2.61** 8.88	**2.77** 9.45	**2.95** 10.1	**3.17** 10.8	**3.41** 11.6	**3.70** 12.6	
24	**1.93** 6.56	**2.01** 6.85	**2.10** 7.17	**2.20** 7.51	**2.31** 7.88	**2.43** 8.30	**2.57** 8.76	**2.72** 9.27	**2.89** 9.86	**3.08** 10.5	**3.31** 11.3	**3.56** 12.1	**3.86** 13.1	
25	**2.01** 6.83	**2.09** 7.14	**2.19** 7.46	**2.29** 7.82	**2.41** 8.21	**2.54** 8.65	**2.68** 9.13	**2.83** 9.66	**1.01** 10.3	**3.21** 11.0	**3.44** 11.7	**3.71** 12.6	**4.01** 13.7	
26	**2.09** 7.11	**2.18** 7.43	**2.28** 7.77	**2.38** 8.13	**2.51** 8.54	**2.64** 9.00	**2.78** 9.50	**2.95** 10.0	**3.13** 10.7	**3.34** 11.4	**3.58** 12.2	**3.86** 13.1	**4.18** 14.2	
27	**2.17** 7.38	**2.26** 7.72	**2.37** 8.06	**2.48** 8.45	**2.60** 8.87	**2.74** 9.35	**2.89** 9.87	**3.06** 10.4	**3.25** 11.1	**3.47** 11.8	**3.72** 12.7	**4.01** 13.6	**4.34** 14.8	
28	**2.25** 7.66	**2.35** 8.00	**2.45** 8.36	**2.57** 8.75	**2.70** 9.20	**2.84** 9.69	**3.00** 10.2	**3.17** 10.8	**3.37** 11.5	**3.60** 12.3	**3.86** 13.1	**4.16** 14.1	**4.50** 15.3	
29	**2.33** 7.93	**2.43** 8.29	**2.54** 8.66	**2.66** 9.08	**2.80** 9.53	**2.94** 10.0	**3.11** 10.6	**3.29** 11.2	**3.49** 11.9	**3.73** 12.7	**3.99** 13.6	**4.30** 14.6	**4.66** 15.9	
30	**2.41** 8.21	**2.52** 8.58	**2.63** 8.96	**2.75** 9.39	**2.89** 9.86	**3.04** 10.4	**3.21** 11.0	**3.40** 11.6	**3.62** 12.3	**3.86** 13.1	**4.13** 14.1	**4.45** 15.2	**4.82** 16.4	
31	**2.49** 8.48	**2.60** 8.86	**2.72** 9.26	**2.84** 9.70	**2.99** 10.2	**3.14** 10.7	**3.32** 11.3	**3.52** 12.0	**3.74** 12.7	**3.99** 13.6	**4.27** 14.5	**4.60** 15.7	**4.98** 17.0	
32	**2.57** 8.75	**2.68** 9.15	**2.81** 9.56	**2.94** 10.0	**3.08** 10.5	**3.25** 11.1	**3.43** 11.7	**3.63** 12.4	**3.86** 13.1	**4.11** 14.0	**4.41** 15.0	**4.75** 16.2	**5.14** 17.5	
33	**2.65** 9.03	**2.77** 9.44	**2.89** 9.86	**3.03** 10.3	**3.18** 10.8	**3.35** 11.4	**3.53** 12.1	**3.74** 12.7	**3.98** 13.5	**4.24** 14.5	**4.55** 15.5	**4.90** 16.7	**5.31** 18.1	
34	**2.73** 9.30	**2.85** 9.72	**2.98** 10.2	**3.12** 10.6	**3.28** 11.2	**3.45** 11.8	**3.64** 12.4	**3.86** 13.1	**4.10** 14.0	**4.37** 14.9	**4.69** 16.0	**5.05** 17.2	**5.47** 18.6	
												½ %		

TABLE 17.—AREAS AND WEIGHTS OF ¼-IN. WIRE SPIRALS

Core diam. (inches)	Areas of equivalent cylinders in square inches given in heavy figures. Weights of wire in pounds per foot length of spiral in light figures												
	Pitch of spiral in inches												
	3″	2⅞″	2¾″	2⅝″	2½″	2⅜″	2¼″	2⅛″	2″	1⅞″	1¾″	1⅝″	1½″
10	**0.515** 1.75	**0.537** 1.83	**0.562** 1.91	**0.588** 2.00	**0.618** 2.10	**0.651** 2.21	**0.686** 2.33	**0.726** 2.47	**0.771** 2.62	**0.824** 2.80	**0.882** 3.00	**0.950** 3.23	**1.03** 3.50
11	**0.566** 1.92	**0.591** 2.01	**0.618** 2.10	**0.646** 2.20	**0.679** 2.31	**0.715** 2.43	**0.754** 2.56	**0.798** 2.72	**0.848** 2.88	**0.905** 3.08	**0.971** 3.30	**1.04** 3.55	**1.13** 3.85
12	**0.617** 2.10	**0.645** 2.19	**0.674** 2.29	**0.705** 2.40	**0.741** 2.52	**0.781** 2.65	**0.823** 2.80	**0.871** 2.96	**0.925** 3.15	**0.987** 3.36	**1.06** 3.60	**1.14** 3.88	**1.23** 4.19
13	**0.668** 2.27	**0.699** 2.38	**0.730** 2.48	**0.764** 2.60	**0.803** 2.73	**0.846** 2.88	**0.892** 3.03	**0.944** 3.21	**1.00** 3.41	**1.07** 3.64	**1.15** 3.90	**1.23** 4.20	**1.33** 4.54
14	**0.720** 2.45	**0.752** 2.56	**0.786** 2.67	**0.823** 2.80	**0.865** 2.94	**0.911** 3.10	**0.961** 3.26	**1.02** 3.46	**1.08** 3.67	**1.15** 3.92	**1.23** 4.20	**1.33** 4.52	**1.44** 1% 4.90
15	**0.771** 2.62	**0.807** 2.74	**0.843** 2.87	**0.882** 3.00	**0.926** 3.15	**0.976** 3.32	**1.03** 3.50	**1.09** 3.71	**1.16** 3.94	**1.23** 4.20	**1.32** 4.50	**1.42** 4.85	**1.54** 5.24
16	**0.823** 2.80	**0.860** 2.92	**0.898** 3.05	**0.940** 3.20	**0.987** 3.36	**1.04** 3.54	**1.10** 3.73	**1.16** 3.95	**1.23** 4.20	**1.31** 4.48	**1.41** 4.80	**1.52** 5.17	**1.64** 5.60
17	**0.874** 2.97	**0.913** 3.10	**0.954** 3.24	**1.00** 3.40	**1.05** 3.56	**1.11** 3.75	**1.17** 3.96	**1.23** 4.19	**1.31** 4.46	**1.40** 4.76	**1.50** 5.10	**1.61** 5.49	**1.74** 5.94
18	**0.926** 3.15	**0.966** 3.29	**1.01** 3.44	**1.06** 3.60	**1.11** 3.78	**1.17** 3.98	**1.24** 4.20	**1.31** 4.45	**1.39** 4.73	**1.48** 5.05	**1.59** 5.40	**1.71** 5.82	**1.85** 6.28
19	**0.978** 3.32	**1.02** 3.47	**1.07** 3.62	**1.12** 3.80	**1.17** 3.98	**1.24** 4.20	**1.30** 4.43	**1.38** 4.69	**1.46** 4.98	**1.56** 5.32	**1.67** 5.70	**1.80** 6.14	**1.95** 6.64
20	**1.03** 3.50	**1.07** 3.65	**1.12** 3.82	**1.17** 4.00	**1.23** 4.20	**1.30** 4.42	**1.37** 4.66	**1.45** 4.93	**1.54** 5.25	**1.64** 5.60	**1.76** 6.00	**1.90** 6.46	**2.06** 7.00
21	**1.08** 3.67	**1.13** 3.84	**1.18** 4.01	**1.23** 4.20	**1.30** 4.40	**1.37** 4.64	**1.44** 4.90	**1.53** 5.18	**1.62** 5.51	**1.73** 5.88	**1.85** 6.30	**1.99** 6.78	**2.16** 7.34
22	**1.13** 3.85	**1.18** 4.02	**1.23** 4.20	**1.29** 4.40	**1.36** 4.62	**1.43** 4.86	**1.51** 5.13	**1.69** 5.43	**1.70** 5.77	**1.81** 6.16	**1.94** 6.60	**2.09** 7.11	**2.27** 7.70
23	**1.18** 4.02	**1.23** 4.20	**1.29** 4.38	**1.35** 4.60	**1.42** 4.83	**1.49** 5.08	**1.58** 5.36	**1.67** 5.67	**1.77** 6.03	**1.89** 6.44	**2.03** 6.90	**2.18** 7.43	**2.37** 8.05
24	**1.24** 4.20	**1.29** 4.38	**1.35** 4.57	**1.41** 4.80	**1.48** 5.04	**1.56** 5.30	**1.65** 5.60	**1.74** 5.92	**1.85** 6.30	**1.97** 6.72	**2.12** 7.20	**2.28** 7.75	**2.47** 8.40
25	**1.29** 4.37	**1.34** 4.57	**1.40** 4.78	**1.47** 5.00	**1.54** 5.25	**1.63** 5.53	**1.71** 5.83	**1.82** 6.17	**1.93** 6.56	**2.06** 7.09	**2.20** 7.50	**2.37** 8.07	**2.57** 8.75
26	**1.34** 4.55	**1.40** 4.75	**1.46** 4.96	**1.53** 5.20	**1.61** 5.46	**1.69** 5.75	**1.78** 6.06	**1.89** 6.42	**2.00** 6.82	**2.14** 7.28	**2.29** 7.80	**2.47** 8.39	**2.67** 9.10
27	**1.39** 4.72	**1.45** 4.94	**1.52** 5.15	**1.59** 5.40	**1.67** 5.67	**1.76** 5.96	**1.85** 6.30	**1.96** 6.66	**2.08** 7.08	**2.22** 7.56	**2.38** 8.10	**2.56** 8.72	**2.78** ½% 9.45
28	**1.44** 4.90	**1.50** 5.11	**1.57** 5.34	**1.64** 5.60	**1.73** 5.88	**1.82** 6.19	**1.92** 6.53	**2.03** 6.91	**2.16** 7.34	**2.30** 7.84	**2.47** 8.40	**2.66** 9.04	**2.88** 9.80
29	**1.49** 5.07	**1.56** 5.30	**1.63** 5.53	**1.70** 5.80	**1.79** 6.09	**1.89** 6.41	**1.99** 6.77	**2.11** 7.16	**2.24** 7.60	**2.38** 8.12	**2.56** 8.70	**2.75** 9.36	**2.98** 10.15

Spirals above and to right of zigzag lines are nearest commercial size fully equal to percentage of core area indicated at end of line.

Weights in above table include the wire only. To this must be added the weights of the spacing bars and the weight of extra turn at ends of spiral.

Table 18.—Areas and Weights of 5/16-in. Wire Spirals

Core diam. (inches)	Areas of equivalent cylinders in square inches given in heavy figures. Weights of wire in pounds per foot length of spiral in light figures. Pitch of spiral in inches													
	3″	2⅞″	2¾″	2⅝″	2½″	2⅜″	2¼″	2⅛″	2″	1⅞″	1¾″	1⅝″	1½″	
10	**0.803** 2.73	**0.838** 2.86	**0.876** 2.98	**0.918** 3.13	**0.964** 3.28	**1.01** 3.46	**1.07** 3.65	**1.13** 3.86	**1.20** 4.10	**1.28** 4.38	**1.38** 4.69	**1.48** 5.05	**1.61** 5.48	
11	**0.884** 3.01	**0.922** 3.14	**0.963** 3.28	**1.01** 3.44	**1.06** 3.61	**1.12** 3.80	**1.18** 4.01	**1.25** 4.25	**1.33** 4.52	**1.41** 4.82	**1.51** 5.16	**1.63** 5.56	**1.77** 6.02	
12	**0.965** 3.28	**1.01** 3.43	**1.05** 3.58	**1.10** 3.75	**1.16** 3.93	**1.22** 4.15	**1.28** 4.39	**1.36** 4.64	**1.44** 4.93	**1.54** 5.26	**1.65** 5.63	**1.78** 6.06	**1.93** 6.57	
13	**1.04** 3.55	**1.09** 3.72	**1.14** 3.88	**1.19** 4.06	**1.25** 4.27	**1.32** 4.50	**1.39** 4.75	**1.47** 5.02	**1.57** 5.34	**1.67** 5.70	**1.79** 6.10	**1.93** 6.57	**2.09** 7.11	1½ %
14	**1.12** 3.83	**1.17** 4.00	**1.23** 4.18	**1.28** 4.38	**1.35** 4.60	**1.42** 4.84	**1.50** 5.11	**1.59** 5.41	**1.69** 5.75	**1.80** 6.13	**1.93** 6.56	**2.08** 7.08	**2.25** 7.67	
15	**1.20** 4.10	**1.26** 4.29	**1.31** 4.48	**1.38** 4.69	**1.45** 4.93	**1.52** 5.19	**1.61** 5.48	**1.70** 5.79	**1.81** 6.16	**1.93** 6.57	**2.07** 7.04	**2.22** 7.59	**2.41** 8.21	
16	**1.29** 4.38	**1.34** 4.57	**1.40** 4.78	**1.47** 5.01	**1.54** 5.26	**1.62** 5.53	**1.71** 5.84	**1.81** 6.18	**1.93** 6.57	**2.06** 7.01	**2.20** 7.51	**2.38** 8.08	**2.57** 8.76	
17	**1.37** 4.65	**1.42** 4.86	**1.49** 5.08	**1.56** 5.32	**1.64** 5.58	**1.72** 5.88	**1.82** 6.21	**1.93** 6.56	**2.05** 6.98	**2.19** 7.45	**2.34** 7.98	**2.52** 8.59	**2.73** 9.31	
18	**1.45** 4.93	**1.51** 5.14	**1.58** 5.37	**1.65** 5.63	**1.73** 5.91	**1.83** 6.22	**1.93** 6.57	**2.04** 6.95	**2.17** 7.39	**2.31** 7.88	**2.48** 8.44	**2.67** 9.10	**2.89** 9.86	
19	**1.53** 5.20	**1.59** 5.43	**1.66** 5.68	**1.74** 5.94	**1.83** 6.24	**1.93** 6.57	**2.03** 6.94	**2.16** 7.34	**2.29** 7.80	**2.44** 8.32	**2.62** 8.92	**2.82** 9.60	**3.05** 10.4	
20	**1.61** 5.47	**1.68** 5.71	**1.75** 5.97	**1.83** 6.26	**1.93** 6.57	**2.03** 6.91	**2.14** 7.30	**2.27** 7.72	**2.41** 8.21	**2.57** 8.76	**2.75** 9.39	**2.97** 10.1	**3.21** 11.0	1 %
21	**1.69** 5.74	**1.76** 6.00	**1.84** 6.27	**1.93** 6.56	**2.02** 6.89	**2.13** 7.26	**2.25** 7.67	**2.38** 8.11	**2.53** 8.62	**2.70** 9.20	**2.89** 9.85	**3.12** 10.6	**3.37** 11.5	
22	**1.77** 6.02	**1.84** 6.29	**1.93** 6.57	**2.02** 6.88	**2.12** 7.22	**2.23** 7.61	**2.35** 8.03	**2.50** 8.49	**2.65** 9.04	**2.83** 9.65	**3.03** 10.3	**3.26** 11.1	**3.53** 12.0	
23	**1.85** 6.29	**1.93** 6.57	**2.02** 6.87	**2.11** 7.19	**2.22** 7.55	**2.33** 7.95	**2.46** 8.40	**2.61** 8.88	**2.77** 9.45	**2.95** 10.1	**3.17** 10.8	**3.41** 11.6	**3.70** 12.6	
24	**1.93** 6.56	**2.01** 6.85	**2.10** 7.17	**2.20** 7.51	**2.31** 7.88	**2.43** 8.30	**2.57** 8.76	**2.72** 9.27	**2.89** 9.86	**3.08** 10.5	**3.31** 11.3	**3.56** 12.1	**3.86** 13.1	
25	**2.01** 6.83	**2.09** 7.14	**2.19** 7.46	**2.29** 7.82	**2.41** 8.21	**2.54** 8.65	**2.68** 9.13	**2.83** 9.66	**1.01** 10.3	**3.21** 11.0	**3.44** 11.7	**3.71** 12.6	**4.01** 13.7	
26	**2.09** 7.11	**2.18** 7.43	**2.28** 7.77	**2.38** 8.13	**2.51** 8.54	**2.64** 9.00	**2.78** 9.50	**2.95** 10.0	**3.13** 10.7	**3.34** 11.4	**3.58** 12.2	**3.86** 13.1	**4.18** 14.2	
27	**2.17** 7.38	**2.26** 7.72	**2.37** 8.06	**2.48** 8.45	**2.60** 8.87	**2.74** 9.35	**2.89** 9.87	**3.06** 10.4	**3.25** 11.1	**3.47** 11.8	**3.72** 12.7	**4.01** 13.6	**4.34** 14.8	
28	**2.25** 7.66	**2.35** 8.00	**2.45** 8.36	**2.57** 8.75	**2.70** 9.20	**2.84** 9.69	**3.00** 10.2	**3.17** 10.8	**3.37** 11.5	**3.60** 12.3	**3.86** 13.1	**4.16** 14.1	**4.50** 15.3	
29	**2.33** 7.93	**2.43** 8.29	**2.54** 8.66	**2.66** 9.08	**2.80** 9.53	**2.94** 10.0	**3.11** 10.6	**3.29** 11.2	**3.49** 11.9	**3.73** 12.7	**3.99** 13.6	**4.30** 14.6	**4.66** 15.9	
30	**2.41** 8.21	**2.52** 8.58	**2.63** 8.96	**2.75** 9.39	**2.89** 9.86	**3.04** 10.4	**3.21** 11.0	**3.40** 11.6	**3.62** 12.3	**3.86** 13.1	**4.13** 14.1	**4.46** 15.2	**4.82** 16.4	
31	**2.49** 8.48	**2.60** 8.86	**2.72** 9.26	**2.84** 9.70	**2.99** 10.2	**3.14** 10.7	**3.32** 11.3	**3.52** 12.0	**3.74** 12.7	**3.99** 13.6	**4.27** 14.5	**4.60** 15.7	**4.98** 17.0	
32	**2.57** 8.75	**2.68** 9.15	**2.81** 9.56	**2.94** 10.0	**3.08** 10.5	**3.25** 11.1	**3.43** 11.7	**3.63** 12.4	**3.86** 13.1	**4.11** 14.0	**4.41** 15.0	**4.75** 16.2	**5.14** 17.5	
33	**2.65** 9.03	**2.77** 9.44	**2.89** 9.86	**3.03** 10.3	**3.18** 10.8	**3.35** 11.4	**3.53** 12.1	**3.74** 12.7	**3.98** 13.5	**4.24** 14.5	**4.55** 15.5	**4.90** 16.7	**5.31** 18.1	
34	**2.73** 9.30	**2.85** 9.72	**2.98** 10.2	**3.12** 10.6	**3.28** 11.2	**3.45** 11.8	**3.64** 12.4	**3.86** 13.1	**4.10** 14.0	**4.37** 14.9	**4.69** 16.0	**5.05** 17.2	**5.47** 18.6	
												½ %		

Table 19.—Areas and Weights of 3/8-in. Wire Spirals

Areas of equivalent cylinders in square inches given in heavy figures
Weights of wire in pounds per foot length of spiral in light figures

Core diam. (inches)	Pitch of spiral in inches													
	3"	2⅞"	2¾"	2⅝"	2½"	2⅜"	2¼"	2⅛"	2"	1⅞"	1¾"	1⅝"	1½"	
15	**1.74** 5.91	**1.81** 6.16	**1.89** 6.44	**1.98** 6.73	**2.08** 7.18	**2.19** 7.46	**2.32** 7.87	**2.46** 8.32	**2.60** 8.85	**2.78** 9.45	**2.97** 10.1	**3.21** 10.9	**3.47** 11.8	
16	**1.85** 6.30	**1.93** 6.57	**2.02** 6.87	**2.12** 7.18	**2.22** 7.55	**2.34** 7.95	**2.47** 8.39	**2.62** 8.88	**2.78** 9.44	**2.96** 10.1	**3.17** 10.8	**3.42** 11.6	**3.70** 12.6	
17	**1.97** 6.69	**2.05** 6.98	**2.14** 7.30	**2.25** 7.63	**2.36** 8.02	**2.48** 8.45	**2.62** 8.92	**2.78** 9.44	**2.95** 10.0	**3.15** 10.7	**3.37** 11.5	**3.63** 12.3	**3.94** 13.4	
18	**2.08** 7.08	**2.17** 7.39	**2.27** 7.73	**2.38** 80.8	**2.50** 8.50	**2.64** 8.94	**2.78** 9.44	**2.94** 10.0	**3.12** 10.6	**3.33** 11.3	**3.57** 12.1	**3.84** 13.1	**4.16** 14.2	
19	**2.20** 7.48	**2.29** 7.81	**2.40** 8.15	**2.51** 8.53	**2.64** 8.97	**2.78** 9.45	**2.93** 9.97	**3.10** 10.5	**3.30** 11.2	**3.52** 12.0	**3.77** 12.8	**4.06** 13.8	**4.40** 14.9	1½%
20	**2.32** 7.87	**2.41** 8.21	**2.52** 8.59	**2.64** 8.98	**2.78** 9.44	**2.92** 9.94	**3.09** 10.5	**3.27** 11.1	**3.47** 11.8	**3.70** 12.6	**3.97** 13.5	**4.27** 14.5	**4.63** 15.7	
21	**2.43** 8.26	**2.53** 8.62	**2.65** 9.02	**2.78** 9.43	**2.92** 9.92	**3.07** 10.4	**3.24** 11.0	**3.43** 11.6	**3.67** 12.4	**3.89** 13.2	**4.16** 14.2	**4.49** 15.2	**4.86** 16.5	
22	**2.55** 8.66	**2.65** 9.01	**2.77** 9.45	**2.91** 9.88	**3.05** 10.4	**3.22** 10.9	**3.39** 11.5	**3.59** 12.2	**3.82** 13.0	**4.07** 13.8	**4.36** 14.8	**4.70** 16.0	**5.09** 17.3	
23	**2.66** 9.05	**2.78** 9.45	**2.90** 9.88	**3.04** 10.3	**3.19** 10.9	**3.36** 11.4	**3.55** 12.1	**3.75** 12.8	**3.99** 13.6	**4.26** 14.5	**4.56** 15.5	**4.91** 16.7	**5.32** 18.1	
24	**2.78** 9.45	**2.90** 9.86	**3.03** 10.3	**3.17** 10.8	**3.33** 11.3	**3.51** 11.9	**3.70** 12.6	**3.92** 13.3	**4.17** 14.2	**4.44** 15.1	**4.76** 16.2	**5.12** 17.4	**5.55** 18.9	
25	**2.89** 9.84	**3.02** 10.3	**3.15** 10.7	**3.30** 11.2	**3.47** 11.8	**3.65** 12.4	**3.86** 13.1	**4.08** 13.9	**4.34** 14.7	**4.63** 15.7	**4.96** 16.9	**5.34** 18.2	**5.79** 19.7	
26	**3.01** 10.2	**3.14** 10.7	**3.28** 11.2	**3.44** 11.7	**3.61** 12.3	**3.80** 12.9	**4.01** 13.6	**4.25** 14.4	**4.51** 15.4	**4.81** 16.4	**5.16** 17.5	**5.55** 18.9	**6.02** 20.4	
27	**3.12** 10.6	**3.26** 11.1	**3.41** 11.6	**3.57** 12.1	**3.75** 12.7	**3.95** 13.4	**4.17** 14.2	**4.41** 15.0	**4.69** 15.9	**5.00** 17.0	**5.36** 18.2	**5.76** 19.6	**6.25** 21.2	
28	**3.24** 11.0	**3.38** 11.5	**3.53** 12.0	**3.70** 12.6	**3.89** 13.2	**4.09** 13.9	**4.73** 14.7	**4.57** 15.5	**4.86** 16.5	**5.19** 17.6	**5.55** 18.9	**5.98** 20.4	**6.48** 22.0	
29	**3.36** 11.4	**3.50** 11.9	**3.66** 12.5	**3.84** 13.0	**4.03** 13.7	**4.24** 14.4	**4.47** 15.2	**4.74** 16.1	**5.04** 17.1	**5.37** 18.3	**5.75** 19.6	**6.20** 21.1	**6.71** 22.8	1%
30	**3.47** 11.8	**3.62** 12.3	**3.79** 12.9	**3.97** 13.5	**4.17** 14.2	**4.39** 14.9	**4.63** 15.7	**4.90** 16.6	**5.21** 17.7	**5.55** 18.9	**5.95** 20.2	**6.41** 21.8	**6.94** 23.6	
31	**3.59** 12.2	**3.74** 12.7	**3.91** 13.3	**4.10** 13.9	**4.31** 14.6	**4.53** 15.4	**4.78** 16.3	**5.06** 16.2	**5.39** 18.3	**5.74** 19.5	**6.15** 20.9	**6.62** 22.5	**7.18** 24.4	
32	**3.70** 12.6	**3.86** 13.1	**4.04** 13.7	**4.23** 14.4	**4.44** 15.1	**4.68** 15.9	**4.94** 16.8	**5.23** 17.8	**5.56** 18.9	**5.93** 20.1	**6.35** 21.6	**6.84** 23.3	**7.41** 25.2	
33	**3.82** 13.0	**3.98** 13.5	**4.17** 14.2	**4.36** 14.8	**4.59** 15.6	**4.82** 16.4	**5.09** 17.3	**5.39** 18.3	**5.73** 19.5	**6.11** 20.8	**6.55** 22.3	**7.05** 24.0	**7.64** 26.0	
34	**3.94** 13.4	**4.11** 14.0	**4.29** 14.6	**4.50** 15.3	**4.72** 16.1	**4.97** 16.9	**5.25** 17.8	**5.56** 18.9	**5.91** 20.1	**6.30** 21.4	**6.75** 23.0	**7.27** 24.7	**7.87** 26.8	
35	**4.05** 13.8	**4.23** 14.4	**4.42** 15.0	**4.63** 15.7	**4.86** 16.5	**5.12** 17.4	**5.40** 18.4	**5.72** 19.4	**6.08** 20.7	**6.48** 22.0	**6.95** 23.6	**7.48** 25.4	**8.10** 27.5	
36	**4.18** 14.2	**4.34** 14.8	**4.54** 15.4	**4.76** 16.2	**5.00** 17.0	**5.26** 17.9	**5.56** 18.9	**5.88** 20.0	**6.25** 21.3	**6.66** 22.7	**7.14** 24.3	**7.69** 26.2	**8.33** 28.3	
37	**4.28** 14.6	**4.47** 15.2	**4.67** 15.9	**4.89** 16.6	**5.14** 17.5	**5.41** 18.4	**5.71** 19.4	**6.05** 20.6	**6.43** 21.9	**6.85** 23.3	**7.34** 25.0	**7.91** 26.9	**8.56** 29.1	
38	**4.40** 15.0	**4.59** 15.6	**4.80** 16.3	**5.03** 17.1	**5.28** 18.0	**5.56** 18.9	**5.87** 19.9	**6.21** 21.1	**6.60** 22.4	**7.04** 23.9	**7.54** 25.6	**8.12** 27.6	**8.79** 29.9	
39	**4.51** 15.4	**4.71** 16.0	**4.92** 16.7	**5.16** 17.5	**5.42** 18.4	**5.70** 19.4	**6.02** 20.5	**6.37** 21.6	**6.77** 23.0	**7.22** 24.6	**7.74** 26.3	**8.33** 28.3	**9.03** 30.7	

Table 20.—Areas and Weights of $\frac{7}{16}$-in. Wire Spirals

Core diam. (inches)	Areas of equivalent cylinders in square inches given in heavy figures Weights of wire in pounds per foot length of spiral in light figures													
	Pitch of spiral in inches													
	3″	2⅞″	2¾″	2⅝″	2½″	2⅜″	2¼″	2⅛″	2″	1⅞″	1¾″	1⅝″	1½″	
15	**2.36** 8.03	**2.47** 8.38	**2.58** 8.76	**2.70** 9.21	**2.84** 9.65	**2.98** 10.1	**3.15** 10.7	**3.34** 11.3	**3.54** 12.0					
16	**2.52** 8.56	**2.63** 8.94	**2.75** 9.35	**2.88** 9.80	**3.02** 10.3	**3.18** 10.8	**3.36** 11.4	**3.56** 12.1	**3.78** 12.8	**4.03** 13.7				
17	**2.68** 9.10	**2.80** 9.50	**2.92** 9.93	**3.06** 10.4	**3.22** 10.9	**3.38** 11.5	**3.57** 12.1	**3.78** 12.8	**4.01** 13.7	**4.28** 14.6	**4.59** 15.6			
18	**2.83** 9.64	**2.96** 10.1	**3.09** 10.5	**3.24** 11.0	**3.40** 11.6	**3.58** 12.2	**3.78** 12.9	**3.99** 13.6	**4.25** 14.5	**4.53** 15.4	**4.86** 16.5	**5.23** 17.8		
19	**2.99** 10.2	**3.12** 10.6	**3.26** 11.1	**3.42** 11.6	**3.59** 12.2	**3.78** 12.8	**3.99** 13.6	**4.22** 14.3	**4.49** 15.3	**4.79** 16.3	**5.13** 17.4	**5.52** 18.8	**5.98** 20.3	
20	**3.15** 10.7	**3.28** 11.2	**3.44** 11.7	**3.60** 12.2	**3.78** 12.8	**3.98** 13.5	**4.20** 14.3	**4.44** 15.1	**4.72** 16.1	**5.04** 17.1	**5.40** 18.3	**5.81** 19.7	**6.29** 21.4	2%
21	**3.30** 11.2	**3.45** 11.7	**3.61** 12.3	**3.78** 12.8	**3.97** 13.5	**4.18** 14.2	**4.41** 15.0	**4.66** 15.8	**4.96** 16.9	**5.29** 18.0	**5.67** 19.3	**6.10** 20.7	**6.61** 22.5	
22	**3.46** 11.8	**3.62** 12.3	**3.78** 12.8	**3.96** 13.5	**4.16** 14.1	**4.38** 14.9	**4.62** 15.7	**4.89** 16.6	**5.19** 17.7	**5.54** 18.8	**5.94** 20.2	**6.39** 21.7	**6.92** 23.7	
23	**3.62** 12.3	**3.78** 12.9	**3.95** 13.4	**4.14** 14.1	**4.35** 14.8	**4.58** 15.6	**4.83** 16.4	**5.11** 17.4	**5.43** 18.5	**5.79** 19.7	**6.21** 21.1	**6.68** 22.7	**7.24** 24.6	
24	**3.78** 12.8	**3.94** 13.4	**4.12** 14.0	**4.32** 14.7	**4.54** 15.4	**4.78** 16.2	**5.04** 17.1	**5.33** 18.1	**5.67** 19.3	**6.04** 20.6	**6.48** 22.0	**6.97** 23.7	**7.55** 25.7	
25	**3.93** 13.4	**4.11** 14.0	**4.29** 14.6	**4.50** 15.3	**4.73** 16.1	**4.98** 16.9	**5.25** 17.9	**5.55** 18.9	**5.90** 20.1	**6.30** 21.4	**6.75** 22.9	**7.26** 24.7	**7.89** 26.8	
26	**4.09** 13.9	**4.28** 14.5	**4.47** 15.2	**4.68** 15.9	**4.92** 16.7	**5.18** 17.6	**5.46** 18.6	**5.77** 19.6	**6.14** 20.9	**6.55** 22.3	**7.02** 23.8	**7.55** 25.7	**8.18** 27.8	1½%
27	**4.25** 14.4	**4.44** 15.1	**4.64** 15.8	**4.86** 16.5	**5.11** 17.3	**5.38** 18.3	**5.67** 19.3	**6.00** 20.4	**6.38** 21.7	**6.80** 23.1	**7.29** 24.8	**7.84** 26.7	**8.50** 28.9	
28	**4.41** 15.0	**4.60** 15.6	**4.81** 16.4	**5.04** 17.1	**5.29** 18.0	**5.57** 18.9	**5.88** 20.0	**6.21** 21.1	**6.61** 22.5	**7.05** 24.0	**7.56** 25.7	**8.13** 27.6	**8.81** 30.0	
29	**4.57** 15.5	**4.76** 16.2	**4.99** 16.9	**5.22** 17.7	**5.48** 18.6	**5.77** 19.6	**6.09** 20.7	**6.44** 21.9	**6.85** 23.3	**7.31** 24.8	**7.83** 26.6	**8.43** 28.6	**9.13** 31.1	
30	**4.72** 16.0	**4.93** 16.8	**5.16** 17.5	**5.40** 18.3	**5.67** 19.3	**5.97** 20.3	**6.31** 21.5	**6.66** 22.6	**7.09** 24.1	**7.56** 25.7	**8.10** 27.5	**8.72** 29.6	**9.45** 32.1	
31	**4.88** 16.6	**5.09** 17.3	**5.33** 18.1	**5.58** 19.0	**5.86** 19.9	**6.17** 20.9	**6.51** 22.2	**6.88** 23.4	**7.32** 24.9	**7.81** 26.6	**8.37** 28.4	**9.01** 30.6	**9.76** 33.2	
32	**5.04** 17.1	**5.26** 17.9	**5.50** 18.7	**5.76** 19.6	**6.05** 20.6	**6.37** 21.6	**6.72** 22.9	**7.10** 24.2	**7.56** 25.7	**8.06** 27.4	**8.64** 29.4	**9.30** 31.6	**10.1** 34.3	
33	**5.19** 17.7	**5.42** 18.4	**5.67** 19.3	**5.94** 20.2	**6.24** 21.2	**6.57** 22.3	**6.93** 23.6	**7.33** 24.9	**7.79** 26.5	**8.31** 28.3	**8.92** 30.3	**9.59** 32.6	**10.4** 35.3	
34	**5.35** 18.2	**5.58** 19.0	**5.84** 19.9	**6.12** 20.8	**6.43** 21.9	**6.77** 23.0	**7.15** 24.3	**7.55** 25.7	**8.03** 27.3	**8.56** 29.1	**9.19** 31.2	**9.89** 33.6	**10.7** 36.4	
35	**5.51** 18.7	**5.75** 19.5	**6.02** 20.4	**6.30** 21.4	**6.62** 22.5	**6.97** 23.7	**7.35** 25.0	**7.77** 26.4	**8.26** 28.1	**8.82** 30.0	**9.45** 32.1	**10.2** 34.6	**11.0** 37.5	
36	**5.66** 19.3	**5.91** 20.1	**6.18** 21.0	**6.48** 22.0	**6.80** 23.1	**7.17** 24.4	**7.56** 25.7	**7.99** 27.2	**8.50** 28.9	**9.07** 30.8	**9.72** 33.0	**10.5** 35.5	**11.3** 38.5	
37	**5.82** 19.8	**6.08** 20.7	**6.35** 21.6	**6.66** 22.6	**6.99** 23.8	**7.37** 25.0	**7.79** 26.4	**8.22** 27.9	**8.74** 29.7	**9.32** 31.7	**10.0** 33.9	**10.7** 36.5	**11.6** 39.6	
38	**5.98** 20.4	**6.24** 21.2	**6.52** 22.2	**6.84** 23.3	**7.18** 24.4	**7.57** 25.7	**7.98** 27.2	**8.44** 28.7	**8.98** 30.5	**9.58** 32.6	**10.3** 34.9	**11.0** 37.5	**12.0** 40.7	
39	**6.13** 20.9	**6.40** 21.8	**6.69** 22.8	**7.02** 23.9	**7.37** 25.1	**7.77** 26.4	**8.19** 27.9	**8.66** 29.5	**9.22** 31.3	**9.83** 33.4	**10.5** 35.6	**11.3** 38.5	**12.3** 41.71	1%

TABLE 21.—AREAS AND WEIGHTS OF ½-IN. WIRE SPIRALS

Core diam. (inches)	Areas of equivalent cylinders in square inches given in heavy figures Weights of wire in pounds per foot length of spiral in light figures													
	Pitch of spiral in inches													
	3″	2⅞″	2¾″	2⅝″	2½″	2⅜″	2¼″	2⅛″	2″	1⅞″	1¾″	1⅝″	1½″	
15	**3.09** 10.5	**3.22** 11.0	**3.36** 11.4	**3.52** 12.0	**3.70** 12.6	**3.90** 13.2								
16	**3.29** 11.2	**3.43** 11.7	**3.59** 12.2	**3.76** 12.8	**3.94** 13.4	**4.15** 14.1								
17	**3.49** 11.9	**3.65** 12.4	**3.82** 13.0	**3.99** 13.6	**4.19** 14.3	**4.42** 15.0	**4.66** 15.9							
18	**3.70** 12.6	**3.86** 13.1	**4.04** 13.7	**4.23** 14.4	**4.44** 15.1	**4.68** 15.9	**4.94** 16.8	**5.23** 17.8						
19	**3.91** 13.3	**4.08** 13.9	**4.27** 14.5	**4.46** 15.2	**4.69** 16.0	**4.94** 16.8	**5.22** 17.7	**5.52** 18.8	**5.86** 20.0					
20	**4.11** 14.0	**4.29** 14.6	**4.49** 15.3	**4.70** 16.0	**4.93** 16.8	**5.19** 17.7	**5.49** 18.6	**5.80** 19.8	**6.17** 21.0	**6.59** 22.4				
21	**4.32** 14.7	**4.50** 15.3	**4.71** 16.0	**4.94** 16.8	**5.18** 17.6	**5.45** 18.5	**5.76** 19.6	**6.09** 20.8	**6.48** 22.0	**6.90** 23.5				
22	**4.52** 15.4	**4.72** 16.1	**4.94** 16.8	**5.16** 17.6	**5.42** 18.5	**5.71** 19.4	**6.03** 20.5	**6.38** 21.7	**6.79** 23.1	**7.24** 24.6	**7.76** 26.4			
23	**4.73** 16.1	**4.94** 16.8	**5.16** 17.6	**5.40** 18.4	**5.68** 19.3	**5.98** 20.3	**6.31** 21.4	**6.67** 22.7	**7.10** 24.1	**7.57** 25.8	**8.10** 27.6	**8.73** 29.9		
24	**4.93** 16.8	**5.15** 17.5	**5.39** 18.3	**5.64** 19.2	**5.93** 20.2	**6.24** 21.2	**6.58** 22.4	**6.96** 23.7	**7.40** 25.2	**7.90** 26.9	**8.46** 28.8	**9.11** 31.0		
25	**5.15** 17.5	**5.36** 18.3	**5.61** 19.1	**5.87** 20.0	**6.17** 21.0	**6.49** 22.1	**6.85** 23.3	**7.25** 24.7	**7.71** 26.2	**8.22** 28.0	**8.82** 30.0	**9.50** 32.3	**10.3** 35.0	
26	**5.35** 18.2	**5.59** 19.0	**5.84** 19.9	**6.11** 20.8	**6.42** 21.8	**6.76** 23.0	**7.13** 24.3	**7.55** 25.7	**8.02** 27.2	**8.56** 29.1	**9.18** 31.2	**9.87** 33.6	**10.7** 36.4	
27	**5.55** 18.9	**5.80** 19.7	**6.06** 20.6	**6.34** 21.6	**6.66** 22.7	**7.01** 23.9	**7.40** 25.2	**7.83** 26.7	**8.33** 28.3	**8.88** 30.2	**9.52** 32.4	**10.2** 34.9	**11.1** 37.8	2%
28	**5.75** 19.6	**6.01** 20.4	**6.29** 21.4	**6.58** 22.4	**6.90** 23.5	**7.28** 24.7	**7.68** 26.1	**8.12** 27.6	**8.64** 29.4	**9.21** 31.3	**9.87** 33.6	**10.6** 36.2	**11.5** 39.2	
29	**5.97** 20.3	**6.22** 21.2	**6.51** 22.2	**6.81** 23.2	**7.15** 24.4	**7.54** 25.6	**7.96** 27.1	**8.41** 28.7	**8.95** 30.4	**9.55** 32.5	**10.2** 34.8	**11.0** 37.5	**11.9** 40.6	
30	**6.17** 21.0	**6.44** 21.9	**6.74** 22.9	**7.05** 24.0	**7.40** 25.2	**7.79** 26.5	**8.23** 28.0	**8.70** 29.6	**9.25** 31.5	**9.87** 33.6	**10.6** 36.0	**11.4** 38.8	**12.3** 42.0	
31	**6.37** 21.7	**6.66** 22.6	**6.95** 23.6	**7.28** 24.8	**7.64** 26.0	**8.05** 27.4	**8.50** 28.9	**8.99** 30.6	**9.56** 32.5	**10.2** 34.7	**10.9** 37.2	**11.8** 40.0	**12.7** 43.3	
32	**6.58** 22.4	**6.87** 23.4	**7.18** 24.4	**7.52** 25.6	**7.89** 26.8	**8.31** 28.3	**8.77** 29.8	**9.28** 31.6	**9.87** 33.6	**10.5** 35.8	**11.3** 38.4	**12.1** 41.3	**13.2** 44.7	
33	**6.78** 23.1	**7.09** 24.1	**7.40** 25.2	**7.75** 26.4	**8.14** 27.7	**8.58** 29.2	**9.05** 30.8	**9.57** 32.6	**10.2** 34.6	**10.9** 36.9	**11.6** 39.5	**12.5** 42.6	**13.6** 46.1	
34	**6.99** 23.8	**7.30** 24.8	**7.62** 25.9	**7.98** 27.2	**8.38** 28.5	**8.83** 30.0	**9.32** 31.7	**9.86** 33.5	**10.5** 35.6	**11.2** 38.0	**12.0** 40.7	**12.9** 43.9	**14.0** 47.6	1½%
35	**7.19** 24.5	**7.51** 25.6	**7.85** 26.7	**8.22** 28.0	**8.63** 29.4	**9.09** 30.9	**9.60** 32.6	**10.1** 34.5	**10.8** 36.7	**11.5** 39.2	**12.3** 41.9	**13.3** 45.2	**14.4** 48.9	
36	**7.40** 25.2	**7.74** 26.3	**8.08** 27.5	**8.46** 28.8	**8.89** 30.2	**9.36** 31.8	**9.88** 33.6	**10.4** 35.5	**11.1** 37.8	**11.8** 40.3	**12.7** 43.2	**13.7** 46.5	**14.8** 50.4	
37	**7.60** 25.9	**7.95** 27.0	**8.30** 28.2	**8.69** 29.6	**9.13** 31.1	**9.60** 32.7	**10.1** 34.5	**10.7** 36.5	**11.4** 38.8	**12.2** 41.4	**13.0** 44.4	**14.0** 47.8	**15.2** 51.8	
38	**7.82** 26.6	**8.16** 27.8	**8.53** 29.1	**8.93** 30.4	**9.38** 31.9	**9.88** 33.6	**10.4** 35.5	**11.0** 37.5	**11.7** 39.9	**12.5** 42.5	**13.4** 45.6	**14.4** 49.1	**15.6** 53.2	
39	**8.02** 27.3	**8.37** 28.5	**8.75** 29.7	**9.16** 31.2	**9.63** 32.7	**10.1** 34.5	**10.7** 36.4	**11.3** 38.5	**12.0** 40.9	**12.8** 43.6	**13.7** 46.8	**14.8** 50.4	**16.0** 54.6	
										1%				

TABLE 22.—AREAS AND WEIGHTS OF 5/16-IN. WIRE SPIRALS

Core diam. (inches)	Areas of equivalent cylinders in square inches given in heavy figures Weights of wire in pounds per foot length of spiral in light figures													
	Pitch of spiral in inches													
	3"	2⅞"	2¾"	2⅝"	2½"	2⅜"	2¼"	2⅛"	2"	1⅞"	1¾"	1⅝"	1½"	
20	**5.21** 17.7	**5.43** 18.5	**5.68** 19.3	**5.95** 20.2	**6.24** 21.2									
21	**5.46** 18.6	**5.70** 19.4	**5.96** 20.3	**6.24** 21.2	**6.55** 22.3	**6.90** 23.5								
22	**5.73** 19.5	**5.97** 20.3	**6.25** 21.3	**6.55** 22.2	**6.86** 23.4	**7.23** 24.6	**7.65** 26.0							
23	**5.99** 20.4	**6.25** 21.2	**6.53** 22.2	**6.84** 23.2	**7.18** 24.4	**7.56** 25.7	**7.99** 27.2	**8.45** 28.7						
24	**6.25** 21.2	**6.52** 22.2	**6.81** 23.2	**7.14** 24.2	**7.50** 25.5	**7.89** 26.8	**8.34** 28.3	**8.83** 29.9	**9.37** 31.9					
25	**6.51** 22.1	**6.79** 23.1	**7.10** 24.2	**7.44** 25.2	**7.81** 26.6	**8.22** 28.0	**8.69** 29.5	**9.20** 31.2	**9.76** 33.2	**10.4** 35.4				
26	**6.77** 23.0	**7.06** 24.0	**7.39** 25.1	**7.74** 26.3	**8.12** 27.6	**8.56** 20.1	**9.05** 30.7	**9.57** 32.4	**10.1** 34.6	**10.8** 36.8				
27	**7.03** 23.9	**7.33** 24.4	**7.67** 26.1	**8.03** 27.3	**8.43** 28.6	**8.88** 30.2	**9.39** 31.9	**9.93** 33.7	**10.5** 35.8	**11.2** 38.2	**12.0** 40.9			
28	**7.29** 24.8	**7.60** 25.8	**7.95** 27.0	**8.33** 28.3	**8.74** 29.7	**9.20** 31.3	**9.73** 33.0	**10.3** 34.9	**10.9** 37.2	**11.7** 39.6	**12.5** 42.5			
29	**7.55** 25.7	**7.88** 26.8	**8.24** 28.0	**8 62** 29.3	**9.06** 30.8	**9.55** 32.4	**10.1** 34.3	**10.6** 36.2	**11.3** 38.5	**12.1** 41.1	**12.9** 44.0	**13.9** 47.4		
30	**7.81** 26.6	**8.14** 27.7	**8.52** 28.9	**8.93** 30.3	**9.37** 31.8	**9.88** 33.5	**10.4** 35.4	**11.0** 37.4	**11.7** 39.8	**12.5** 42.4	**13.4** 45.5	**14.4** 49.0		
31	**8.06** 27.4	**8.41** 28.6	**8.80** 29.9	**9.22** 31.3	**9.68** 32.7	**10.2** 34.6	**10.8** 36.6	**11.4** 38.7	**12.1** 41.2	**12.9** 43.9	**13.8** 47.0	**14.9** 50.5	**16.1** 54.9	
32	**8.32** 28.3	**8.69** 29.5	**9.09** 30.9	**9.52** 32.3	**10.0** 33.9	**10.5** 35.8	**11.1** 37.8	**11.8** 39.9	**12.5** 42.5	**13.3** 45.3	**14.3** 48.5	**15.4** 52.2	**16.6** 56.6	2%
33	**8.58** 29.2	**8.97** 30.4	**9.37** 31.9	**9.82** 33.3	**10.3** 35.0	**10.8** 36.9	**11.4** 39.0	**12.1** 41.1	**12.9** 43.8	**13.7** 46.7	**14.7** 50.0	**15.8** 53.8	**17.2** 58.4	
34	**8.84** 30.1	**9.23** 31.3	**9.65** 32.8	**10.1** 34.3	**10.6** 36.0	**11.2** 38.0	**11.8** 40.2	**12.5** 42.4	**13.3** 45.2	**14.1** 48.2	**15.2** 51.6	**16.3** 55.4	**17.7** 60.2	
35	**9.10** 30.9	**9.50** 32.3	**9.95** 33.8	**10.4** 35.3	**10.9** 37.1	**11.5** 39.1	**12.1** 41.4	**12.8** 43.6	**13.7** 46.5	**14.5** 49.6	**15.6** 53.1	**16.8** 57.1	**18.2** 61.9	
36	**9.38** 31.9	**9.79** 33.2	**10.2** 34.8	**10.7** 36.4	**11.2** 38.2	**11.8** 40.3	**12.5** 42.5	**13.2** 44.9	**14.1** 47.8	**15.0** 51.0	**16.1** 54.7	**17.3** 58.8	**18.7** 63.7	
37	**9.63** 32.7	**10.0** 34.1	**10.5** 35.7	**11.0** 37.4	**11.5** 39.2	**12.2** 41.4	**12.8** 43.7	**13.6** 46.1	**14.4** 49.2	**15.4** 52.4	**16.5** 56.2	**17.8** 60.4	**19.2** 65.5	
38	**9.90** 33.6	**10.3** 35.1	**10.8** 36.7	**11.3** 38.4	**11.9** 40.3	**12.5** 42.5	**13.2** 44.9	**14.0** 47.4	**14.8** 50.5	**15.8** 53.8	**16.9** 57.7	**18.3** 62.0	**19.8** 67.3	
39	**10.2** 34.5	**10.6** 36.0	**11.1** 37.7	**11.6** 39.4	**12.2** 41.4	**12.8** 43.6	**13.5** 46.0	**14.3** 48.6	**15.2** 51.8	**16.2** 55.2	**17.4** 59.2	**18.7** 63.6	**20.3** 69.0	
						1%							1½%	

Spirals above and to right of zigzag lines are nearest commercial size fully equal to percentage of core area indicated at end of line.

Weights in above table include the wire only. To this must be added the weights of the spacing bars and the weight of the extra turn at ends of spiral.

MEMBERS SUBJECT TO DIRECT AXIAL TENSION

In reinforced concrete members subject to direct tension the stress will be uniform over the section and the tensile strength of the concrete may not be counted as a dependable design factor. The section may and probably will be cracked entirely through when the full design load comes upon the member. All the stress must therefore be considered as taken by the longitudinal reinforcement.

65. Formula.—The only formula necessary for the design of such members is

$$f_s = P/A_s \tag{87}$$

in which P = tension or load on member.

A_s = area of longitudinal reinforcement.

f_s = tensile unit stress in longitudinal reinforcement.

66. Design of Tension Members.—The design of such members is extremely simple and needs little explanation. The steel stress in such members is commonly limited to 12,000 lb. per sq. in. The cracks in the concrete should be distributed at such frequent intervals that each will be of very small size. The simplest way to insure thi result is to use deformed bars of small diameter. Not all deformed bars have merit for this purpose and a properly designed bar should be specified and none other accepted. The cracks may be made fine and well distributed by placing wire mesh *with welded or integral intersections* just under the fireproofing shell of the member. This has value also in case of a severe fire in holding this shell together and its use is recommended in the general case even though deformed bars are also used.

Most designers feel that reinforced concrete tension members should be very conservatively designed. They are used in trusses and for hangers generally and it is not uncommon, in stair hangers for instance, to find designed stresses of only 4,000 to 5,000 lb. per sq. in. in the steel, thus assuring safety under extreme fire conditions or after very extensive corrosion has taken place. With such low stresses the corrosion would cause noticeable spalling of the concrete long before any danger of collapse was present.

The critical points in the design of most tension members lie at their extremities where they receive or deliver their load. For ordinary stair hangers a detail is commonly used making provision that the bent ends bear upon steel cross rods which in turn must engage a large volume of concrete. In column hangers structural steel is generally used in all cases where these hangers are attached to steel trusses at the top. *Bond on straight bars* should never be the sole nor even the main reliance for the end anchorage of tension members, but a positive mechanical anchorage should invariably be provided.

MEMBERS SUBJECT TO BENDING AND DIRECT COMPRESSION

It has previously been pointed out that very few columns are subject to axial compression only. A wall column in a reinforced concrete building is practically always subject to bending as well as compression, and while members should be so arranged as to minimize this bending it cannot be entirely avoided. Certainly in the usual beam-and-slab type of building and in all flat slab buildings, columns should be designed as members subject to bending and direct compression. It

is also true that beams and slabs very commonly take direct stress as well as bending. In stair slabs it is always true. The wind load is responsible for direct compression in other members. Too little attention has been paid to the actual condition of beams, slabs, and columns in this respect and we may expect to see less dependence (to an undetermined degree) on factor of safety and more careful design in the future. In arch rings and rigid-frame construction where the design takes account of such facts as these, the designer is certainly entitled to reduce his factor of safety on the working stresses below what would be proper in much present-day building design. In general, the direct stress is compression, for, except in certain members of concrete trusses, the framing of concrete structures is arranged so as to avoid direct tension in the members. The following discussion will be confined to cases of bending and compression. Formulas for cases in which the direct stress is tension may be derived in the same general manner as for cases where the direct stress is compression.

67. Design of Plain Concrete Piers for Eccentric Loading.—To design a plain concrete pier or caisson for eccentric loading, it is first necessary to assume the dead weight of the pier and find the total load and also the change in the point of application of the resultant due to the addition of the weight of the pier. If the resultant is inclined to the axis of the member, the component N parallel to the axis must also be determined. Next determine the distance c of the point of application of the rsultant from the gravity axis of the assumed section. The maximum fiber stress, f_c, in the concrete is determined by the usual formula for homogeneous material, Formula (88).[1] If the point of application of the resultant lies inside the "core" of the section, i.e., if c is less than $t/6$, no other figures are necessary. If it does not so lie, there will be tension on the section. In plain concrete construction a greater tension than $0.025f_c'$, or a maximum of 50 lb. per sq. in., should not be allowed. When this stress is exceeded, a reinforced concrete member must be used. The formula for computing the extreme fiber stresses is:

$$\left.\begin{array}{l} f_c \\ f_c'' \end{array}\right\} = \frac{N}{A} \begin{array}{l} + \\ - \end{array} \frac{Ntc}{2I_c} \qquad (88)$$

68. Bending and Compression on Reinforced Concrete Sections.—Stresses due to bending and compression on reinforced concrete sections may be determined by the methods of flexure for concrete beams modified to fit the conditions of the problem under consideration. Rectangular and round sections will be considered.

69. Notation.—The formulas for determining stresses in reinforced concrete members subject to bending and direct compression require certain additional symbols as follows:

$A = bt$ = area of concrete section.

A' = area of steel near face of member most highly stressed in compression.

A_s = area of steel near face of member least highly stressed in compression.

$A_o = A_1 + A_2$ = total steel area, with symmetrically placed reinforcement.

A_t = area of transformed section

[1] See also Sec. 1, p. 139.

e' = distance from point of application of N to the center of the tension steel.
f_c'' = compressive stress in extreme fiber of concrete at opposite face of member from f_c.
c_1 = distance from gravity axis to face most highly stressed in compression.
c_2 = distance from gravity axis to face least highly stressed in compression.
d' = distance from c.g. of steel area A_1 to nearest face of beam.
d_2 = distance from c.g. of steel area A_2 to nearest face of beam.
f_s = stress in steel near face of member least highly stressed in compression.
f_s' = stress in steel near face of member most highly stressed in compression.
I_t = moment of inertia of transformed section.
I_c = moment of inertia of concrete section about gravity axis.
I_s = moment of inertia of steel section about gravity axis.
$M = N_c$ = moment on section.
N = component of R normal to section.
N_c = total compression in concrete.
Q, L and Z = expressions reduced to tabular values to reduce labor of computation, see eqs. (134), (123) and (109).
$p' = \frac{A'}{bt}$ = steel ratio in face most highly stressed in compression.
$p = \frac{A_s}{bt}$ = steel ratio in face least highly stressed in compression.
p_o = ratio of the sum of A_s and A' to bt.
r = distance from c.g. of steel area A' to gravity axis with symmetrically placed reinforcement.
r = radius of circular section to center of reinforcement.
R = resultant force at any section.
R_1 = a design factor for Case II, circular section, $= \frac{N_c}{\pi r^3}$.
t = total depth of member.
x = distance from line of action of N_c to center line of circular section.

Figures 17 to 28 inclusive illustrate some of the above symbols.

70. Bending and Direct Compression—Rectangular Sections.[1]—Three general cases of bending and direct compression will be discussed. They are: Case I—Compression over the entire section, reinforcement on both faces of section; Case II—Tension on a portion of the section, reinforcement on both faces of section; Case III—Tension on a portion of the section, reinforcement on tension face only.

70a. Case I. Compression over the Entire Section—Unsymmetrical Reinforcement.—Figure 17*a* shows a rectangular section of a member and its reinforcement. The fiber stresses in concrete and steel in this section may be determined by reducing the steel area to an equivalent concrete area, which is

[1] Diagrams and derivations of formulas by GEORGE A. HOOL and W. S. KINNE.

assumed as applied in the plane of the reinforcing steel. This new section is known as a *transformed section.* Figure 17*b* shows the transformed section for Fig. 17*a*.

When the shape of the transformed section has been determined, the properties of the section, such as its area, position of the gravity axis, the moment of inertia of the section, and the distances to the extreme-fibers may be calculated. Substituting these values in eq. (88), p. 527, for the conditions shown in Fig. 17, the stresses in concrete and steel are as follows:

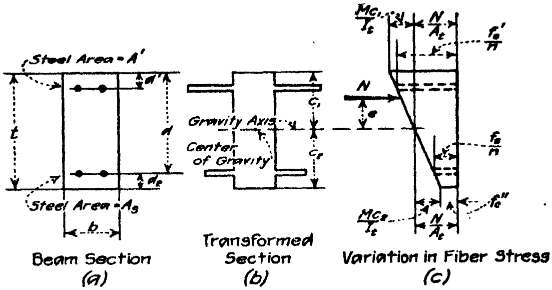


FIG. 17.—Stress distribution under bending and direct stress—Case I.

$$f_c = \frac{N}{A_t} + \frac{Mc_1}{I_t} \tag{89}$$

$$f_c'' = \frac{N}{A_t} - \frac{Mc_2}{I_t} \tag{90}$$

$$f_s = n\left(\frac{N}{A_t} + \frac{M(c_1 - d')}{I_t}\right) \tag{91}$$

$$f_s' = n\left(\frac{N}{A} - \frac{M(c_2 - d_2)}{I_t}\right) \tag{92}$$

where A_t and I_t are respectively the area and the moment of inertia of the transformed section and $M = Ne$ is the moment on the section. Note that the steel stresses in eqs. (91) and (92) are n times the concrete stresses in the plane of the steel.

In making calculations of fiber stress on any given section, it is generally more convenient to express the properties of the section directly in terms of the concrete and steel areas. Thus if A_t denotes the area of section including concrete area equivalent to the steel area, we have

$$A_t = bt + n(A' + A_s) \tag{93}$$

Strictly speaking $(n - 1)$ should be used in place of n, but the approximate formula has been used here for convenience. Formula (93) reduces to

$$A_t = bt\,[1 + n(p' + p)] \tag{94}$$

The distance from the face most highly stressed to the center of gravity of the transformed section is (by moments about the top of section, Fig. 17*b*)

$$c_1 = \frac{bt\frac{t}{2} + nA'd' + nA_sd}{bt + n(A' + A_s)} = \frac{\frac{t}{2} + np'd' + npd}{1 + n(p' + p)} \tag{95}$$

It can be shown that the moment of inertia of the section, including the concrete area equivalent to the steel area, taken about the gravity axis of the section is

$$I = I_c + nI_s \tag{96}$$

when I_c = *moment of inertia of concrete area and* I_s = *moment of inertia of steel area. Expressed in terms of the dimensions shown on Fig. 17,*

$$I_c = \frac{1}{3}bc_1^3 + \frac{1}{3}bc_2^3 = \frac{b}{3}(c_1^3 + c_2^3) \tag{97}$$

and

$$I_s = A'n(c_1 - d')^2 + nA_s(c_2 - d_2)^2 \tag{98}$$

Therefore

$$I_t = \frac{b}{3}(c_1^3 + c_2^3) + n[A'(c_1 - d')^2 + A_2(c_2 - d_2)^2] \tag{99}$$

Formulas for fiber stress may be derived by substituting in eqs. (89) to (92) inclusive values of I_t, A_t, c_1 and c_2 as found above. However, these formulas are cumbersome, and it will generally be found simplest to determine the properties of the section from eqs. (93) to (99) inclusive, and substitute these values in (89) to (92).

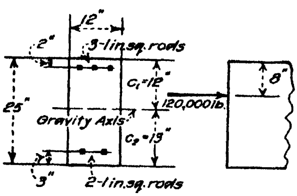


FIG. 18.

Illustrative Problem.—Determine the fiber stresses in steel and concrete for the conditions shown in Fig. 18. Assume n = 15.

From eq. (93)

$$A = (12)(25) + 15(3 + 2) = 375 \text{ sq. in.}$$

The steel ratios are:

$$p' = \frac{A'}{bt} = \frac{3}{(12)(25)} = 1.00 \text{ per cent and } p = \frac{A_s}{bt} = \frac{2}{(12)(25)} = 0.667 \text{ per cent}$$

From eq. (95)

$$c_1 = \frac{12.5 + 15[(0.0100)(2) + (0.00667)(22)]}{1 + 15(0.0100 + 0.00667)} = 12.0 \text{ in.}$$

and

$$c_2 = t - c = 25 - 12.0 = 13.0 \text{ in.}$$

From eq. (99)

$$I = \frac{12}{3}[(12)^3 + (13)^3] + 15[3.00(12 - 2)^2 + 2.00(13 - 3)^2] = 23{,}200 \text{ in.}^4$$

Since c_1 = 12.0 in., the eccentricity of application of the load is 12.00 − 8.00 = 4.00 in., and $M = Ne$ = (120,000)(4.00) = 480,000 in.-lb. The concrete and steel stresses are as follows:

Concrete at top of section; by eq. (89)

$$f_c = \frac{120{,}000}{375} + \frac{(480{,}000)(12.00)}{23{,}200} = 568 \text{ lb. per sq. in.}$$

Steel near top of section; by eq. (90)

$$f_s' = 15\left[\frac{120{,}000}{375} + \frac{(480{,}000)(10.00)}{23{,}200}\right] = 7{,}900 \text{ lb. per sq. in.}$$

Steel near bottom of section; by eq. (91)

$$f_s = 15\left[\frac{120{,}000}{375} + \frac{(480{,}000)(10.00)}{23{,}200}\right] = 1{,}695 \text{ lb. per sq. in.}$$

Concrete at bottom of section; by eq. (92)

$$f_c'' = \frac{120{,}000}{375} - \frac{(480{,}000)(13.00)}{23{,}200} = 51 \text{ lb. per sq. in.}$$

Reinforcement Symmetrical.—When the steel areas on the two faces of the section are equal and are placed at equal distances from the faces of the section,

comparatively simple formulas may be derived for fiber stresses in concrete and steel. Figure 19*a* shows the section under consideration. Let A_0 be the ***total*** steel area in the section, equally divided between the two faces of the beam and placed at a distance d' from each face. For the given conditions the gravity axis is at the center of the section, as shown. Other properties of the section which may readily be derived from Fig. 19*a* are as follows:

$$A_t = bt(1 + np_0) \tag{100}$$

$$I_t = I_c + nI_s = \frac{bt^3}{12} + nA_0r^2 = \frac{bt}{12}(t^2 + 12np_0r^2) \tag{101}$$

$$c_1 = c_2 = \frac{t}{2} \text{ and } r = (c_1 - d') = (c_2 - d_2) = \left(\frac{t}{2} - d'\right) \tag{102}$$

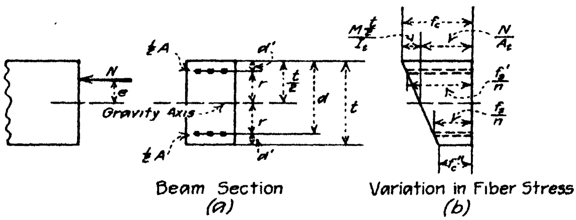


FIG. 19.—Stress distribution—Case I—symmetrical reinforcement.

After eq. (88), p. 527, we may write

$$f_c = \frac{N}{A_t} \pm \frac{M\frac{t}{2}}{I_t}$$

Since $M = Ne$,

$$f_c = N\left(\frac{1}{A_t} \pm \frac{e\frac{t}{2}}{I_t}\right) \tag{103}$$

On substituting proper values in eq. (103), the extreme fiber stresses in the concrete are found to be

$$f_c = N\left(\frac{1}{bt(1 + np_0)} + \frac{e\frac{t}{2}}{\frac{bt}{12}(t^2 + 12np_0r^2)}\right)$$

which may be written

$$f_c = \frac{N}{bt}\left(\frac{1}{1 + np_0} + \frac{e}{t} \cdot \frac{6}{1 + 12np_0\left(\frac{r}{t}\right)^2}\right) \tag{104}$$

and

$$f_c'' = \frac{N}{bt}\left(\frac{1}{1 + np_0} - \frac{e}{t} \cdot \frac{6}{1 + 12np_0\left(\frac{r}{t}\right)^2}\right) \tag{105}$$

The stresses in the steel, in terms of the concrete stresses, are

$$f_s' = n\left[f_c - \frac{d'}{t}(f_c - f_c'')\right] \tag{106}$$

$$f_s = n\left[f_c'' + \frac{d'}{t}(f_c - f_c'')\right] \tag{107}$$

By referring to Fig. 19*b* it can be seen that the stress in the steel is always less than nf_c. Thus if f_c is within its allowable value, then also will the steel be safely stressed.

In deriving the formulas given in this case, it was assumed that the stress over the entire section is compressive. The variation in stress across the section, however, depends upon the eccentricity of the load N. For the case under consideration, the limiting condition will occur when f_c'' of Fig. 19*b* is zero. The value of e for $f_c'' = 0$ may be determined by equating the right-hand member of eq. (105) to zero and solving for e, from which

$$e = \frac{1 + 12np_0\left(\frac{r}{t}\right)^2}{(1 + np_0)} \frac{t}{6}$$

From this equation, the limiting ratio of eccentricity to width of section for which compression exists over the entire section is

$$\frac{e}{t} = \frac{\left[1 + 12np_0\left(\frac{r}{t}\right)^2\right]}{6(1 + np_0)} \tag{108}$$

If in any case the existing eccentric ratio e/t is less than the value given by eq. (106) the problem may be solved by the formulas for Case I. If the existing eccentric ratio exceeds the value given by eq. (106) the formulas of Case II must be used.

Table 18 gives values of the limiting eccentric ratio e/t for $n = 12$ and 15 for various values of steel embedment.

TABLE 18.—VALUES OF e/t FOR SYMMETRICAL REINFORCEMENT

n	d'/t	Values of p_0								
		0.005	0.0075	0.01	0.0125	0.015	0.0175	0.02	0.0225	0.025
15	0.05	0.183	0.191	0.198	0.204	0.210	0.216	0.222	0.227	0.232
	0.10	0.177	0.182	0.186	0.190	0.194	0.198	0.202	0.205	0.208
	0.15	0.172	0.175	0.177	0.179	0.181	0.183	0.185	0.187	0.188
	0.20	0.168	0.168	0.168	0.169	0.169	0.170	0.170	0.170	0.1706
12	0.05	0.180	0.186	0.192	0.198	0.203	0.208	0.213	0.218	0.222
	0.10	0.175	0.179	0.183	0.187	0.191	0.194	0.197	0.200	0.202
	0.15	0.171	0.173	0.175	0.177	0.179	0.181	0.182	0.184	0.185
	0.20	0.167	0.168	0.168	0.168	0.169	0.169	0.169	0.170	0.170

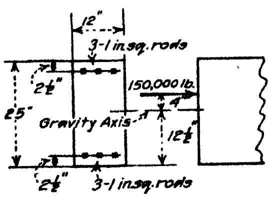


FIG. 20.

If the existing eccentric ratio is *less* than the value in Table 18, the problem comes under Case I, and if greater than given in the table, Case II must be used.

Illustrative Problem.—Determine the fiber stresses in concrete and steel for the conditions shown in Fig. 20. Assume $n = 15$.

Since $\frac{e}{t} = \frac{4}{25} = 0.16$ is less than the value given in Table 18 for the given conditions, the problem may be solved by the formulas for Case I

The steel ratio is $p_0 = \frac{6}{(12)(25)} = 0.02.$

From eq. (104)

$$f_c = \frac{150{,}000}{(12)(25)}\left[\frac{1}{1 + (15)(0.02)} + \frac{4.0}{25} \cdot \frac{6}{1 + (12)(15)(0.02)(0.40)^2}\right]$$

$$= 500(0.769 + 0.610) = 690 \text{ lb. per sq. in.}$$

Also, from eq. (105)

$$f_c'' = 500(0.769 - 0.610) = 79.5 \text{ lb. per sq. in.}$$

The steel stresses are

by eq. (106)

$$f_s' = 15[690 - \frac{1}{10}(690 - 79.5)] = 9{,}430 \text{ lb. per sq. in.}$$

by eq. (107)

$$f_s = 15[79.5 + \frac{1}{10}(690 - 79.5)] = 2{,}110 \text{ lb. per sq. in.}$$

Diagrams.—The equations derived under this case contain too many variables to permit the construction of diagrams for the solution of problems. However, if it be assumed that n, the ratio of the moduli E_s and E_c is constant, and if it be assumed that the steel is embedded to a certain percentage of the depth in all sections, the number of variables is reduced so that diagrams may be constructed. Thus, if we assume $n = 15$, and if we assume that the steel is embedded in the concrete one-tenth of the total depth from each surface, so that $r = 0.4t$, eq. (105) becomes

$$f_c = \frac{N}{bt}\left[\frac{1}{1 + 15p_0} + \frac{e}{t} \cdot \frac{6}{1 + 28.8p_0}\right]$$

or if the expression in brackets is denoted by Z

$$f_c = \frac{N}{bt}Z \tag{109}$$

and eq. (108) becomes

$$\frac{e}{t} = \frac{1 + 28.8p_0}{6(1 + 15p_0)} \tag{110}$$

Diagrams 18 to 23 inclusive give values of Z for various values of P_0, $\frac{e}{t}$ and $\frac{d'}{t}$, and for $n = 12$ and 15. The termination of the curves is determined in Diagram 22 by eq. (110) and by similar equations in the other diagrams. For values of $\frac{e}{t}$ to the right of the dotted line terminating the curves, Case I does not apply—that is, there is tension in the concrete and Case II must be employed.

Values of f_c'', f_s and f_s', if desired, may be determined from the value of f_c. From Fig. 19b it can be seen that

$$\frac{N}{A_t} = \tfrac{1}{2}(f_c + f_c'')$$

Solving for f_c'', we have

$$f_c'' = 2\frac{N}{A_t} - f_c \tag{111}$$

where A_t is given by eq. (100) and N by the conditions of the problem. Values of f_s and f_s' may be determined from eqs. (106) and (107)

Illustrative Problem.—Determine the fiber stresses in concrete and steel for the conditions shown in Fig. 20. Use diagrams and assume $n = 15$.

For the given conditions

$$\frac{e}{t} = \frac{4}{25} = 0.16 \qquad \text{Since } \frac{d'}{t} = \frac{2.5}{25} = \frac{1}{10}$$

use Diagram 22, from which $Z = 1.38$

From eq. (109)

$$f_c = \frac{N}{bt} Z = \frac{(150{,}000)(1.38)}{(12)(25)} = 690 \text{ lb. per sq. in.}$$

From eq. (111)

$$f_c'' = 2\frac{N}{A_t} - f_c$$

where A_t as given by eq. (100) is

$$A_t = (12)(25)[1 + (15)(0.02)] = 390 \text{ sq. in.}$$

Hence

$$f_c'' = \frac{(2)(150{,}000)}{390} - 690 = 80 \text{ lb. per sq. in.}$$

The steel stresses may be determined as in the preceding problem. Note the close check on results obtained by formulas and diagrams.

70b. Case II. Tension on Part of Section—Reinforcement on Both Faces of Section.—When the tension on the extreme fibers of the section is within the safe strength of the concrete, the method and formulas of Case I may

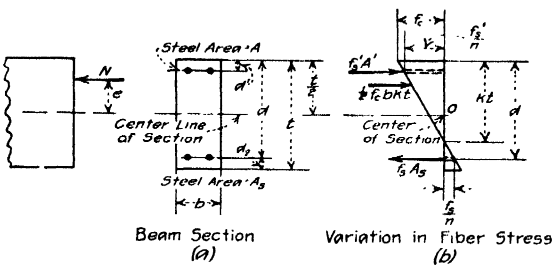


FIG. 21.—Stress distribution under bending and direct stress—Case II.

be employed. When the safe tensile strength of the concrete is exceeded, a method of analysis based on the theory of flexure for concrete beams must be used. In this analysis it is assumed that all tension is taken by steel, no reliance being placed on concrete in carrying tension.

Unsymmetrical Reinforcement.—Figure 21*a* shows a rectangular section and its reinforcement. The assumed variation in fiber stress is shown in Fig. 21*b*. From Fig. 21*b* it can be seen that there are four unknowns which must be determined. These are the fiber stresses f_c, f_s, and f_s' and kt, the distance from the compression face of the beam to the neutral axis.

From Fig. 21*b* it can readily be seen that

$$f_s' = nf_c\left(1 - \frac{d_1}{kt}\right) \tag{112}$$

and

$$f_s = nf_c\left(\frac{d}{kt} - 1\right) \tag{113}$$

These equations are derived from a consideration of similar triangles. Since forces on the section are in equilibrium, the summation of fiber stresses times the

area on which they act must be equal to N, the resultant thrust on the section. Then

$$N = \tfrac{1}{2}f_c bkt + f_s'A' - f_s A_s \tag{114}$$

Also, the moments about any point must be zero for equilibrium. Taking moments about O, Fig. 21b, we have

$$M = \tfrac{1}{2}f_c kt^2 b\left(\tfrac{1}{2} - \frac{k}{3}\right) + A'f_s'\left(\frac{t}{2} - d'\right) + A_s f_s\left(d - \frac{t}{2}\right) \tag{115}$$

Equations (112) to (115) inclusive give four independent condition equations, which, when solved as a group of simultaneous equations, will give the desired values of fiber stresses and distance to the neutral axis. It is possible, of course, to derive formulas for these unknowns. However, these formulas are cumbersome and difficult to use when obtained. It will generally be found best to substitute known quantities, as far as possible, in eqs. (112) to (115) and solve the resulting equations.

The solution for the desired unknowns may be expedited by reducing the number of unknowns. This can be done by substituting values of f_s' and f_s from eqs. (112) and (113) in eqs. (114) and (115). From eq. (114) we have

$$N = \frac{f_c}{kt}\left(\tfrac{1}{2}bk^2t^2 + nkt(A' + A_s) - n(A'd' - A_s d)\right) \tag{116}$$

and from (115)

$$M = \frac{f_c}{kt}\Bigg(-\tfrac{1}{6}k^3t^3b + \tfrac{1}{4}k^2t^3b + nkt\left[A'\left(\frac{t}{2} - d'\right) - A_s\left(d - \frac{t}{2}\right)\right] - n\left[A'd'\left(\frac{t}{2} - d'\right) - A_s d\left(d - \frac{t}{2}\right)\right]\Bigg) \tag{117}$$

On substituting known quantities as far as possible, the simultaneous solution of these equations will result in a cubic equation in k, from which the value of k may be determined. Substituting this value of k in eq. (116), the value of f_c is readily determined. Having f_c and k, eqs. (112) and (113) may be solved for f_s' and f_s. All unknowns may thus be completely determined.

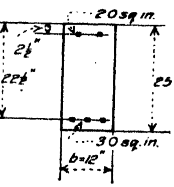


FIG. 22.

Illustrative Problem.—Determine the fiber stresses in steel and concrete for the section shown in Fig. 22 for $M = 600{,}000$ in.-lb. and $N = 60{,}000$ lb. Assume $n = 15$.

For the conditions shown in Fig. 22, $t = 25$ in., $b = 12$ in., $A' = 2$ sq. in., $A_s = 3$ sq. in., $d' = 2.5$ in., and $d = 22.5$ in. From eq. (116)

$$60{,}000 = \frac{f_c}{kt}(3{,}750k^2 + 1{,}875k - 1{,}087.5)$$

and from eq. (117)

$$600{,}000 = \frac{f_c}{kt}(-31{,}250k^3 + 46{,}875k^2 - 3{,}750k + 9{,}375)$$

On equating these equations and reducing, we have the cubic equation

$$k^3 - 0.30k^2 + 0.72k - 0.65 = 0$$

Solving this equation by the method explained on p. 537, we have $k = 0.67$. Substituting this value of k and $t = 25$, in eq. (116) in the form given above, we have

$$60{,}000 = \frac{f_c}{(0.67)(25)}(1{,}680 + 1{,}255 - 1{,}087.5)$$

from which

$$f_c = 546 \text{ lb. per sq. in.}$$

From eqs. (112) and (113)

$$f_s' = (15)(546)\left[1 - \frac{2.5}{(0.67)(25)}\right] = 6{,}990 \text{ lb. per sq. in.}$$

and

$$f_s = (15)(546)\left[\frac{22.5}{(0.67)(25)} - 1\right] = 2{,}780 \text{ lb. per sq. in.}$$

Reinforcement Symmetrical.—Figure 23 shows a beam section with equal steel areas embedded at the same distances from the surfaces of the section. For

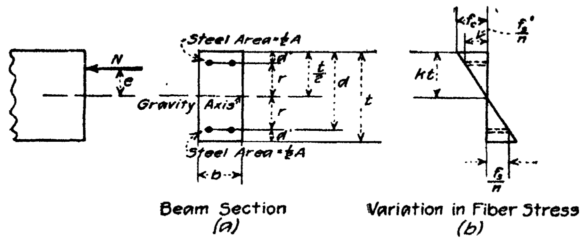


FIG. 23.—Stress distribution—Case II—symmetrical reinforcement.

symmetrical sections, eqs. (112) and (113) remain unchanged, but all other equations become much simpler. With distances as shown on Fig. 23, eqs. (116) and (117) become

$$N = \frac{f_c bt}{2}\left(\frac{k^2 + 2np_0k - np_0}{k}\right) \tag{118}$$

$$M = f_c bt^2\left[\frac{np_0}{k}\left(\frac{r}{t}\right)^2 + \frac{k}{12}(3 - 2k)\right] \tag{119}$$

To determine the value of k, solve eqs. (118) and (119) for f_c and equate the resulting expressions. Noting that $M = Ne$, we have the cubic equation

$$k^3 - 3\left(\frac{1}{2} - \frac{e}{t}\right)k^2 + 6np_0\frac{e}{t}k = 3np_0\left[\frac{e}{t} + 2\left(\frac{r}{t}\right)^2\right] \tag{120}$$

The solution of this cubic equation for the value of k is given in textbooks on higher algebra. A solution by cut and try methods is given in the illustrative problems at the end of this article.

When k has been determined from eq. (120), the value of f_c may be determined from either eq. (118) or (119). Generally, eq. (119) is preferable, from which

$$f_c = \frac{M}{bt^2\left[\frac{np_0}{k}\left(\frac{r}{t}\right)^2 + \frac{k}{12}(3 - 2k)\right]} \tag{121}$$

The fiber stresses in steel are given by eqs. (112) and (113), with values of k and f_c as given by eqs. (120) and (121).

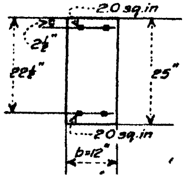


FIG. 24.

Illustrative Problem.—Determine the stresses in steel and concrete for the section shown in Fig. 24 for $M = 600{,}000$ in.-lb. and $N = 60{,}000$ lb. Assume $n = 15$.

For the conditions shown in Fig. 24, $t = 25$ in.; $b = 12$ in., $A' = A_s = 2$ sq. in., $D' = 2.5$ in., $d = 22.5$ in., and $r = 10$ in. Therefore

$$\frac{r}{t} = \frac{10}{25} = 0.4,\ e = \frac{600{,}000}{60{,}000} = 10 \text{ in.},\ \frac{e}{t} = \frac{10}{25} = 0.4,\ p_0 = \frac{4}{300} = 0.0133$$

Substituting values in eq. (120), we have the cubic equation

$$k^3 - 0.30k^2 + 0.48k - 0.432 = 0$$

This cubic equation may be solved by the method given below.

Cubic equations resulting from substitution in eq. (120) will generally be found to contain one real root whose value lies between 0 and 1, and two imaginary roots. An exact solution for these roots is generally difficult and beyond the reach of the ordinary engineer. Since we are concerned only with the value of the real root, which from the conditions of the problem, must lie between 0 and 1, the following cut and try solution may be used. This solution is based on the fact that the substitution in an equation of one of its roots gives that equation a zero value. Hence by assuming values of k, it is possible finally to find that value of k which will give the above cubic equation a zero value.

Assume first that $k = 0.5$. For $k = 0.5$ the value of the above equation is found to be -0.142. Since a negative value was obtained by this substitution, it is evident that the assumed value of k was too small, for the positive quantity produced by the first three terms was not great enough to neutralize the negative quantity indicated by the last term of the equation. Therefore, the desired root must lie between 0.5 and 1. On trying $k = 0.7$, the value of the equation was found to be $+0.10$. Hence the desired root is between 0.5 and 0.7. By this process it was found that $k = 0.629$ gave a zero value and was therefore the desired root.

A simple method of evaluating equations is given in textbooks on mathematics. The substitutions for $k = 0.5$ and $k = 0.629$ are given below:

k^3	k^2	k	Absolute term	
1	−0.30	+0.48	−0.432	+0.5
	+0.50	+0.10	+0.290	
	+0.20	+.058	−0.142	
1	−0.300	+0.480	−0.432	+0.629
	+0.629	+0.207	+0.432	
	+0.329	+0.687	0	

The process is as follows: Write out the coefficients of the several terms in order. Under the coefficient of k^2 place the assumed value of k. Add the terms *algebraically*. Multiply this sum by the assumed value of k (thus $0.329 \times 0.629 = 0.207$) and place the product under the coefficient of k. Repeat the operation as shown above. The sum which appears under the absolute term is the value of the equation for the assumed k. Thus for $k = 0.5$, the value is -0.142, and for $k = 0.629$, the value is zero. With a little practice cubic equations of this type may be solved rapidly and accurately. Generally two places in the result are sufficient.

This method may be applied to any cubic equation for the solution of the real roots. The explanation given above applies for the determination of the real positive root which lies between 0 and 1. Cubic equations resulting from substitution in eq. (120) or in eqs. (116) and (117) are of this form.

Substituting $k = 0.629$ and $\frac{r}{t} = 0.4$ in eq. (121) we have

$$f_c = \frac{600{,}000}{(12)(25)^2\left[\frac{(15)(0.0133)(0.4)^2}{0.629} + \frac{3(0.629) - 2(0.629)^2}{12}\right]} = \frac{600{,}000}{(12)(25)^2(0.142)}$$

from which

$$f_c = 560 \text{ lb. per sq. in.}$$

From eqs. (112) and (113)

$$f_s' = (15)(560)\left[1 - \frac{2.5}{(0.629)(25)}\right] = 7{,}080 \text{ lb. per sq. in.}$$

and

$$f_s = (15)(560)\left[\frac{22.5}{(0.629)(25)} - 1\right] = 3{,}620 \text{ lb. per sq. in.}$$

Diagrams.—The solution of problems in bending and compression for symmetrical sections under Case II is greatly expedited by the use of diagrams based on eqs. (120) and (121). In eq. (120), if certain values be assumed for n and for the steel embedment, there results an equation for k in terms of the steel ratio p_0 and the eccentric ratio $\frac{e}{t}$. From this equation, curves may be plotted giving values of k for various values of p_0 and $\frac{e}{t}$. Diagrams 24 to 30 inclusive give values of k for $n = 12$ and 15; $d'/t = 0.5$, 0.10, and 0.15; and for various values of p_0 and $\frac{e}{t}$

Equation (121) for f_c may be placed in the form

$$f_c = \frac{M}{bt^2L} \tag{122}$$

where

$$L = \left[\frac{np_0}{k}\left(\frac{r}{t}\right)^2 + \frac{k}{12}(3 - 2k)\right] \tag{123}$$

Diagrams 27 and 31 give values of L for various values of the terms involved.

The method or procedure in solving problems under Case II by means of the diagrams is as follows: (1) Determine k from the proper diagram; (2) find L from Diagram 27; (3) solve eq. (122) for f_c; (4) find unit stresses in steel from eqs. (112) and (113). The illustrative problem which follows gives the calculations in detail.

Illustrative Problem.—Determine the stresses in steel and concrete for the section shown in Fig. 24 for $M = 600{,}000$ in.-lb. and $N = 60{,}000$ lb. Assume $n = 15$ and solve the problem by means of diagrams.

As before,

$$p_0 = \frac{4}{300} = 0.0133,\ e = \frac{600{,}000}{60{,}000} = 10 \text{ in.}$$

and

$$\frac{e}{t} = \frac{10}{25} = 0.4$$

From Diagram 29 ($d' = 0.1t$) with $p_0 = 0.0133$ and $\frac{e}{t} = 0.4$ we find

$$k = 0.629$$

From Diagram 31 with $k = 0.629$ and $p_0 = 0.0133$, we find

$$L = 0.142$$

Then from eq. (122)

$$f_c = \frac{600{,}000}{(12)(25)^2(0.142)} = 580 \text{ lb. per sq. in.}$$

Since the steel stresses are determined by the same formulas as in the preceding problem, the substitutions will not be given here.

70c. Case III. Tension over Part of Section—Reinforcement on Tension Face Only.[1]—Figure 25 shows the assumed conditions. This case may be analyzed by methods similar to those used for Case II. Formulas for fiber stresses in steel and concrete and for value of k may be determined from those of Case II by modifying the conditions to fit Case III. Thus from eqs. (107) and (117) with $t = d$, $e + \frac{d}{2} = e'$, $A' = 0$, and $A_s = pbd$, we have

[1] Diagrams and derivation of formulas by George A. Hool and W. S. Kinne.

$$N = \frac{bdf_c}{k}\left[\frac{k^2}{2} - np(1-k)\right] \tag{124}$$

and

$$M = Ne = \frac{f_c bd^2}{k}\left[-\frac{1}{6}k^3 + \frac{1}{4}k^2 + \frac{1}{2}np(1-k)\right] \tag{125}$$

From eqs. (124) and (125) we have

$$k^3 + 3k^2\left(\frac{e'}{d} - 1\right) - 6np(1-k)\frac{e}{d} = 0 \tag{126}$$

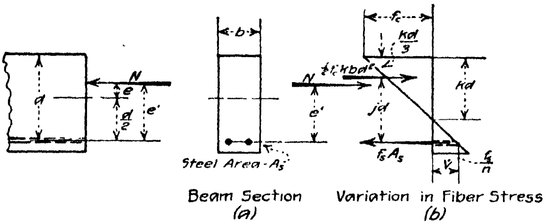


FIG. 25.—Stress distribution under bending and direct stress—Case III.

Values of k may be determined from eq. (126). From Fig. 25,

$$j = 1 - \frac{k}{3}$$

The stress in the concrete is

$$f_c = \frac{2Ne'}{kj\,bd^2} \tag{127}$$

This expression is obtained from eqs. (124) and (125), noting that

$$e = e' - \frac{d}{2}$$

The stress in the steel, in terms of the concrete stress, is

$$f_s = \frac{nf_c(1-k)}{k} \tag{3}$$

Illustrative Problem.—Determine the stresses in steel and concrete on section a-b of the retaining wall of Fig. 26. Assume $n = 15$.

The normal force, N, on the section is the weight of the vertical wall. With concrete at 150 lb. per cu. ft., $N = (12)(150) = 1{,}800$ lb. Assuming N as applied at the center of the vertical wall and taking moments about the steel, we have

$$M = (2{,}400)(4,(12)(1{,}800)(4.5) - 123{,}100 \text{ in.-lb.}$$

Hence

$$e' = \frac{M}{N} = \frac{123{,}100}{1{,}800} = 68.5 \text{ in. and } \frac{e'}{d} = \frac{68.5}{10.5} = 6\ 53$$

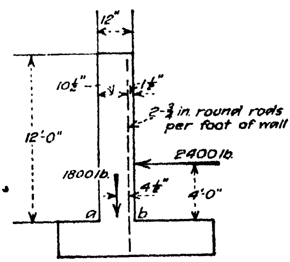


FIG. 26.

For the reinforcement shown in Fig. 26, the steel ratio is

$$p = \frac{(2)(0.442)}{(12)(10.5)} = 0.00702$$

Substituting these values in eq. (126) we have the following cubic equation for k:

$$k^3 + 16.59k^2 + 4.13k - 4.13 = 0$$

Solving this equation by the method given on p. 537, we find

$$k = 0.385 \quad j = 1 - \frac{0.385}{3} = 0.872$$

and from eq. (127),

$$f_c = \frac{(2)(1,800)(6.53)}{(0.385)(0.872)(12)(10.5)^2} = 530 \text{ lb. per sq. in.}$$

From eq. (3) the stress in the steel is

$$f_s = (15)(530)\frac{0.615}{0.385} = 12,700 \text{ lb. per sq. in.}$$

Diagrams.—Problems under Case III may be solved by means of Diagrams 32 and 33. These diagrams were found to be more convenient than the form used in the preceding cases. Values of K, which appear on the lower right hand margin of the diagrams, are derived from eq. (127), which may be written

$$K = \frac{1}{2} f_c kj = \frac{Ne'}{bd^2} = \frac{M}{bd^2} \tag{128}$$

Diagrams 32 and 33 may be used for investigating existing designs to determine fiber stresses, or they may be used for the design of beams to fit certain given conditions. The use of the diagrams will be explained by means of the illustrative problems which follow.

Illustrative Problem.—Determine the stresses in steel and concrete on section *a-b* of the retaining wall of Fig. 26. Assume $n = 15$ and use diagrams.

From the preceding problem, we have

$$p = 0.00702; \quad \frac{e'}{d} = 6.53; \text{ and } M = 123,100 \text{ in.-lb.}$$

From eq. (128)

$$K = \frac{M}{bd^2} = \frac{123,100}{(12)(10.5)^2} = 93.2$$

Entering Diagram 33 with a value $p = 0.00702$ on the lower right hand margin and tracing vertically to a value of $\frac{e'}{d} = 6.53$, then horizontally to the left to a point vertically over $K = 93.2$ we find

$$f_s = 12,700 \text{ lb. per sq. in. and } f_c = 530 \text{ lb. per sq. in.}$$

Illustrative Problem.—Design the vertical wall of the retaining wall of Fig. 26 for working stresses of $f_s = 16,000$ lb. per sq. in. and $f_c = 600$ lb. per sq. in. Assume $n = 15$.

Since the size of the wall is not known, assume its weight to be 1,800 lb. applied 4½ in. in front of the steel. Then $M = (2,400)(4)(12) + (1,800)(4.5) = 123,100$ in.-lb. For the given working stresses the left hand part of Diagram 33 gives $K = 87.5$. Then from eq. (128)

$$d = \frac{M}{bK} = \frac{123,100}{(12)(87.5)} = 10.8 \text{ in.}$$

The assumed weight of wall and position of its center should now be checked against the actual conditions. If the actual and assumed values do not check, repeat the above operation until a check is reached. As the above value is so close to the conditions shown in Fig. 26, it will be accepted.

The eccentric ratio is

$$\frac{e'}{d} = \frac{M}{Nd} = \frac{123,100}{(1,800)(10.8)} = 6.33$$

Following across Diagram 33 to the right from the intersection of the given f_s and f_c curves to a value of $\frac{e'}{d} = 6.33$, and then following vertically downward to the lower right-hand margin, we find $p = 0.0060$. The steel area required per foot of wall is then

$$A_s = (0.0060)(12)(10.8) = 0.778 \text{ sq. in.}$$

71. Bending and Direct Compression—Circular Sections.[1]—Two general cases of bending and direct compression on circular sections will be considered. These are: Case I—Compression over the entire section; and Case II—Tension on part of the section.

71a. Case. I. Compression over Entire Section.—The extreme fiber stresses in concrete and steel may be determined by a method similar to that used for Case I of rectangular sections, modified to meet the conditions for circular sections. Figure 27 shows the section under consideration. It will be assumed, in deriving the formulas for fiber stress, that the steel reinforcement forms a continuous sheet which is equivalent in area to the given reinforcement. If A_0 = total steel area and p_0 = steel ratio, we have

$$p_0 = \frac{A_0}{\pi r^2}$$

where r = radius of section to center of steel. The area of concrete outside the steel reinforcement will be neglected.

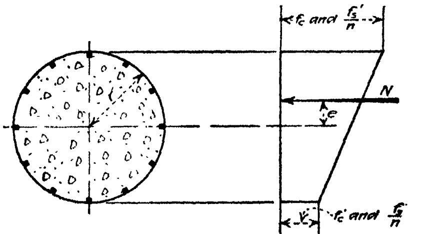


FIG. 27.—Stress distribution in eccentrically loaded circular cored column—Case I.

A general expression for extreme fiber stress in concrete derived by the methods used for eq. (103) modified to fit the conditions for the circular section of Fig. 27, is

$$\left.\begin{matrix} f_c \\ f_c'' \end{matrix}\right\} = N\left(\frac{1}{A_t} \pm \frac{er}{I_t}\right) \tag{129}$$

In eq. (129)

$$A_t = \text{area of section}^2 = \pi r^2[1 + (n - 1)p_0]$$

and

I_t = moment of inertia of section =

$$I_c + (n - 1)I_s = \frac{\pi r^4}{4} + p_0(n - 1)\frac{\pi r^4}{2} = [0.25 + 0.5(n - 1)p_0]\,\pi r^4$$

Placing these values in eq. (129) we have

$$f_c = \frac{N}{\pi r^2}\left[\frac{1}{1 + (n - 1)p_0} + \frac{e}{r}\cdot\frac{1}{0.25 + 0.5(n - 1)p_0}\right] \tag{130}$$

and

$$f_c'' = \frac{N}{\pi r^2}\left[\frac{1}{1 + (n - 1)p_0} - \frac{e}{r}\cdot\frac{1}{0.25 + 0.5(n - 1)p_0}\right] \tag{131}$$

The steel stresses in terms of the concrete stresses are

$$f_s' = nf_c$$

and (132)

$$f_s = nf_c'$$

Illustrative Problem.—A round column with a 20-in. core reinforced with ten 1-in. square rods sustains a load of 200,000 lb. applied 2 in. off center. Determine the maximum unit stress in the concrete. Assume $n = 15$.

[1] Based on a solution devised by MR. C. S. WHITNEY, Structural Engineer, Milwaukee, Wis.

[2] Note that the more exact expression for equivalent concrete area is used in the following discussion

For the given conditions

$$N = 200{,}000;\ p_0 = \frac{10}{314.2} = 0.0318,\ r = 10 \text{ in., and } \frac{e}{r} = \frac{2}{10} = 0.20$$

From eq. (130)

$$f_c = \frac{200{,}000}{(3.142)(10)^2}\left[\frac{1}{1 + (15 - 1)0.0318} + 0.2\frac{1}{0.25 + 0.5(15 - 1)0.0318}\right]$$

from which

$$f_c = 710 \text{ lb. per sq. in.}$$

Diagrams.—Equation (130) may be written in the form

$$f_c = \frac{N}{\pi r^2} Q \tag{133}$$

where

$$Q = \left[\frac{1}{1 + (n - 1)p_0} + \frac{e}{r}\cdot\frac{1}{0.25 + 0.5(n - 1)p_0}\right] \tag{134}$$

Diagram 34 gives values of the right-hand member of eq. (134) for various values of the several terms.

To determine f_c, note from Fig. 27 that

$$\frac{1}{2}(f_c + f_c') = \frac{N}{A_t}$$

from which

$$f_c' = 2\frac{N}{A_t} - f_c \tag{135}$$

where $A_t = \pi r^2[1 + (n - 1)p_0]$ and N is given by the conditions of the problem. Equations (132) give values of f_s' and f_s.

Illustrative Problem.—A round column with a 20-in. core, reinforced with 10 sq. in. of steel, sustains a load of 200,000 lb. applied 2 in. off center. Determine the maximum unit stress in the concrete. Assume $n = 15$ and use diagrams.

For the given conditions,

$$\frac{10}{}\ \cdot 0.0318,\ r = 10 \text{ in., and } \frac{e}{r} = \frac{2}{10} = 0.20$$

From Diagram 34

$$\frac{\pi r^2 f_c}{N} = 1.12$$

Therefore

$$f_c = \frac{1.12N}{\pi r^2} = \frac{(1.12)(200{,}000)}{314.16} = 712 \text{ lb. per sq. in.}$$

71*b*. Case II. Tension over Part of Section.—Figure 28 shows the section under consideration and the assumed variation in fiber stress across the

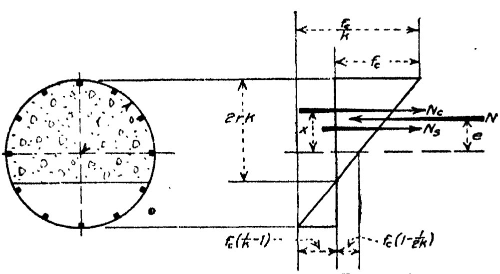


Fig. 28.—Stress distribution in eccentrically loaded circular cored column—Case II.

section. When the eccentricity of loading is so great that there is tension on a part of the section, a direct solution by formulas cannot readily be made because the expression for N_c (the resultant of the compression in the concrete) and x (the distance from N_c to the neutral axis) cannot be expressed by simple formulas.

However, the problem may be solved by carrying the derivation of the formulas as far as the statement of the general condition equations. Then, instead of completing the solution of these equations in the usual manner, values of certain of the variables may be assumed and the value of the remaining variables determined. By this process, repeated for various values of the variables, data may be obtained from which curves may be plotted giving the desired relations between all of the variables. This will now be done for the case under consideration.

Since the resultant of forces acting on the section must be zero, we have from Fig. 28

$$N = N_c + N_s$$

where N_c = total compression in concrete and N_s = resultant of steel stresses. From the fiber stress diagram, it can be seen that the average stress in the steel is n times the stress in the concrete at the center of the section—that is, it is equal to $nf_c\left(1 - \frac{1}{2k}\right)$. Hence

$$N_s = p_0\pi r^2 nf_c\left(1 - \frac{1}{2k}\right)$$

Then

$$N = N_c + p_0\pi r^2 nf_c\left(1 - \frac{1}{2k}\right)$$

which may be written

$$N = f_c\pi r^2\left[\frac{N_c}{f_c\pi r^2} + np_0\left(1 - \frac{1}{2k}\right)\right] \tag{136}$$

The term N_c in eq. (136) represents the volume of the concrete fiber stress wedge of Fig. 28. It can readily be shown that

$$N_c = f_c\pi r^2 K_1$$

where K_1 is an abstract ratio which may be expressed as a function of k. Hence

$$\frac{N_c}{f_c\pi r^2} = K_1$$

Values of K_1 are given in Diagram 35. The expression for K_1 is very complex; it will not be given here.

Since the section is in equilibrium under the forces shown in Fig. 28, moments about the center line of the section must be zero. We then have

$$M = Ne = N_c x + M_s$$

where M_s = moment of steel stresses about the center of section, N_c = total compression in concrete, and x = distance from line of action of N_c to center of section. To determine the value of M_s, let f_s' be the extreme unit fiber stress in the steel due to bending only. From Fig. 28 it can be seen that f_s' is equal to n times the difference between f_c and the stress in the concrete at the center of the section, or

$$f_s' = \frac{nf_c}{2k}$$

Also, from the usual formula for flexure in beams, we have

$$f_s' = \frac{M_s r}{I_s}$$

where I_s = moment of inertia of steel = $p_0\frac{\pi r^4}{2}$

Hence

$$f_s' = \frac{2M_s r}{p_0 \pi r^4}$$

Equating this value of f_s' to the one given above, we have

$$M_s = f_c \pi r^3 \frac{np_0}{4k}$$

Placing this value of M_s in the above equation for M, there results

$$M = Ne = N_c x + f_c \pi r^3 \frac{np_0}{4k}$$

which may be written

$$M = Ne = f_c \pi r^3 \left(\frac{N_c x}{f_c \pi r^3} + \frac{np_0}{4k}\right) \tag{137}$$

The term $N_c x$ is the moment of the resultant concrete stress about the center of the section. It can readily be shown that

$$N_c x = f_c \pi r^3 K_2$$

where K_2 is an abstract ratio and a function of k. Values of K_2 are given in Diagram 35.

Letting $\frac{Ne}{\pi r^3} = R_1$, we have from eq. (137)

$$\frac{R_1}{f_c} = \frac{Ne}{\pi r^3 f_c} = \left(\frac{N_c x}{f_c \pi r^3} + \frac{np_0}{4k}\right) \tag{138}$$

From eqs. (136) and (137) we may write

$$\frac{r}{e} = \frac{Nr}{Ne} = \frac{\left[\frac{N_c}{f_c \pi r^2} + np_0\left(1 - \frac{1}{2k}\right)\right]}{\left[\frac{N_c x}{f_c \pi r^3} + \frac{np_0}{4k}\right]} \tag{139}$$

and from eqs. (138) and (139) we may derive the expression

$$\frac{R_1}{f_c} = \frac{\left[\frac{N_c}{f_c \pi r^2} + np_0\left(1 - \frac{1}{2k}\right)\right]}{\left(\frac{r}{e}\right)} \tag{140}$$

It will not be advisable to carry the direct solution of the equations beyond this point. Curves may be plotted from eqs. (139) and (140) by means of which the concrete stress may be determined for any given value of $\frac{r}{e}$. To calculate values from which these curves may be plotted, assume n as a constant and give p_0 any given value. Assume some value of k and insert this value in eq. (139), together with values of $\frac{N_c}{f_c \pi r^2}$ and $\frac{N_c x}{f_c \pi r^3}$ obtained from Diagram 35. Substitute $\frac{r}{e}$ and k in eq. (140) to obtain $\frac{R_1}{f_c}$. Plot values of $\frac{r}{e}$ and $\frac{R_1}{f_c}$ for the given value of p_0. The curves given on the right-hand side of Diagrams 36 and 37 were obtained by this process. These curves will give the extreme fiber stress in the concrete for any value of eccentricity which produces tension on the section.

The maximum tensile stress in the steel, f_s, is shown on Fig. 28 to be

$$f_s = nf_c\left(\frac{1}{k} - 1\right) \tag{3}$$

Solving eq. (3) for f_c and equating the resulting expression to the value of f_c obtained from eq. (138) we have

$$\frac{f_s}{R_1} = \frac{\pi r^3 f_s}{N_c e} = \frac{n\left(\frac{1}{k} - 1\right)}{\left(\frac{N_c x}{f_c \pi r^3} + \frac{n p_0}{4k}\right)} \tag{141}$$

From eqs. (141) and (139) we may write

$$\frac{f_s}{R_1} = \frac{n\left(\frac{1}{k} - 1\right)\left(\frac{r}{e}\right)}{\frac{N_c}{f_c \pi r^2} + n p_0\left(1 - \frac{1}{2k}\right)} \tag{142}$$

By a process similar to the one described above, eqs. (139) and (142) were used to plot the curves given on the left-hand side of Diagrams 36 and 37. Stresses in the steel on the tension face may be determined from these curves for any value of eccentricity which will produce tension on the section. The stress in the steel on the compression side is $f_s' = nf_c$.

Illustrative Problem.—A round column with a 20-in. core reinforced with 10 sq. in. of steel, sustains a load of 200,000 lb. applied 4 in. off center. Determine the maximum unit stresses in steel and concrete. Assume $n = 15$.

For the given conditions,

$$r = 10 \text{ in.};\ p_0 = \frac{10}{314.2} \text{ and } \frac{r}{e} = \frac{10}{4} = 2.50$$

From Diagram 37,

$$\frac{R_1}{f_c} = 0.270 \text{ and } \frac{R_1}{f_s} = 5.5$$

Now

$$R_1 = \frac{N_c}{\pi r^3} = \frac{(200{,}000)(2)}{(3.1416)(10)^3} = 254$$

Hence

$$f_c = \frac{R_1}{0.270} = \frac{254}{0.270} = 942 \text{ lb. per sq. in.} \qquad f_s = 5.5$$

$$R_1 = (5.5)(254) = 13{,}950 \text{ lb. per sq. in.}$$

and

$$f_s' = nf_c = (15)(942) = 14{,}100 \text{ lb. per sq. in.}$$

72. Designing for Bending and Direct Compression.—Most of the illustrative problems given in this section have been reviews of designs where the construction was known and the stresses to be determined. The more common problem of design starts with known stresses and seeks to establish economical dimensions of the concrete and the reinforcement. The design of members for bending and direct compression is liable to be an extensive cut-and-try process for beginners, but experience soon enables a designer to make the two (or, at most, three) assumptions very accurately. Formulas have been given in this section under Cases I and II of rectangular sections for the particular condition of symmetrically placed reinforcement and also for the general condition without such limitations. Diagrams are available for these two cases, for symmetrically placed reinforcement only. The designer is therefore in a position where he must make a somewhat difficult decision between the easier but limited solution by the aid of diagrams and the more difficult solution by formulas which permit of much more economical use of the materials. It is obvious that the use of symmetrically placed reinforcement will not lead to economical design under the average conditions. In cases where the eccentric moment may act in any of several directions, the use

of symmetrically placed reinforcement is the proper solution. Under Case III of rectangular sections the diagrams are general and should always be used to save labor.

For circular sections under bending and direct compression, design diagrams have been developed only for symmetrical reinforcement, because of the fact that such conditions commonly occur with interior columns of buildings where the eccentric load may be applied in different directions at various times. For the more exceptional case where the eccentric moment acts in a fixed plane, the designer is probably justified in using symmetrical reinforcement with some waste. If the condition is repeated, as with long rows of round columns carrying crane brackets, the labor of accurate design by cut-and-try methods is justified. If a design with symmetrical reinforcement is made first, the number of trials can be reduced to one or two, and the work is not excessive.

73. Steps to be Taken in Design.—For all cases of rectangular and circular sections under bending and direct compression the first step in design is to make an assumption as to the weight of the member. This weight will act either to produce an added moment or to add to the direct load. This assumed weight must therefore be combined with the given applied forces and moments and the magnitude, direction and point of application of the resultant force on the principal design sections determined. From this the magnitude of the component, N, parallel to the axis of the member is found and the distance, e, from its point of application to the gravity axis is then determined. The final step in each case is to check this initial assumption of weight and to repeat the design operation with a revised weight if necessary. The intermediate design steps for the various cases follow:

Rectangular Sections—Case I, symmetrical reinforcement, using diagrams:

(*a*) Assume values of p_0, d'/t, and Z and with these values enter Diagrams 18, 19, 20, 21, 22 or 23 (to conform with values of n and d'/t), and determine the value of e/t.

(*b*) From this value of e/t and the known value of e determine the depth of the member, t.

(*c*) From eq. (109), in which f_c, N, Z and t are now known, determine b.

The dimensions and proportions of the beam will indicate whether further trials are needed to secure a satisfactory design.

Rectangular Sections—Case I, non-symmetrical reinforcement, using formulas:

(*a*) Work out design with symmetrical reinforcement.

(*b*) Determine f_c'' by Formula (89). If f_c'' is not greatly different from f_c, accept the design with symmetrical reinforcement.

(*c*) If f_c'' is small compared with f_c, assume a new section of member with a minimum (say ¼ of 1 per cent) amount of reinforcement on the f_c'' side and the same or somewhat greater reinforcement on the f_c side, as was found with symmetrical reinforcement.

(*d*) Compute values of A_t, I_t, and c_1 for this section and solve for f_c by eq. (89).

If f_c is too high or much too low new assumptions are necessary. Compare the final design with that having symmetrical reinforcement, for economy.

Rectangular Sections—Case II, symmetrical reinforcement, using diagrams:

(*a*) Assume values of p_0, d'/t and L and with these values enter Diagrams 24, 25, 26, 28, 29 or 30 (to conform to values of n and d'/t), and determine k.

(*b*) With these values of k and p_0 enter Diagram 27, or 31, and determine e/t.

(*c*) From this value of e/t and the known value of e, determine t.

(*d*) From eq. (122), in which f_c, M, and L are now known, determine b.

The dimensions and proportions of the member will indicate whether further trials are necessary to secure a satisfactory design.

Rectangular Sections—Case II, non-symmetrial reinforcement, using formulas:

(*a*) Work out design for symmetrical reinforcement.

(*b*) From this design assume a new trial section and reinforcement. In general, economy will result from reducing the compressive steel area and substituting concrete.

(*c*) Compute values of k by eqs. (116) and (117).

(*d*) Compute f_c from eq. (117).

(*e*) Compute f_s from eq. (113).

Rectangular Sections, Case III:

(*a*) Assume d which will also fix e' and the value of e'/d.

(*b*) Enter Diagram 32 or 33 with the values of f_s, f_c and e'/d, and determine K_1 and p.

(*c*) From eq. (128), in which K_1, M, and d are now known, determine b.

Circular Sections, Case I:

(*a*) Assume core radius, r, which determines value of e/r.

(*b*) Compute the value of $\frac{\pi r^2 f_c}{N}$, and with this value enter Diagram 34, and determine the value of p.

Circular Sections, Case II:

(*a*) Assume core radius, r, which determines values of r/e,

$$R_1\left(=\frac{Ne}{\pi r^3}\right),\ \frac{R_1}{f_c}\ \text{and}\ \frac{f_s}{R_1}$$

(*b*) Enter Diagram 36 or 37 and determine the value of p.

In using the various diagrams and, in particular, in using Diagrams 27 and 31, care must be taken to use the value of n consistently throughout.

74. Tying of the Steel to Prevent Buckling.—The use of high percentages of compressive reinforcement in members subject to bending and direct compression involves special consideration of the tying of the steel to prevent buckling. The ordinary column ties, ¼-in. round rods at 8-in. centers, are not sufficient for all conditions. For rectangular sections, in which ties are usual, a good rule to follow is to provide a sectional area of ties in a length of 1 ft. equal to 10 per cent of the area of the compressive reinforcement at the section of maximum stress. Ties should be spaced not over 8 in. on centers at this section and the spacing may be increased as the stress decreases in beams. In columns a spacing of 8 in. should not be exceeded and adjustments for decreased stress may be made by decreasing the size of bar until the minimum size of ¼-in. round is reached. For circular sections under bending and direct compression, a spiral will commonly be present which will amply provide against buckling. In the exceptional case where no spiral is used in the column design one should be provided when bending is involved. A spiral for this purpose may have a spacing of 6 to 8 in. as a maximum. The percentage of spiral should not be less than ⅛ of the percentage of vertical steel.

DIAGRAM 18.

BENDING AND DIRECT STRESS—CASE I.

Symmetrical reinforcement, $d' = 0.05t$, $n = 12$.

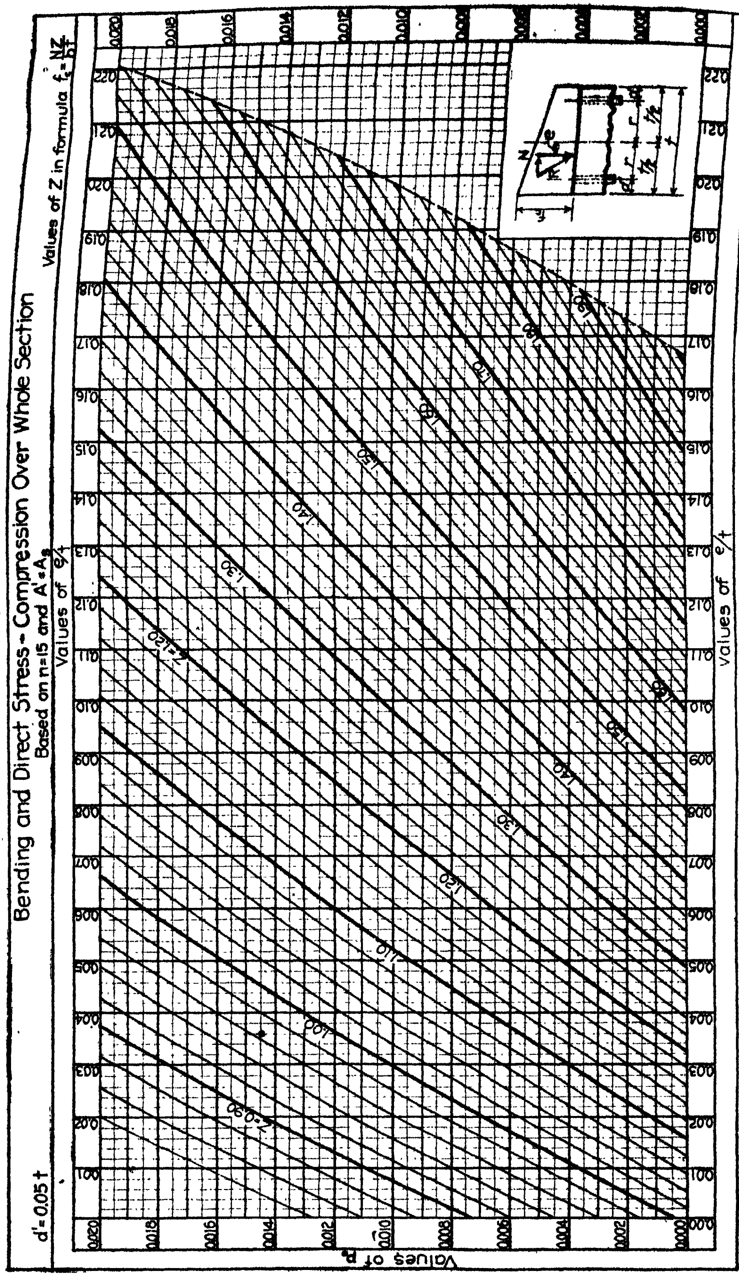


DIAGRAM 19.
BENDING AND DIRECT STRESS—CASE I.
Symmetrical reinforcement, $d' = 0.10t$, $n = 12$.

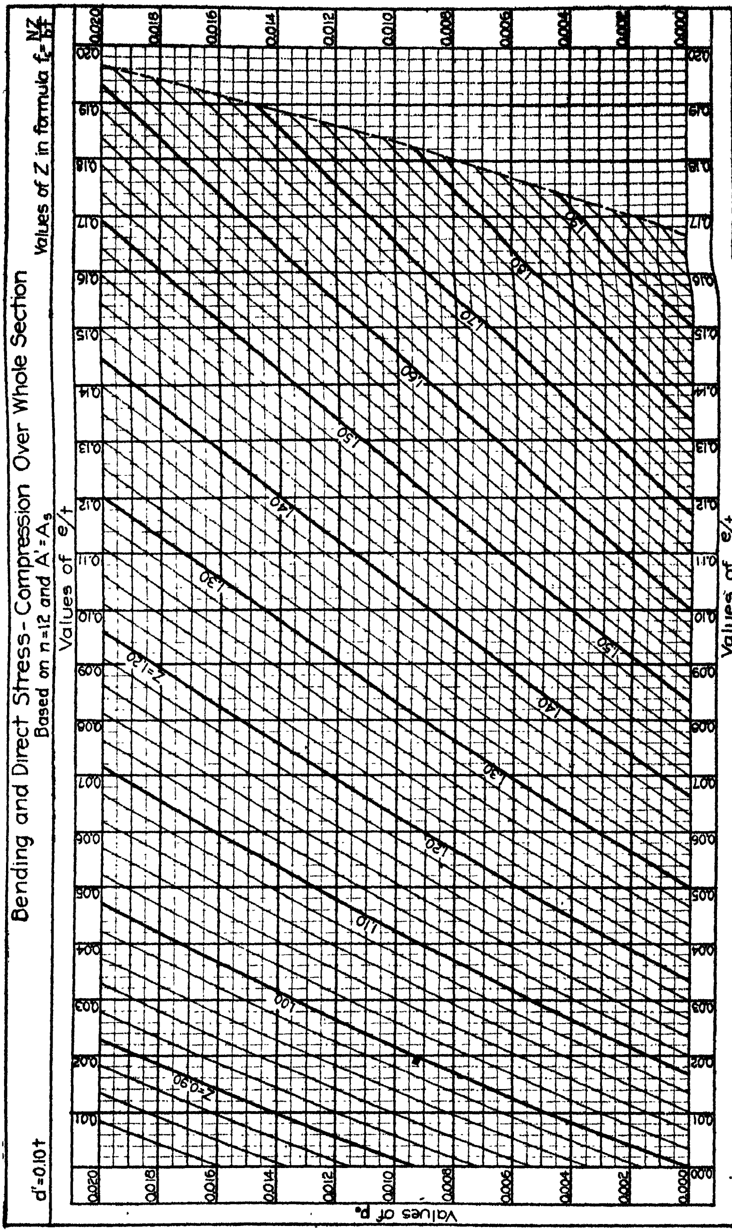


Diagram 20.

Bending and Direct Stress—Case I.

Symmetrical reinforcement, $d' = 0.15t$, $n = 12$.

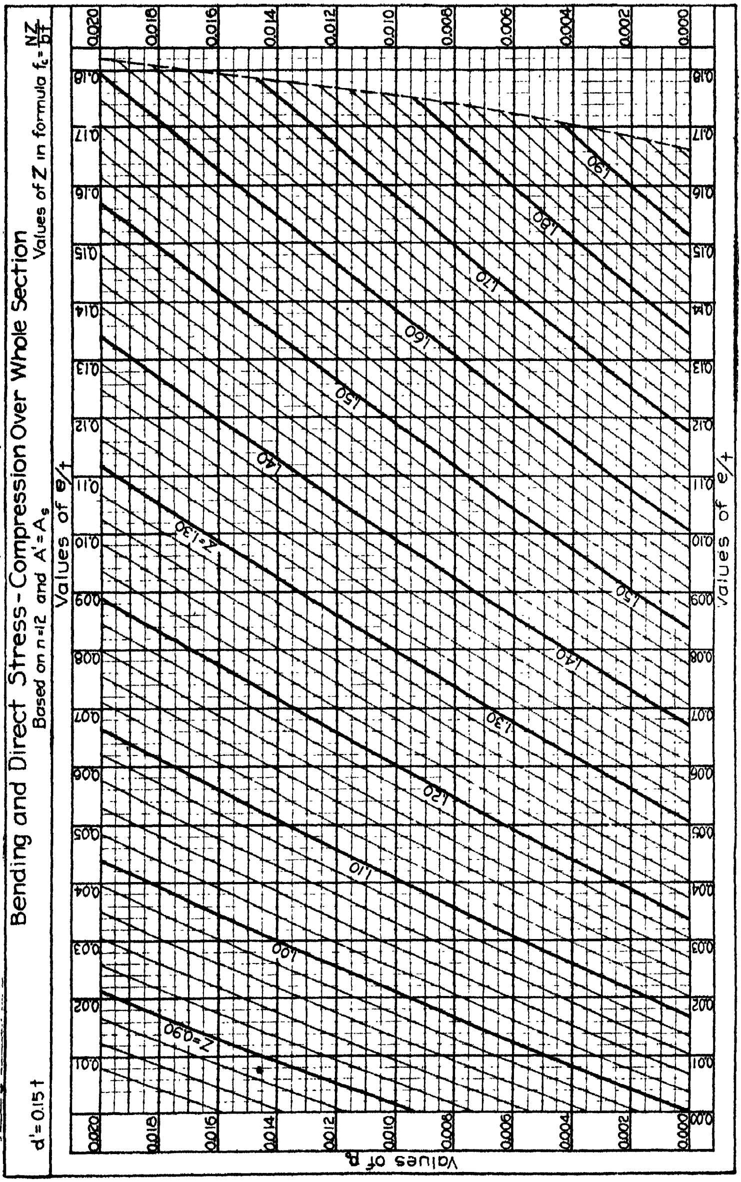


Diagram 21.

Bending and Direct Stress—Case I.

Symmetrical reinforcement, $d' = 0.05t$, $n = 15$.

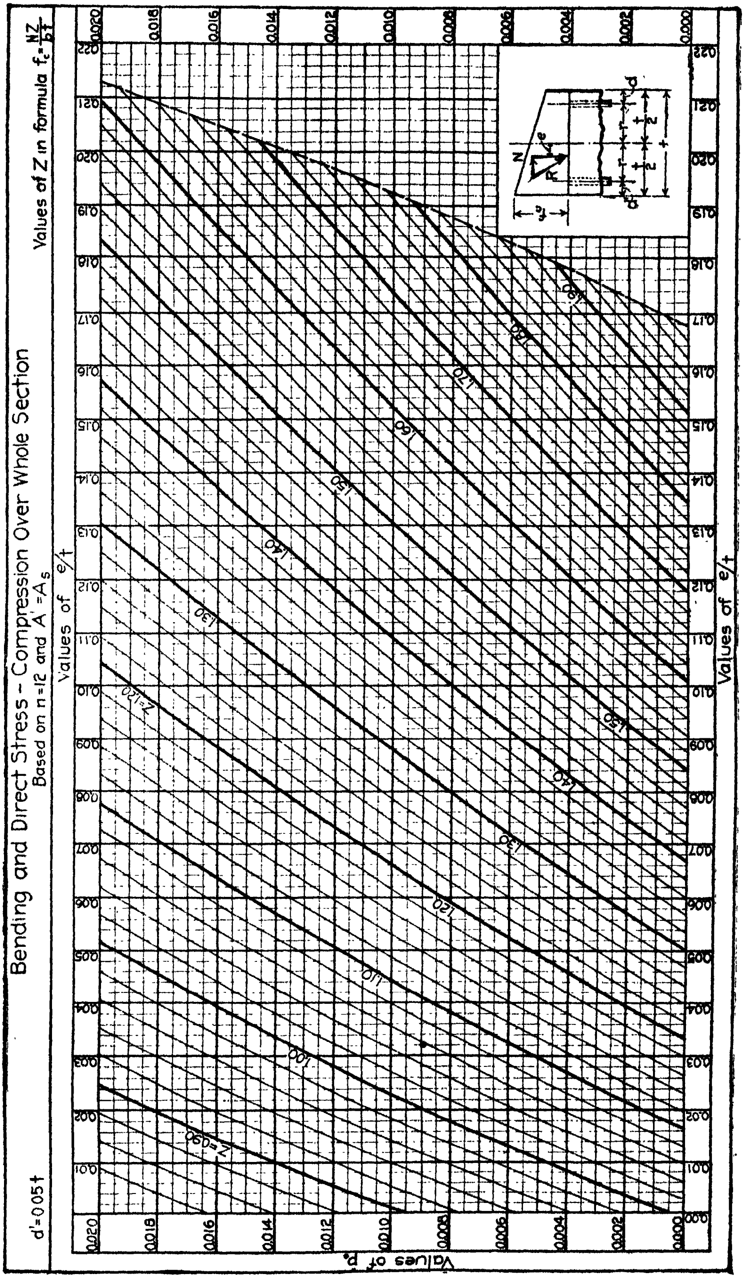


DIAGRAM 22.

BENDING AND DIRECT STRESS—CASE I.

Symmetrical reinforcement, $d' = 0.10t$, $n = 15$.

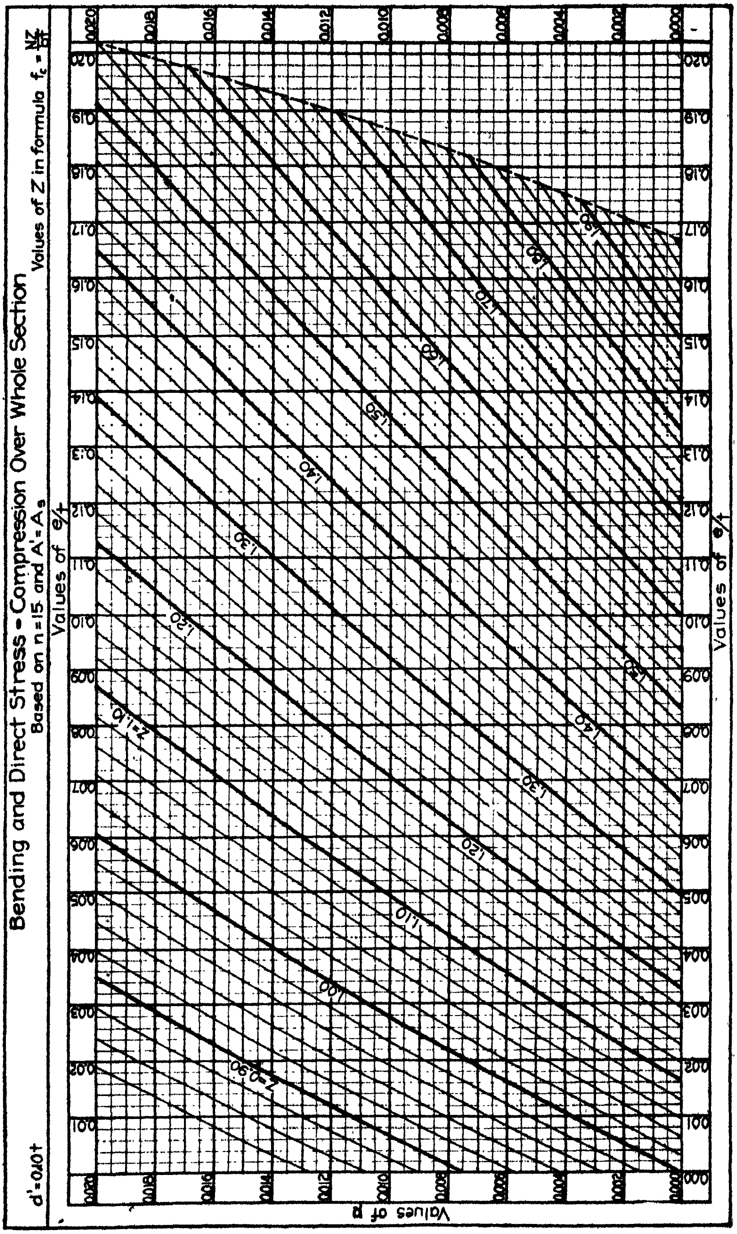


Diagram 23.

Bending and Direct Stress—Case I.

Symmetrical reinforcement, $d' = 0.15t$, $n = 15$.

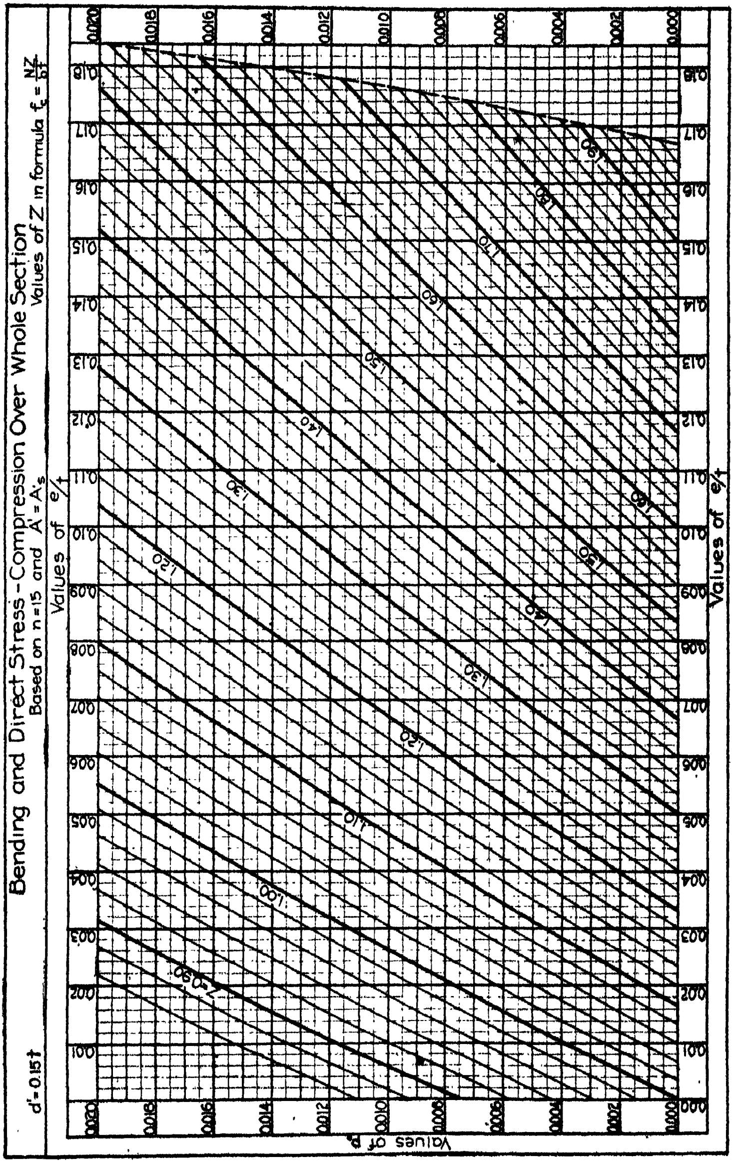


DIAGRAM 24.

BENDING AND DIRECT STRESS—CASE II.

Symmetrical reinforcement, $d' = 0.05t$, $n = 12$.

NOTE.—To be used in connection with Diagram 27.

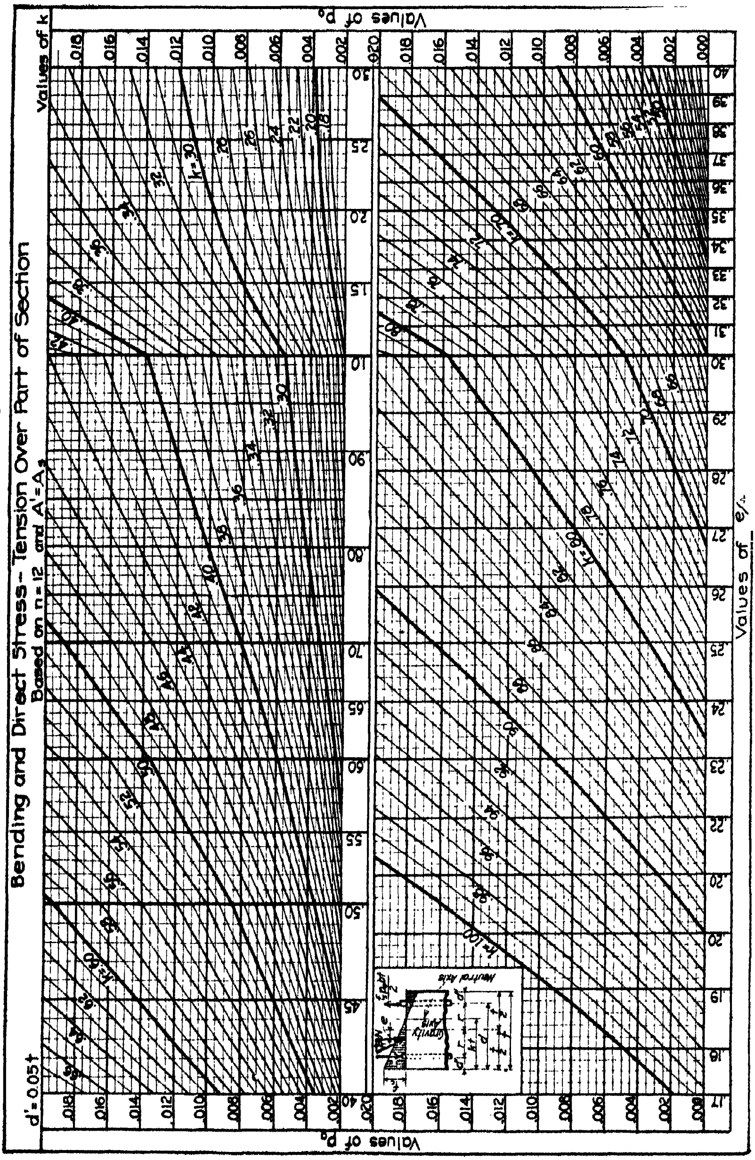


DIAGRAM 25.

BENDING AND DIRECT STRESS—CASE II.

Symmetrical reinforcement, $d' = 0.10t$, $n = 12$.

NOTE.—To be used in connection with Diagram 27.

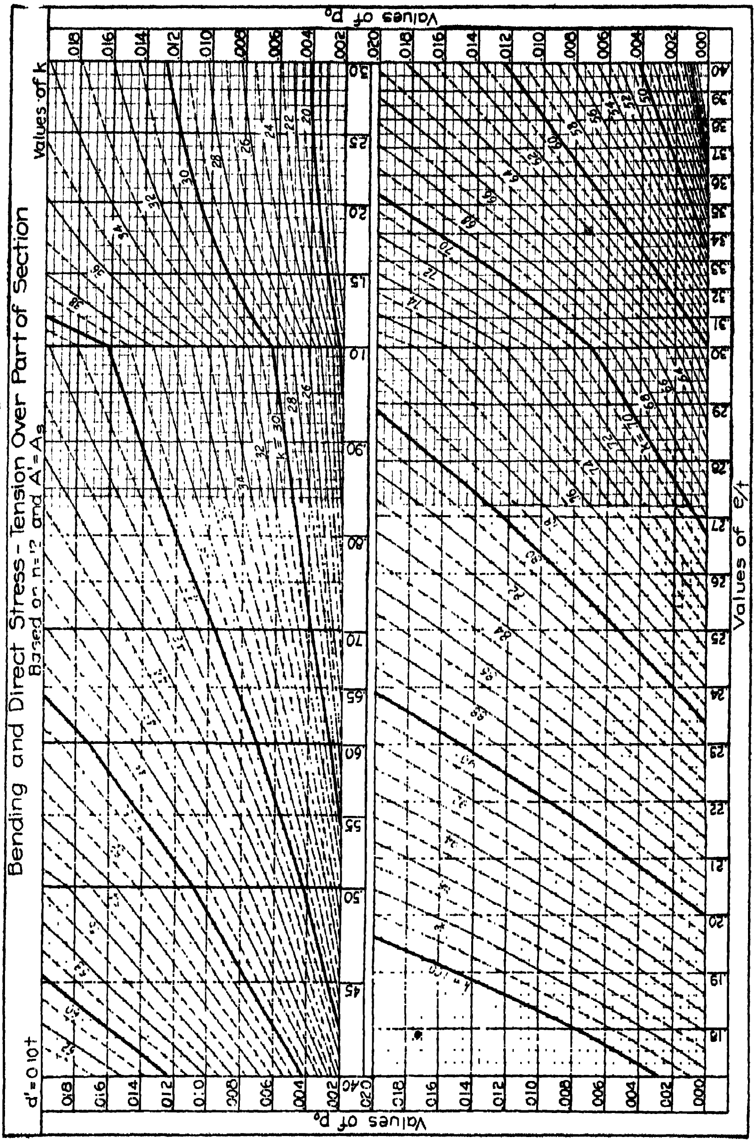


DIAGRAM 26.

BENDING AND DIRECT STRESS—CASE II.

Symmetrical reinforcement, $d' = 0.15t$, $n = 12$.

NOTE.—To be used in connection with Diagram 27.

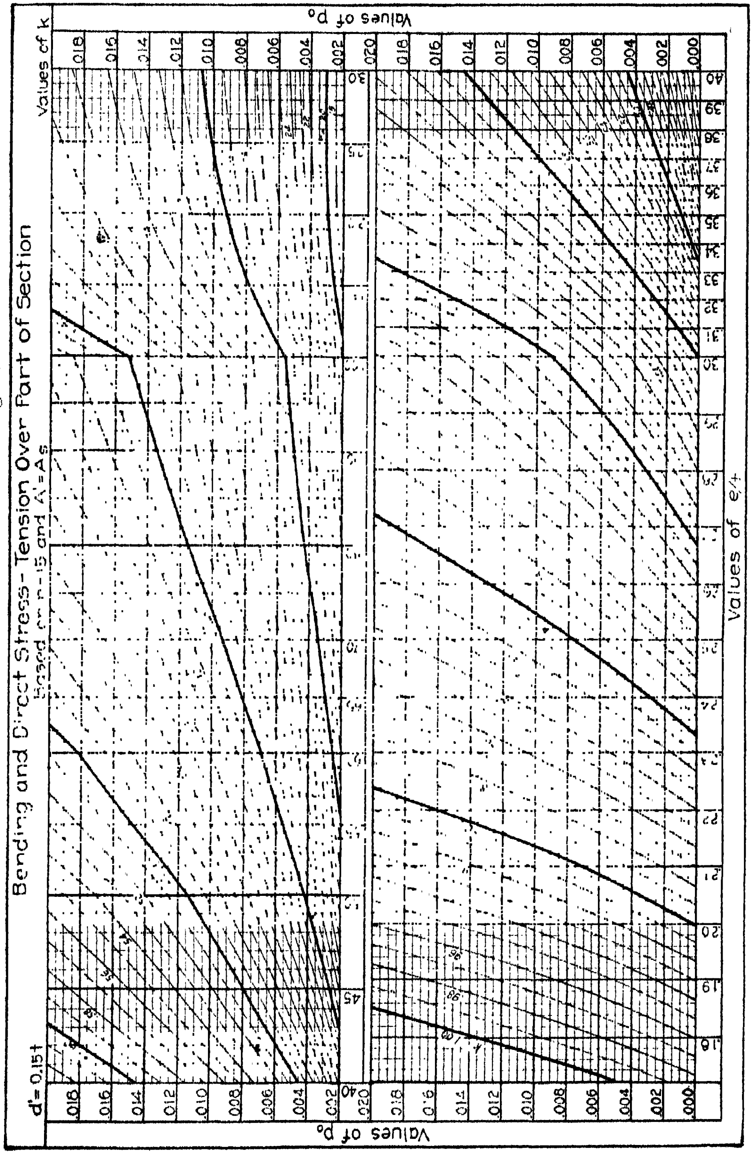


DIAGRAM 27.

BENDING AND DIRECT STRESS—CASE II.

Symmetrical reinforcement, $d' = 0.10t$, $n = 12$.

For $d' = 0.05t$, divide p_0 by 0.790 before entering diagram.

For $d' = 0.15t$, divide p_0 by 1.306 before entering diagram.

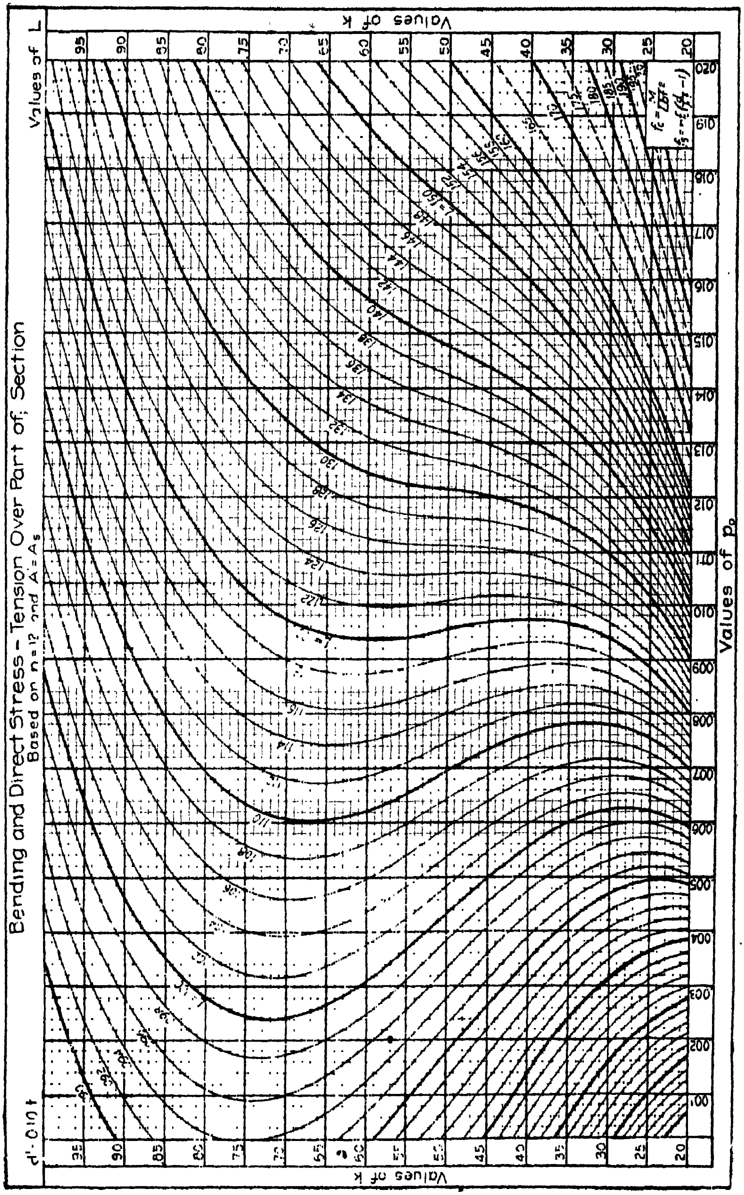


Diagram 28.

Bending and Direct Stress—Case II.

Symmetrical reinforcement, $d' = 0.05t$, $n = 15$.

Note.—To be used in connection with Diagram 31.

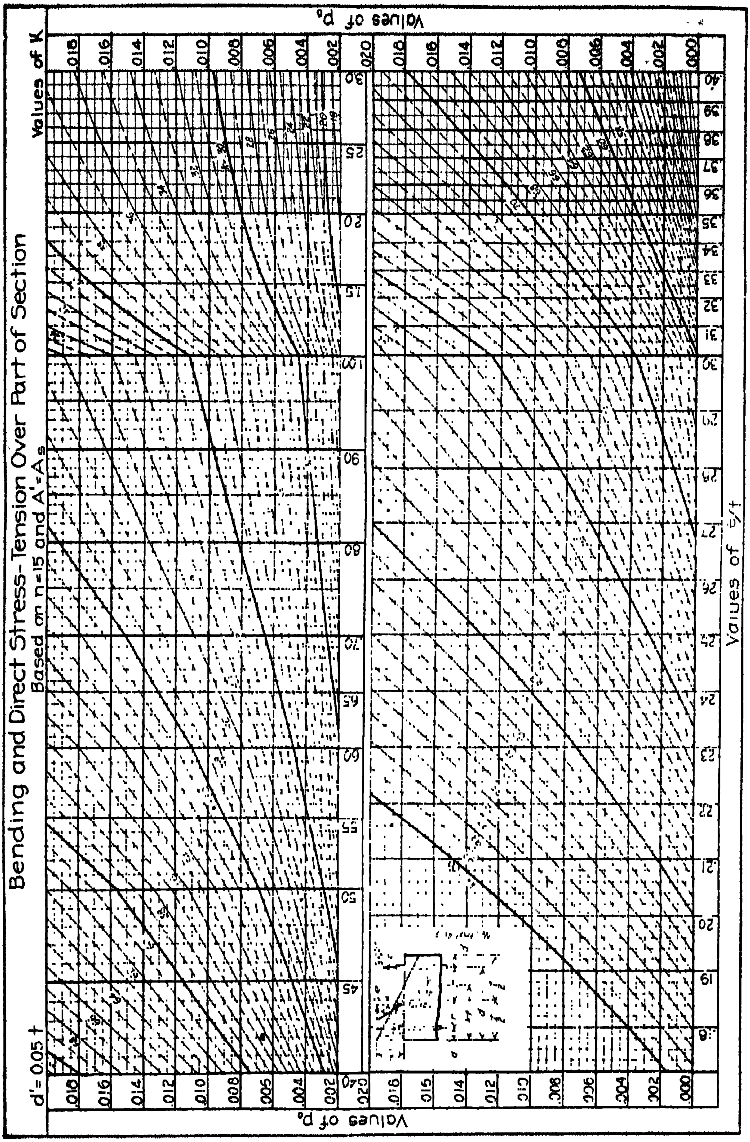


Diagram 29.
Bending and Direct Stress—Case II.
Symmetrical reinforcement, $d' = 0.10t$, $n = 15$.
Note.—To be used in connection with Diagram 31.

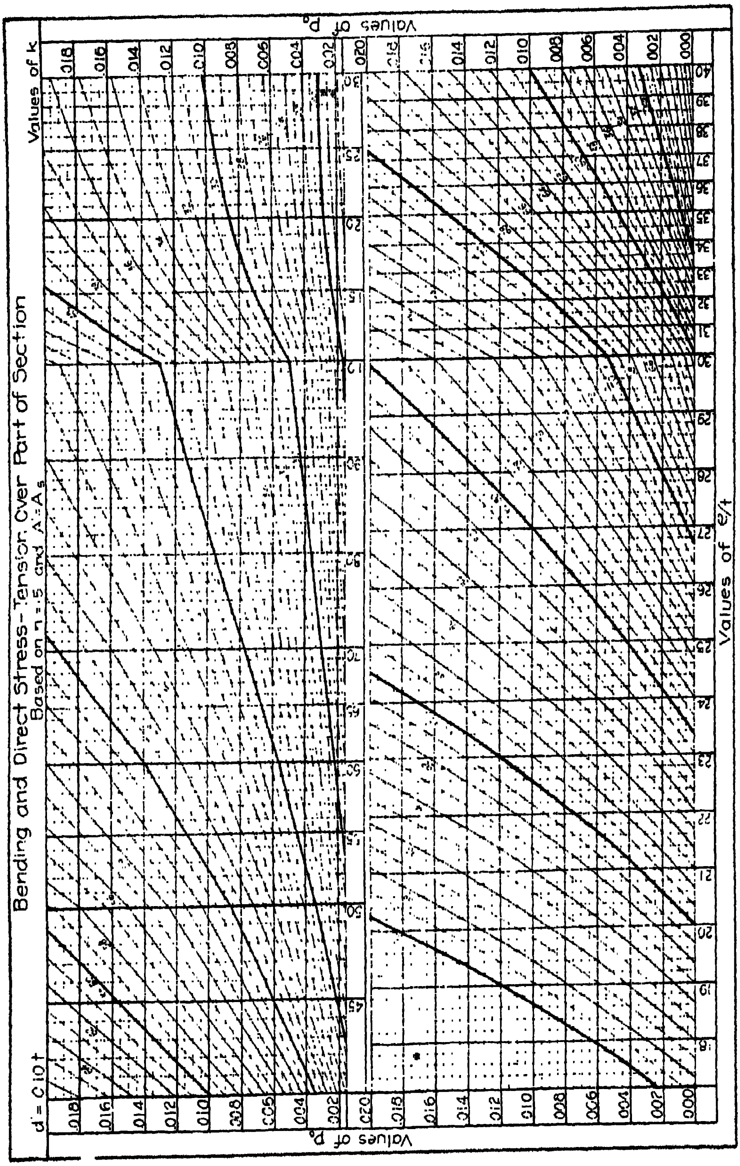


DIAGRAM 30.

BENDING AND DIRECT STRESS—CASE II.

Symmetrical reinforcement, $d' = 0.15t$, $n = 15$.

NOTE.—To be used in connection with Diagram 31.

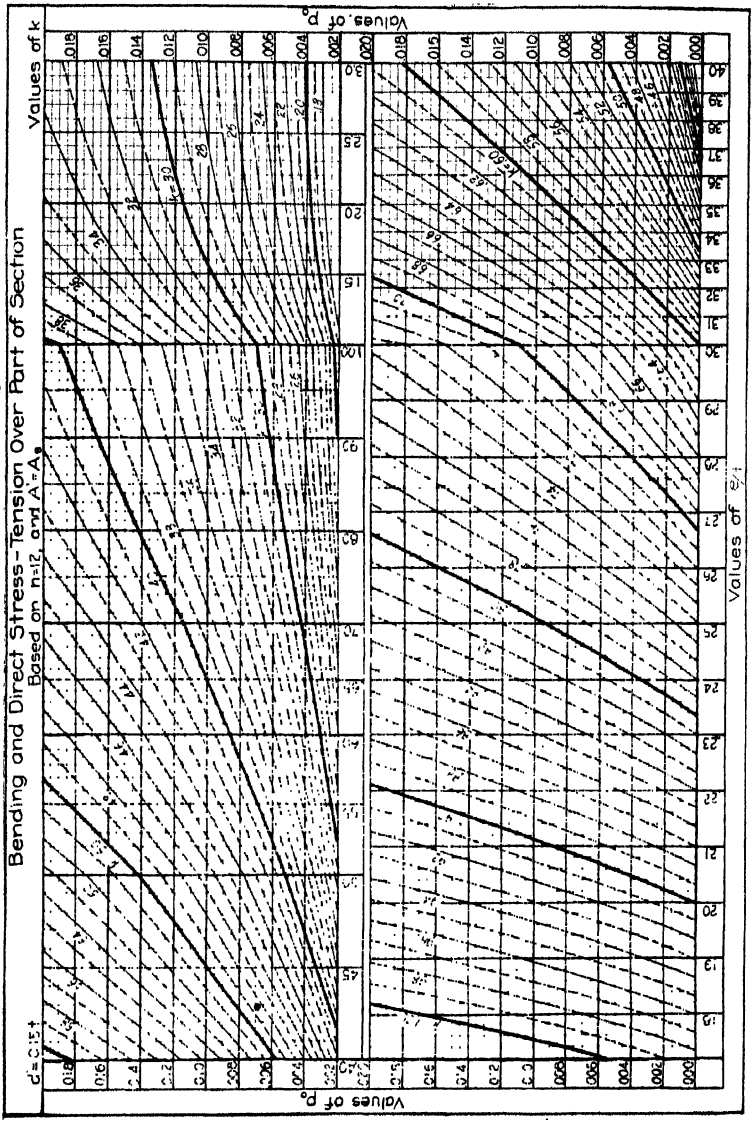


DIAGRAM 31.

BENDING AND DIRECT STRESS—CASE II.

Symmetrical reinforcement, $d' = 0.10t$. $n = 15$.

For $d' = 0.05t$, divide p_0 by 0.790 before entering diagram.

For $d' = 0.15t$, divide p_0 by 1.306 before entering diagram.

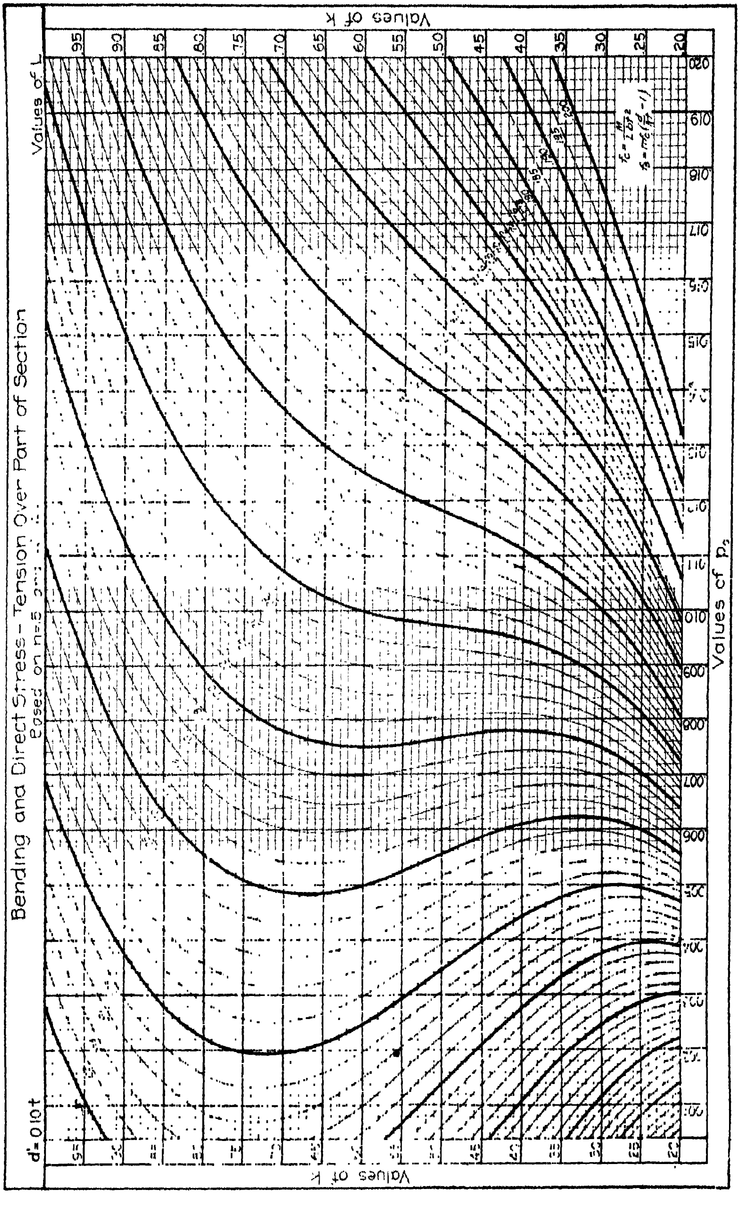


DIAGRAM 32.

Bending and Direct Stress – Steel in Tension Face Only – Tension Over Part of Section

Based on n=12

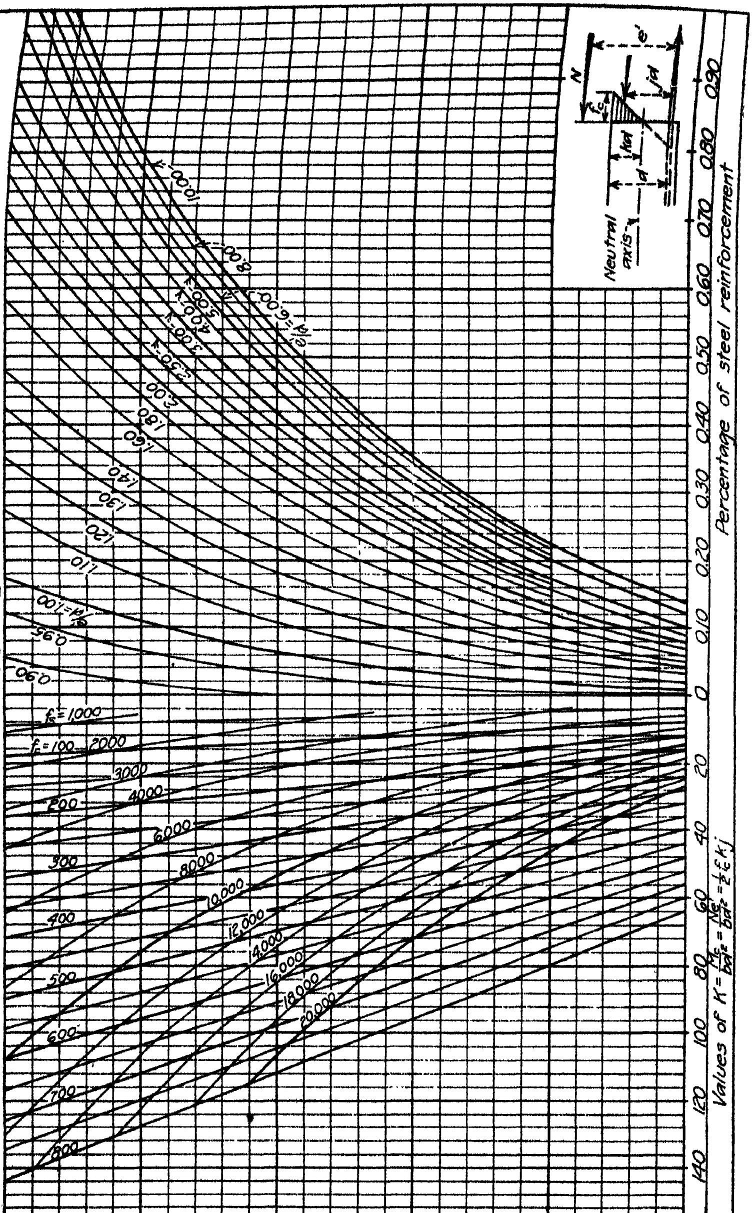


DIAGRAM 33.

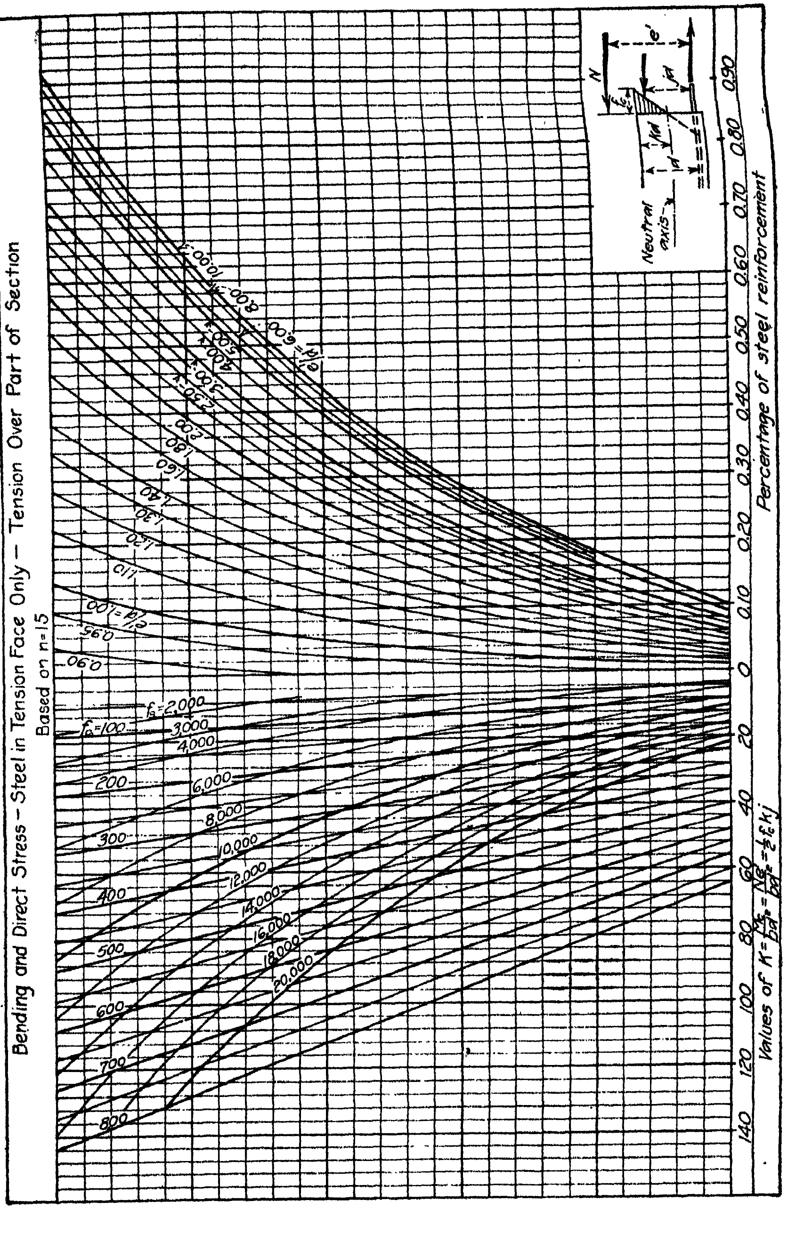

Bending and Direct Stress – Steel in Tension Face Only – Tension Over Part of Section
Based on n=15
Percentage of steel reinforcement
Values of $K' = \frac{M_s}{bd^2} = \frac{Ne}{bd^2} = \frac{1}{2} f_c k j$
0.90 0.80 0.70 0.60 0.50 0.40 0.30 0.20 0.10 0
20 40 60 80 100 120 140
e'/d = 6.00
8.00
10.00
5.00
4.00
3.00
2.50
2.00
1.80
1.60
1.40
1.30
1.20
1.10
e/d = 1.00
0.95
0.90
f_s = 2,000
3,000
4,000
6,000
8,000
10,000
12,000
14,000
16,000
18,000
20,000
f_c = 100
200
300
400
500
600
700
800
Neutral axis
N
e'
kd
jd
d

DIAGRAM 34.
BENDING AND DIRECT STRESS—ROUND COLUMNS—CASE I.

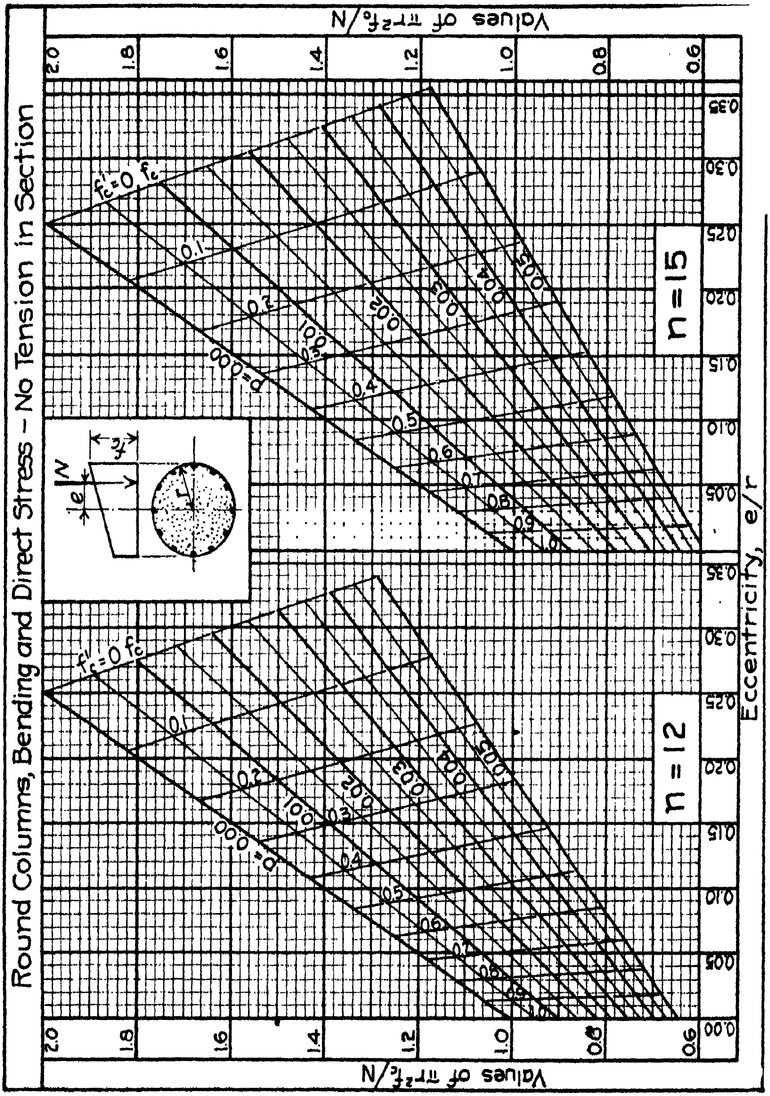
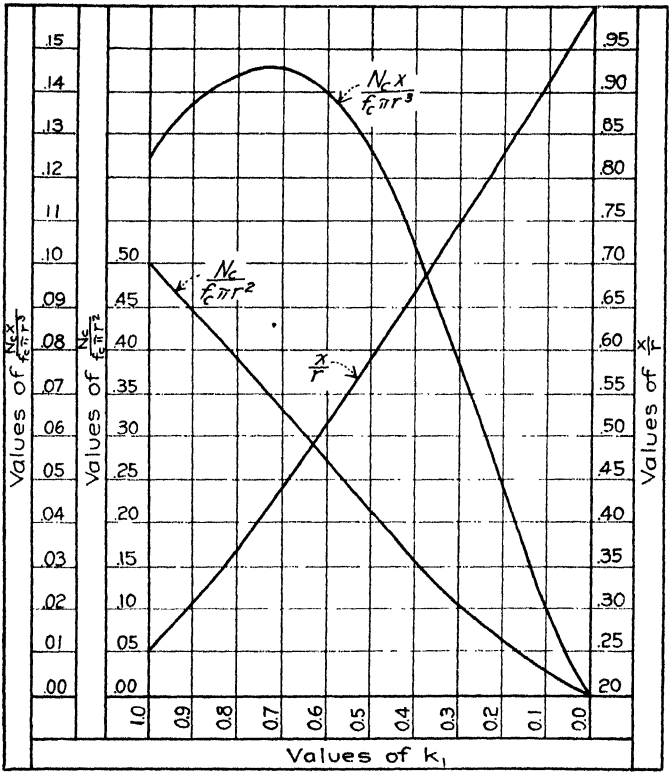


DIAGRAM 35.

BENDING AND DIRECT STRESS—ROUND COLUMNS—CASE II.



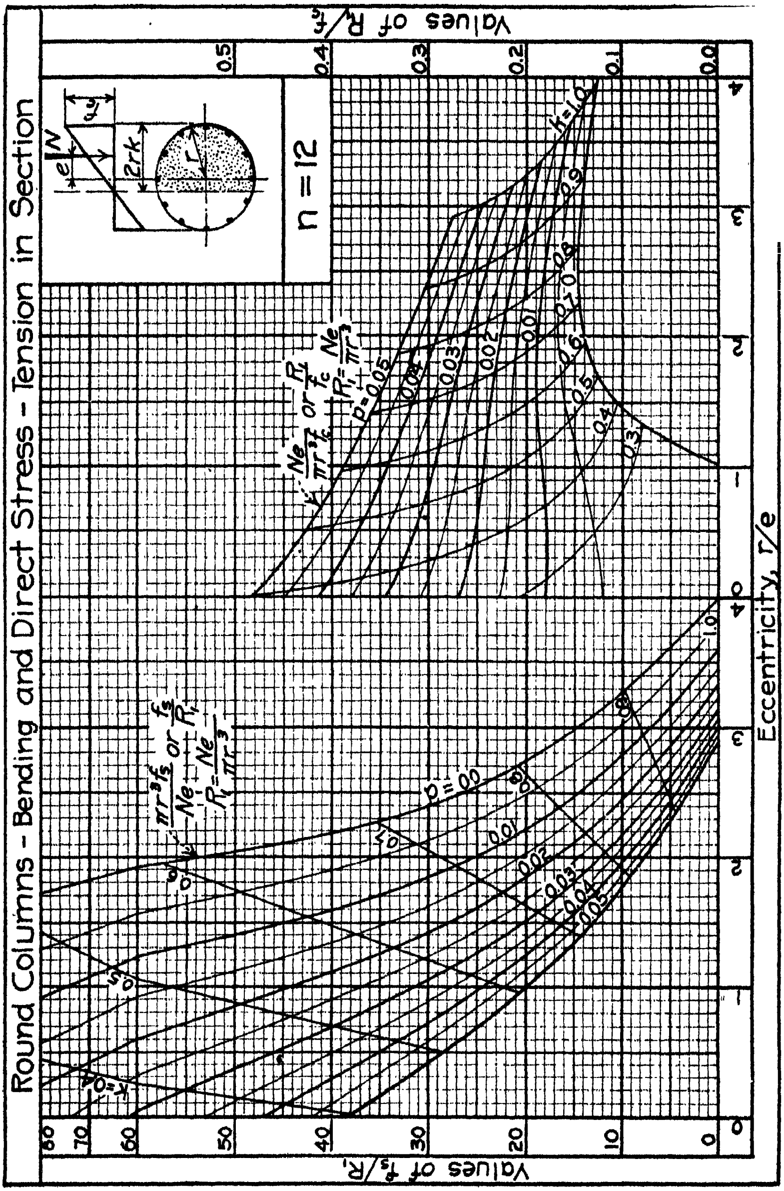
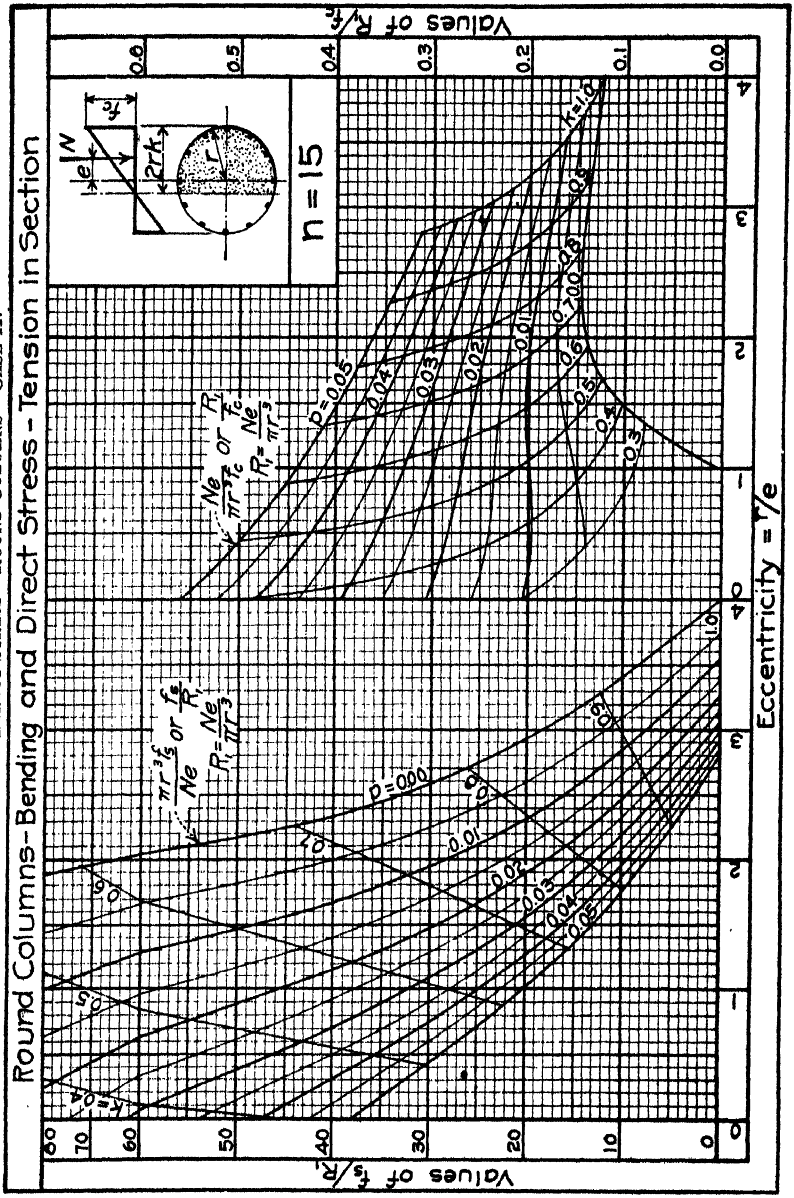


DIAGRAM 36.
BENDING AND DIRECT STRESS—ROUND COLUMNS—CASE II.

DIAGRAM 37.

BENDING AND DIRECT STRESS—ROUND COLUMNS—CASE II.



FOOTINGS

Balanced cantilever slabs are most commonly used in foundation work to distribute the load from a wall or column to the soil. When used as a wall footing, the cantilever projects in two balancing directions and such members are figured in the same manner as slabs by taking beam strips 12 in. wide as a design unit. Owing to the very heavy loading and short cantilever overhang the condition as to shear and bond is unusually severe in such members. Many designers provide hooked ends for all footing bars, to reduce the danger of failure through slipping to a minimum, and they keep the bar sizes small. The other common type of cantilever foundation slab projects in all directions from the column it supports. The design of such a footing considered as a structural member is the particular design problem discussed in the following articles. Almost all other types of spread foundations are designed as beam-members.

75. Notation.—The symbols used in the formulas for the design of square column footings are as follows:

a = width of pier or column supported. If column is round, a is taken as the side of a square of equal area.
A_s = effective steel area in one direction.
b = dimension of base of footing.
c = distance from face of pier to edge of footing.
d = depth from top of footing to c.g. of steel.
e = dimension of flat top of footing.
M = moment in one direction.
t = total thickness of footing.
t_1 = thickness of prismatic portion of footing.
w = column load divided by b^2.

Figure 29 illustrates the symbols given above.

76. Formulas.—The formulas for the design of square column footings follow:
For footing and pier monolithic

$$M = \frac{w}{2}(a + 1.2c)c^2 \tag{143}$$

For footing and pier of separate construction

$$M = \frac{wb^2}{12}(b - a) \tag{144}$$

$$A_s = \frac{M}{f_s jd} \tag{145}$$

$$f_c = \frac{2M}{kjed^2} \tag{146}$$

$$v = \frac{w[b^2 - (a + 2d)_2]}{3.5(a + 2d)d_1} \tag{147}$$

At edge of pier, Formula (17), p. 433, becomes

$$u = \frac{w(c^2 + ac)}{\Sigma ojd} \tag{148}$$

The basis of Formula (143) was first advanced by Prof. Talbot in discussion of footing tests made under his direction. It may be derived from Fig. 29 by writing moments about *mn* of the load on the area *himn* and of the load on the two areas *ghm* and *ijn*, considering that the load on the two triangles acts at $\frac{6}{10}$ of the distance *hm* instead of at their centers of gravity. The basis for taking this lever arm and for writing moments about *mn* instead of about the center line of the footing lies in the test data. For members in which the column is not monolithic

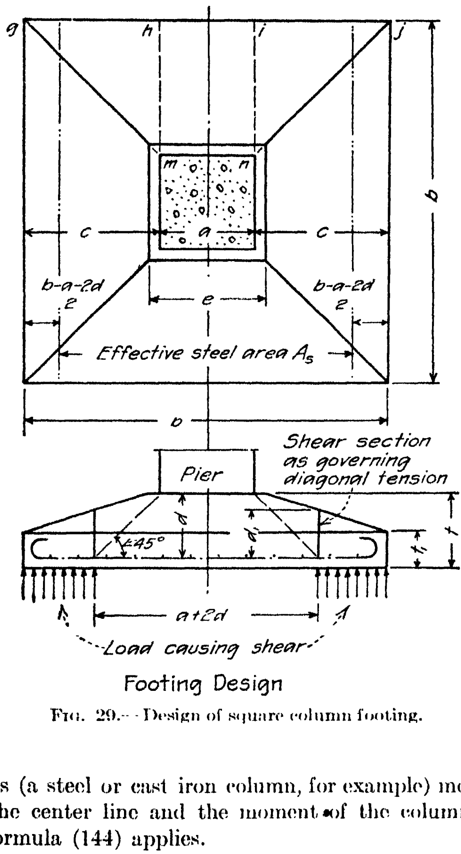


FIG. 29.—Design of square column footing.

with the footings (a steel or cast iron column, for example) moments should be written about the center line and the moment of the column load deducted. For this case Formula (144) applies.

77. Steps to be Taken in Design.—The steps in the design of a square column footing are as follows:

(*a*) From the column load and the allowable pressure on the soil determine the dimension *b* of the footing. For this computation an estimated weight of the footing per square foot must be deducted from the soil pressure. The weight of the footing does not enter into the computations otherwise, as its weight passes directly to the soil without affecting the moment or shear measurably.

(b) Compute w (= column load divided by l^2). Compute M by eq. (143) or (144).

(c) Assume a value of d and compute the shearing unit stress by eq. (147). Revise the assumption of d until an allowable shear on plain concrete is obtained (or design stirrups if depth of footing is limited).

(d) Compute A_s by eq. (145) and determine the size and number of bars making up the effective steel area. In the remaining width of footing provide the same size bars at the same spacing.

(e) Compute the bond by eq. (148), taking Σ_0 as the sum of the perimeters of all bars making up the effective steel area A_s.

(f) Compute the width e by eq. (146). Make the flat top of the footing not less than e, and not less than (a + 8 in.) in width.

78. Effective Width.—The steel close to the edge of footings takes less stress than that directly under the pier. In the tests referred to above it was found that a total tension equivalent to the actual total of the varying tensions in all bars was found by considering all the steel within a certain *effective width* to be stressed as highly as the bars directly under the pier. This effective width was found to be: The width of the pier plus the depth of the footing on either side plus ½ of the remaining width of the footing. The 1921 J. C. report permits the steel outside the effective width to be spaced at twice the interval found within the effective width, but this was not Prof. Talbot's recommendation. In his tests the bars were spaced uniformly across the entire width of the footing.

79. Shape of Footings.—Isolated column footings may have flat tops, sloping tops (as in Fig. 29) or stepped tops. The limiting contour for sloping footings is governed by the shear. Stepped footings should be so proportioned as to completely envelop the minimum sloped footing. The vertical depth at the edge of sloped footings, t_1, is commonly made 12 in. For the practical operation of pouring sloping footings without forms a comparatively dry concrete is used, and the slope may be as steep as 3:5 without causing any difficulty in the field. Stepped footings and flat topped footings require forms.

80. Punching Shear in Footings.—No attention has been paid thus far to punching shear, on which a limit is placed by many specifications. The best practice is to consider that the resistance to punching shear is so great that such stress is always far less critical than shear as governing diagonal tension. For the latter, Prof. Talbot's tests indicate that the critical section may properly be taken at a distance from the face of the column or pier equal to the depth, d, to the steel. This section is shown by a heavy line on Fig. 29. The load causing shear on this section is the load from the section to the edge of the footing. Formula (147) is derived on this basis.

81. Use of Web Reinforcement.—Web reinforcement is not generally used in isolated column footings as it is not economical. Occasionally a condition occurs where the foundation conditions require a shallow footing and web reinforcement in a few footings becomes desirable. In such cases a section cut by a square prism centered on the column center is taken and the shearing stresses and web reinforcement computed on the four planes of this section as for a beam section. Successive sections of varying size completely determine the shear design.

82. Diagram for Determining Depth of Footing.—For fixed proportions of a to b and for any given column load the depth to the steel, d, remains practically

constant for all soil pressures in common use. Figure 30 is based on the assumption that the column of whatever size will rest upon a pier whose side is at least ¼ of the side of the base of the footing. From this figure the value of d for design may be selected with the assurance that the shearing stress as governing diagonal tension on the section shown in Fig. 29 will not exceed 40 or 60 lb. per sq. in. as noted on the diagram. For any other assumption as to the shear section a similar diagram may be prepared.

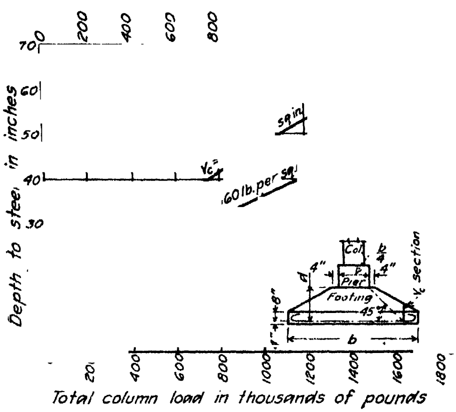


FIG. 30.—Approximate depths of square column footings.

Illustrative Problem.—To design an isolated column footing resting on soil good for a design load of 3,500 lb. per sq. ft. and supporting a 24-in. square column, loaded to 400,000 lb. Use $f_c = 700$, $f_s = 16{,}000$, $n = 15$ and $v_c = 40$ for plain concrete with straight bars, or 60 for plain concrete with hooked bars.

The solution of this problem is given on Design Sheet 18. The following notes apply to that sheet: (a) In Formula (147) note that the value of $(a + 2d)$ in the numerator is used in feet because the load, w, is in lb. per sq. ft. In the denominator $(a + 2d)$ is taken in inches as the stress v_c is desired in pounds per square inch; (b) a design using a smaller d and increasing v_c to 60 lb. per sq. in. would be economical as it would save excavation as well as concrete. The bond unit stress, however, would tend to be very much higher and very small bars would be necessary; (c) the effective width is 109 in. and the total width of footing 134 in. The total steel area in the entire width of footing is therefore $^{134}/_{109}$ times 7.20 sq. in.

DESIGN SHEET 18

Load on soil (total) = 3,500′ □′
Approximate weight of footing = 270

Available for col. load = 3,230

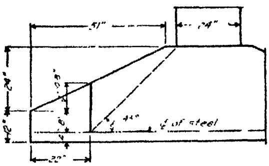
$$b^2 = \frac{400{,}000}{3{,}230} = 124\,\square' \quad b = 11.18. \quad \text{Make footing } 11'\text{-}2'' \text{ square}$$

$$w = \frac{400{,}000}{(11.16)^2} = 3{,}200\,\#/\square'$$

Assume $d = 30''$ $a = 2.0' = 24''$ $c = \dfrac{11.16 - 2.0}{2} = 4.58' = 55''$ $t_1 = 12''$ $t = 30 + 4 = 34''$ $t - t_1 = 22''$ $(a + 2d) = 24 + 60 = 84'' = 7.0'$ $d_1 = 8 + \left(\dfrac{25}{51}\right)(22'') = 18.8''$

By Formula (147)[a]

$$v_c = \frac{3{,}200[(11.16)^2 - (7.0)^2]}{(3.5)(88)(18.8)} = 41.5\,\#/\square'$$



This is O.K. with hooked rods [b]
By eq. (143)

$$M = \frac{3{,}200}{2}[2 + (1.2)(4.58)](4.58)^2 = 252{,}000'\# = 3{,}020{,}000''\#$$

By eq. (145)

$$A_s = \frac{3{,}020{,}000}{(16{,}000)(\tfrac{7}{8})(30)} = 7.20\,\square'' = 16 - \tfrac{3}{4}''\phi \text{ at } 7\tfrac{1}{4}'' \text{ o.c.}$$

Effective width $= 7.0' + \dfrac{11.16 - 7.0}{2} = 9.08' = 9'-1'' = 109''$

For total width of footing steel in each direction $= 20 - \tfrac{3}{4}''\phi$, hooked ends[c]
By Formula (148)

$$u = \frac{3{,}200[(4.58)^2 - 9.16]}{(16)(2.356)(\tfrac{7}{8})(30)} = 97.7\,\#/\square'' \quad \text{O.K. with hooked ends}$$

By Formula (146)

$$c = \frac{(2)(3{,}020{,}000)}{(700)(0.396)(0.868)(30)^2} = 28''$$

Make flat top of footing $24'' + 8'' = 32''$ □

APPENDIX A

GENERAL NOTATION[1]

For all materials except reinforced concrete:

f = unit fiber stress.
v = unit shearing stress (horizontal or vertical).
V = total shear.
c = distance from neutral axis to extreme fiber.
b = breadth of rectangular section.
d = depth of section.
A = area of section.
I = moment of inertia.
r = radius of gyration.
S = section modulus.
M = bending moment or resisting moment.
l = span or length.
L = span or length.
F or P = concentrated load or total stress in a member.
w = uniformly distributed load per unit of length.
W = total uniformly distributed load.
R = reactions at supports or resultant of forces.
E = modulus of elasticity.
y = deflection at any point in a beam
Δ = total deformation or deflection at any point in a beam.
δ = unit deformation.
e = eccentricity.

For reinforced concrete:

(*a*) Rectangular Beams and Slabs

f_s = tensile unit stress in steel.
f_c = compressive unit stress in extreme fiber of concrete.
f_c' = ultimate compressive strength of concrete at age of 28 days, based on tests of 6- × 12-in. or 8- × 16-in. cylinders made in accordance with A.S.T.M. specifications.
E_s = modulus of elasticity of steel.
E_c = modulus of elasticity of concrete.
$n = \frac{E_s}{E_c}$. For values of f_c in flexure not over 900 lb. per sq. in., n is commonly taken as 15.
M = moment of resistance, or bending moment in general.
A_s = steel area.
b = breadth of beam (generally taken as 12 in. in case of slabs).
d = depth of beam to center of steel.
k = ratio of depth of neutral axis to depth, d.
z = depth from compressive face to resultant of compressive stresses.
j = ratio of lever arm of resisting couple to depth, d.
$jd = d - z$ = arm of resisting couple.
p = steel ratio $= \frac{A_s}{bd}$.
l or L = span length of beam or slab.

(*b*) T-beams

b = width of flange.
b' = wideh of stem.
t = thickness of flange.

[1] Notation not found in this appendix appears in text where used.

(c) Beams Reinforced for Compression

A' = area of compressive steel.

p' = steel ratio for compressive steel.

f_s' = compressive unit stress in steel.

C = total compressive stress in concrete.

C' = total compressive stress in steel.

d' = depth of center of compressive steel.

z = depth to resultant of C and C'.

(d) Shear, Bond and Web Reinforcement

V = total shear at any section.

V' = total shear at any section carried by the web reinforcement.

v = maximum shearing unit stress at any section.

u = bond stress per unit area of bar.

o = circumference or perimeter of bar.

Σo = sum of the perimeters of all tension bars at any section.

s = spacing of web members measured at the neutral axis and in the direction of the longitudinal axis of the beam.

a = spacing of web reinforcement bars measured perpendicular to their direction.

A_v = total cross-sectional area of web reinforcement within a distance of "a," or total area of all bars bent up in any one plane.

f_v = tensile unit stress in web reinforcement.

α = angle between web bars and longitudinal bars.

v_c = allowable shearing stress on plain concrete.

N_s = number of stirrups at one end of member.

(e) Flat Slabs

b = side of square drop.

c = base diameter of the largest right cone or pyramid which lies entire within the column and the column capital, whose vertex angle is 90°, and whose base is 1½ in. below the bottom of the slab or the bottom of the drop, if a drop is present.

L = side of square panel.

l_2 = that side of any panel, which is at right angles to the section for which moments are desired.

l_1 = that side of any panel which is parallel to the section for which moments are desired.

M_0 = total moment in one direction on critical sections of one column strip and one middle strip.

= arithmetical sum of $+M_c$, $-M_c$, $+M_m$ and $-M_m$ in one direction.

$+M_c$ = positive moment at center of column strip.

$-M_c$ = negative moment across panel and capital edge on column strip.

$-M_m$ = negative moment across panel edge on middle strip.

$+M_m$ = positive moment at center of middle strip.

q = distance from center of column to center of gravity of semi-periphery of column capital, divided by c. For round column capitals $q = \frac{1}{3}$; for square capitals $q = \frac{1}{4}$, for octagonal capitals $q = \frac{2}{3}$.

t_1 = thickness of slab.

t_2 = thickness of slab and drop combined.

t_c = thickness of slab at center, with panelled ceiling.

w = load per square foot of panel including weight of slab.

W = total load in one panel including weight of slab.

(f) Columns

A = cross-sectional area of member, exclusive of any portion used solely for protective cover.

A_s = area of longitudinal steel.

$p = \dfrac{A_s}{A}$.

$A_c = A(1 - p)$ = net area of concrete.

P = total safe axial load (including weight of column).

h = unsupported height of column.

APPENDIX B

GENERAL PROPERTIES OF SECTIONS

1. Area of a Section.—The area of standard rolled sections may be determined from the rolling mill handbooks. For the section shown in Fig. 1*a*, the area is readily determined by dividing the figure into simple figures, in this case the rectangles A, B and C and the four triangles D. In the case of the built-up section of Fig. 1*b*, the total area is readily found by summing up the areas of the plates and angles. These areas may be taken from any rolling mill handbook.

2. Statical Moment of an Area.—Let Fig. 2 represent any area. The statical moment of this area about any axis, as OX, is the moment of each element of this area about the given axis. Assume the area to be divided into strips parallel to the given axis. Such a strip is represented by 1–2 of Fig. 2. Let b = length of this strip, dy = width of strip perpendicular to the given axis, y = perpendicular distance from center of strip to the given axis, and Q = statical moment of entire area about the given axis.

The area of the strip 1–2 is bdy and its statical moment about the axis OX is $bydy$. For the entire area, Q = sum of all such values as $bydy$, that is

$$= \int_{y_2}^{y_1} bydy \tag{1}$$

To apply eq. (1) to a given area, the width of section must be expressed as a function of y and the resulting equation integrated between the given limits.

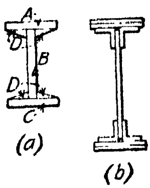


FIG. 1.

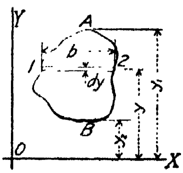


FIG. 2.

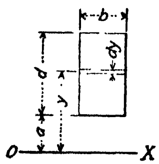


FIG. 3.

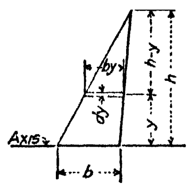


FIG. 4.

Consider the rectangle of Fig. 3. Required the statical moment of the figure about an axis OX at a distance a below the bottom of the figure. From eq. (1), noting that the width of section is constant, we may write

$$Q = b\int_0^{(d+a)} ydy = \frac{b}{2}(d^2 + 2ad) = bd\left(a + \frac{d}{2}\right)$$

But $bd = A$ = area of section. Hence

$$Q = A\left(a + \frac{d}{2}\right) \tag{2}$$

That is, the statical moment about the given axis is equal to the area of the section multiplied by the distance from the axis to a point half way across the section.

For the triangle of Fig. 4,

$$Q = \int_0^h b_y\, ydy$$

when b_y = width of section at any point $= \frac{b}{h}(h - y)$

Hence

$$Q = \int_0^h \frac{b}{h}(h - y)ydy = \frac{bh^2}{6}$$

But $\frac{bh}{2}$ = area of triangle. Thence

$$Q = \text{Area times } \tfrac{1}{3} \text{ of height of triangle above axis} \tag{3}$$

The statical moment of an area is sometimes called the *first moment* of the area.

3. Center of Gravity of an Area.—The center of gravity of an area is the point at which the entire area must be concentrated in order that the product of the area times the distance from this point to a given axis may be equal to the statical moment of the area about the given axis.

Let A represent the area of the section shown in Fig. 5, and let y_g represent the distance from an axis OX to the center of gravity of the section, assumed as located at the point $c.g.$ From the above definition

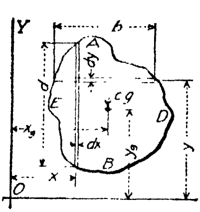


FIG. 5.

$$Ay_g = \int_B^A bydy$$

Since $A = \int_B^A bdy$, we have in general

$$y_g = \frac{\int_B^A bydy}{\int_B^A bdy} \tag{4}$$

In the same manner for an axis OY

$$x_g = \frac{\int_E^D dxdx}{\int_E^D ddx} \tag{5}$$

Equations (4) and (5) give the coordinates of the center of gravity of the section of Fig. 5 with respect to an origin at O.

In these equations the denominators each represent the area of the section. The limits of integration indicate the extreme values of x and y for the section.

From eqs. (4) or (5), it can be seen that the statical moment of any area about an axis through its center of gravity is equal to zero, for under the assumed conditions x_g or y_g must be zero. Since the denominators of eqs. (4) and (5) represent the area of the section, which cannot be zero for a real area, y_g or x_g can be zero only when the numerators of these equations are equal to zero. But these numerators represent the statical moment about the axis in question. Therefore, the *statical moment of an area is zero for an axis through its center of gravity.* This relation is of value in the work to follow.

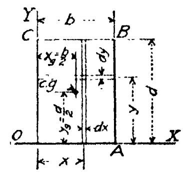


FIG. 6.

To apply eqs. (4) and (5) to any given figure, the dimensions of the section must be expressed as functions of x and y and the integrations performed, as indicated. For the rectangle of Fig. 6, assume a set of axes OX and OY through the sides OC and OA. Let $c.g.$ represent the required center of gravity. The distance from the OX axis to $c.g.$ as given by eq. (4), is

$$y_g = \frac{\int_0^d bydy}{\int_0^d bdy} = \frac{\frac{bd^2}{2}}{bd} = \frac{d}{2}$$

In the same way, the distance from the OY axis to $c.g.$ as given by eq. (5) is

$$x_g = \frac{\int_0^b dxdx}{\int_0^b d\,dx} = \frac{b}{2}$$

The point represented by these coordinates is the center of the section. Hence, the center of gravity of a rectangle is at the center of the section.

For the right angle triangle of Fig. 7*a*, the axes are taken along the right angle sides. From eq. (4), the distance from the OX axis to $c.g.$ is

$$y_g = \frac{\int_0^d \frac{b}{d}(y - d)y\,dy}{\int_0^d \frac{b}{d}(y - d)dy} = \frac{d}{3}$$

The distance from the OY axis to $c.g.$ is found to be $\frac{b}{3}$. Therefore for a right triangle,

the center of gravity is located at a distance from the bases equal to ⅓ of the altitude of the triangle. Note that if the bases of the triangle are bisected and lines are drawn to the opposite vertices of the triangle, the intersection of these lines coincides with the *c.g.* of the triangle. Figure 7*b* shows the coordinates of the *c.g.* for oblique triangles.

When the section is very complicated or the outline very irregular, the above method cannot readily be applied. In such cases approximate methods of integration may be used to advantage. Thus in Fig. 8, the irregular area may be divided into small strips

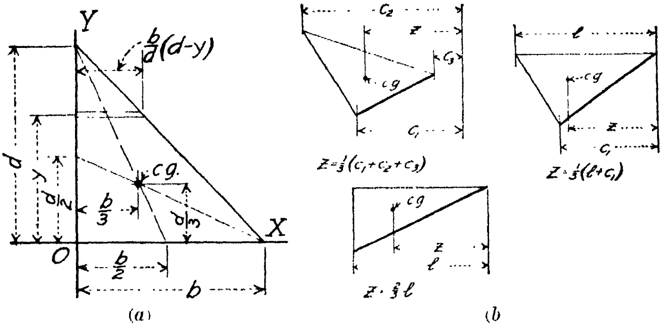


Fig. 7.

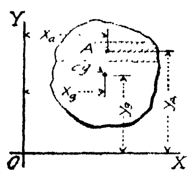


Fig. 8.

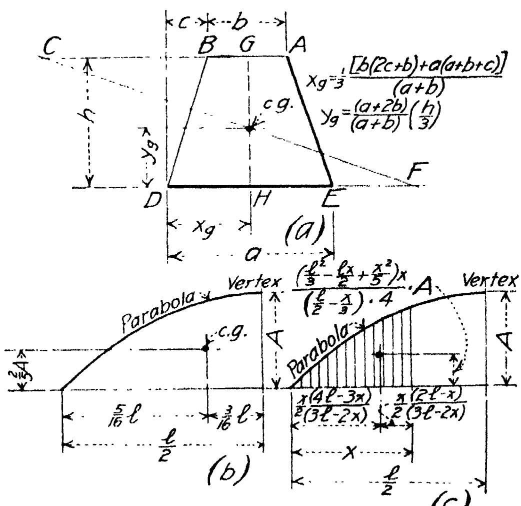


Fig. 9.

representing rectangles, triangles or other simple areas. At the center of gravity of each of these small areas apply a force which is proportional to the area of the strip. Scale or calculate the distances from these centers of gravity to any axes, as OX and OY. If A represent the area of any strip and x_A and y_A the distance from the center of gravity of the area A to the given axis, then

$$x_0 = \frac{\Sigma A x_A}{\Sigma A}$$

and

$$y_0 = \frac{\Sigma A y_A}{\Sigma A}$$

Figure 9 shows the location of the centers of gravity of a few simple figures. A convenient graphical method for locating the center of gravity of a trapezoid is shown on Fig. 9a. The construction is as follows: Produce the top of the figure AB to a point C such that $BC = DE$ = width of base. In the opposite direction produce the base DE to a point F such that $EF = AB$ = width of top. Connect C and F. Bisect the upper and lower faces, locating points G and H. Connect G and H. The intersection of lines CF and GH coincides with the center of gravity of the section.

4. Moment of Inertia.—The moment of inertia of an area with respect to any axis is the sum of the products formed by multiplying each element of the area by the square of its distance from the given axis. Let Fig. 10 represent any area, and let it be required to determine a general expression for the moment of inertia of this area with respect to any axis, as OX. Divide the area into strips, 1–2, parallel to the axis OX. If dy = width of each strip, the area of a strip is bdy. Let I represent the moment of inertia of the given area. Then, from the above definition, the moment of the entire area is

$$I = \int_{y_2}^{y_1} by^2dy \tag{6}$$

To apply eq. (6) to a given area, the width b must be expressed as a function of y and the integration performed as indicated.

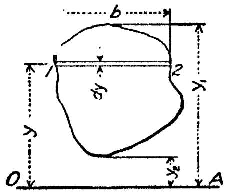


Fig. 10.

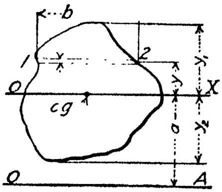


Fig. 11.

The moment of inertia of an area as given by eq. (6), is a quantity of the dimensions distance to the fourth power, for bdy represents an area which has the dimensions distance to the second power, and y^2 is also distance to the second power.

5. Moments of Inertia for Parallel Axes (*Parallel Axes Theorem*).—A very useful and simple relation may be obtained between the moment of inertia of an area for an axis through its center of gravity and any other parallel axis at a distance a from the gravity axis. In Fig. 11 let OX represent an axis through the center of gravity of the section (sometimes called a *gravity axis*) and let OA be any axis parallel to OX and at a distance a from OX. Let y, the distance from any element of area 1–2, to an axis, be referred to OX. Then from the above definition, the moment of inertia about axis OA, which will be denoted by I_A is

$$I_A = \int_{y_2}^{y_1} b(a + y)^2dy$$

Expanding this expression, noting that a is a constant, we have

$$I_A = a^2\int_{y_2}^{y_1} bdy + 2a\int_{y_2}^{y_1} bydy + \int_{y_2}^{y_1} by^2dy$$

In this equation, $\int_{y_2}^{y_1} bdy = A$ = area of section; $\int_{y_2}^{y_1} bydy = Q$ = statical moment of area about an axis through its center of gravity, which is equal to zero; and $\int_{y_2}^{y_1} by^2dy = I_x$ = moment of inertia of area about OX, the gravity axis of the section. Therefore, the above equation may be written

$$I_A = I_x + Aa^2 \tag{7}$$

Equation (7) is very useful when an area may be divided into smaller areas for which the properties are known. Also, eq. (7) shows that the moment of inertia of a section for an axis through its center of gravity is less than the value for any other axis, for moving the axis away from the center of gravity increases the moment of inertia, as indicated by the positive value for Aa^2.

6. Moments of Inertia for Inclined Axes (*Inclined Axes Theorem*).—In Fig. 12, let OX and OY be any pair of rectangular axes through the point O, and let OU and OV be another pair of rectangular axes through O but at an angle α from the first axis. Let dA represent any element of area, whose coordinates with respect to the OX, OY axes are x and y, and u and v with respect to the OU, OV axes.

The moment of inertia of the area about the OU axis, which will be denoted by I_u is

$$I_u = \int v^2 dA$$

and about the OV axis for which I_v denotes the moment of inertia, we have

$$I_v = \int u^2 dA$$

In these equations the limits of integration must cover the entire section.

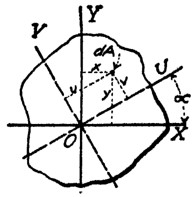


FIG. 12.

From Fig. 12 it can be seen that in terms of x and y, the values of u and v are

$$\left.\begin{aligned} u &= x\cos\alpha + y\sin\alpha \\ v &= y\cos\alpha - x\sin\alpha \end{aligned}\right\} \tag{8}$$

and

Substituting these values of u and v in the above equations, and expanding the terms, we have

$$I_u = \int \cos^2\alpha y^2 dA - 2\int \sin\alpha\cos\alpha\, xy dA + \int \sin^2\alpha x^2 dA \tag{9}$$

and

$$I_v = \int \cos^2\alpha\, x^2 dA + 2\int \sin\alpha\cos\alpha xy dA + \int \sin^2\alpha y^2 dA \tag{10}$$

In eqs. (9) and (10), $\int x^2 dA$ and $\int y^2 dA$ represent respectively, I_y and I_x, the moments of inertia of the section about the OY and OX axes. The term $\int xy dA$ is known as the *product of inertia* of the section. It is the sum of all the products obtained by multiplying each element of area by the product of its distances from the OX and OY axes. The product of inertia will be denoted by J_{xy}, the subscripts indicating the axes for which the product of inertia is taken. Note that J_{xy} is also a term whose dimensions are distance to the fourth power.

Equations (9) and (10) may then be written

$$I_u = I_x \cos^2\alpha - 2J_{xy}\sin\alpha\cos\alpha + I_y\sin^2\alpha \tag{11}$$

and

$$I_v = I_y\cos^2\alpha + 2J_{xy}\sin\alpha\cos\alpha + I_x\sin^2\alpha \tag{12}$$

By means of eqs. (11) and (12) it is possible to find the moments of inertia for axes OU and OV when the moments of inertia and product of inertia for the axes OX and OY are known.

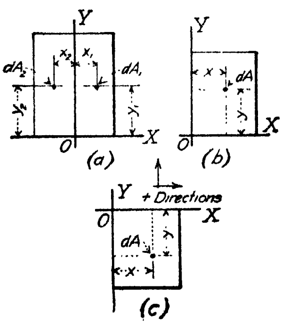


FIG. 13.

A useful relation between the moments of inertia for the two pairs of axes may be obtained by adding eqs. (11) and (12). Noting that $\cos^2\alpha + \sin^2\alpha = 1$, we have

$$I_u + I_v = I_x + I_y \tag{13}$$

The term J_{xy}, the product of inertia, which appears in eqs. (11) and (12), may have positive, negative or zero values, depending upon the location of the coordinate axes. In this respect it differs from the moment of inertia, which has only a positive value due to the fact that the distance to each area is squared.

Figure 13 shows the effect of the position of the coordinate axes on the sign of J_{xy}. Figure 13*a* shows a rectangle with the X axis along the base and the Y axis through the center of the figure. For every area dA_1 on the right of the Y axis there is a corresponding area dA_2 on the left. If positive directions with respect to an origin at O are taken as upward and to the right, it is evident that the product of inertia, $J_{xy} = \int xy dA = O$. In Fig. 13*b*, both x and y are positive, and J_{xy} = a positive quantity. For the conditions shown in Fig. 13*c*, the x-distances are positive while the y-distances are negative, and

J_{xy} = a negative quantity. In general, if one of a pair of axes is an axis of symmetry for a section, the product of inertia is zero for that pair of axes.

7. Products of Inertia for Parallel Axes.—Let it be required to find the relation between the products of inertia for the figure of Fig. 14 with respect to the pairs of axes OX, OY and OU, OV. For the conditions shown, the product of inertia with respect to the OU, OV axes is

$$J_{uv} = \int(b + x)(a + y)dA$$
$$= \int abdA + \int bydA + \int axdA + \int xydA$$

In this expression, $\int dA = A$ = area of section, $\int xydA = J_{xy}$ = product of inertia with respect to the OX, OY axes, and $\int ydA$ and $\int dxA$ are respectively, the statical moments of the area for the X and Y axes. If the OX, OY axes are assumed as passing through the center of gravity of the figure, the above statical moments are zero. The above equation then becomes

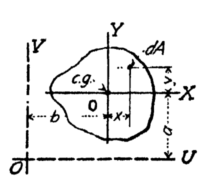


FIG. 14.

$$J_{uv} = J_{xy} + Aab \tag{14}$$

That is, the product of inertia for any pair of axes OU, OV, with respect to a pair of parallel axes through the center of gravity of the figure is equal to product of inertia for the gravity axes plus the area of the figure times the product of the coordinates of the center of gravity of the figure with respect to the OU, OV axes. Due attention must be paid to signs in calculating the several quantities.

8. Principal Axes and Principal Moments of Inertia.—From eq. (11) or (12), it can be seen that the moment of inertia of a section varies with the angle α. To determine the maximum value of the moment of inertia, as given by eq. (11), place $\frac{dI_u}{d\alpha}$ equal to zero and solve for the value of α. If α_0 is the value of this angle, we have

$$\tan 2\alpha_0 = \frac{2J_{xy}}{I_y - I_x} \tag{15}$$

There are two angles which answer the conditions imposed by this equation, one in the first and the other in the second quadrant, and furthermore, these angles differ in value by 90 deg. On substituting values of α_0 as given by eq. (15) in eq. (11), two values of I_u will be derived. By the methods of the calculus, it can be shown that one of these is the maximum value of I_u and the other is the minimum value. These moments of inertia are known as the *principal moments of inertia* for the section and the axes for which they occur are known as the *principal axes* of the section. A similar operation performed on eq. (12) will give results in which the values are the reverse of the above. Note that the maximum and minimum values of moment of inertia occur for axes which are 90 deg. apart.

The product of inertia of an area for a principal axis can readily be shown to be equal to zero. Thus for the axis OU of Fig. 12, we have

$$J_{uv} = \int uvdA$$

On substituting values of u and v as given by eq. (8), we have

$$J_{uv} = J_{xy} \cos 2\alpha + \tfrac{1}{2} \sin 2\alpha(I_x - I_y)$$

When OU is a principal axis, the angle α has the value given by eq. (15). Substituting values of this angle in the above equation it will be found that J_{uv} is equal to zero as stated above.

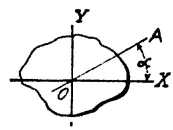


FIG. 15.

Let OX and OY of Fig. 15 be the principal axes of a section, and let I_x and I_y be the principal moments of inertia. Let OA represent any other axis at an angle α from OX. Remembering that the product of inertia for principal axes is zero, we may write from eq. (11)

$$I_A = I_x \cos^2 \alpha + I_y \sin^2 \alpha \tag{16}$$

Equation (16) is a general equation for moment of inertia about any axis in terms of the moments of inertia for the principal axes.

9. Radius of Gyration.—The radius of gyration of an area is the distance from a given axis to a point at which the entire area of the section must be applied in order that the product of the area times the square of this distance to the given axis may be equal to the moment of inertia of the section about the given axis. If r = radius of gyration of

the section as defined above, A = area of section, and I = moment of inertia about the given axis, we have

$$I = Ar^2$$

or

$$= \sqrt{\frac{I}{A}} \tag{17}$$

For a rectangle of width b and depth d, $I = \frac{bd^3}{12}$ and $A = bd$. From eq. (17)

$$= \sqrt{\frac{bd^3}{12bd}} = d\sqrt{\frac{1}{12}} = 0.289d$$

If r_A, r_x and r_y represent the radii of gyration for the corresponding axes of Fig. 15, eq. (16) may be written in the form

$$r_A{}^2 = r_x{}^2 \cos^2 \alpha + r_y{}^2 \sin^2 \alpha \tag{18}$$

By analytical geometry, it can be shown that eq. (18) represents an ellipse with semi-major and minor axes of r_y and r_x respectively, as shown in Fig. 16. The ellipse of Fig. 16 is known as the *inertia* ellipse for the area of Fig. 15.

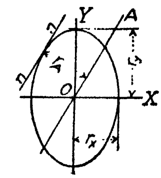


FIG. 16.

If n–n is a tangent to the ellipse parallel to any axis OA through the center of the ellipse, it can be shown that the perpendicular distance from the axis OA to the tangent n–n is equal to the radius of gyration of the section of Fig. 15. The construction shown in Fig. 16 offers a convenient method for determination of moment of inertia.

10. Section Modulus.—In the general formula for resisting moment of beams appears the term $S = \frac{I}{c}$ = moment of inertia of section about the neutral axis divided by the distance from the neutral axis to an extreme fiber of the section. This quantity is known as the *section modulus* for the beam cross-section. It is a quantity of dimensions distance to the third power.

For a rectangle of width b and depth d, we have

$$S = \frac{\frac{bd^3}{12}}{\frac{d}{2}} = \frac{bd^2}{6}$$

Values of section modulus for rolled shapes are given in rolling mill handbooks.

APPENDIX C

DEFLECTION OF BEAMS

Elastic Curve Method

1. General Equation of Elastic Curve for Bending.—Let $OABL$ of Fig. 1a show (greatly exaggerated) the bent position of the neutral axis of a beam which, before the application of the loads, was represented by the straight line OL. This deformation of the beam is due to positive bending moment.

If points A and B of Fig. 1a represent two adjacent points on the neutral axis of the bent beam, the normals at these sections will meet in a point F, which is the center of curvature for these points. To locate points on the beam with respect to a given point, let O, a point at one end of the beam, be taken as an origin. Any convenient point will do as well. Let x and y respectively be the horizontal and vertical distances from the origin O to the center of any element AB, and assume that x and y are positive when measured to the right and downward respectively, as shown by the arrows in Fig. 1. It is evident that the angle between the tangents to the curve at A and B is equal to the angle between the radii at these points, as shown in Fig. 1a.

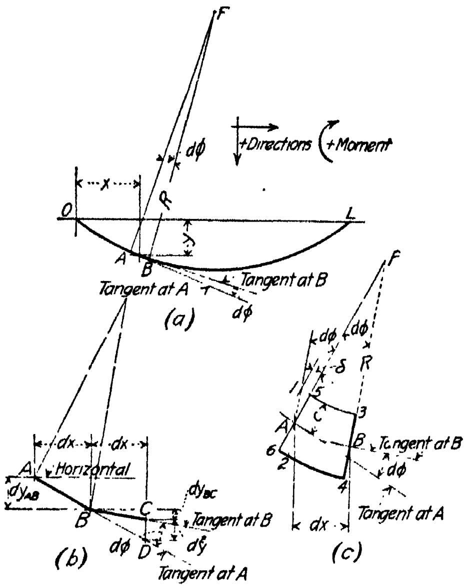


Fig. 1.

In Fig. 1b let AB and BC show two adjacent elements of the neutral axis, and assume that their projections on a horizontal axis are each equal to dx. Let the vertical projections of these elements be represented by dy_{AB} and dy_{BC}. From Fig. 1b, it is evident that the difference between dy_{AB} and dy_{BC} is a measure of the change in deflection across the two elements AB and BC. Let d^2y denote the change in vertical projection of the two elements. If the deflection is small compared to the dimensions of the beam section, which is the usual case, we may consider $AB = BC = dx$ and $\angle BCD = 90$ deg. Hence, since $d\phi$ is a very small angle,

$$d^2y = -dx \, . \, d\phi$$

The minus sign is used in this equation because, as shown in Fig. 1b the angle between element BC and a horizontal axis is less than the corresponding angle for element AB. To conform to the direction notation given above, increasing values of x and y result in increasing angles between successive elements. Since a decreasing angle exists for the conditions shown in Fig. 1b, a minus sign must be used in the above equation.

From Figs. 1a or b, $d\phi = dx/R$, where R = radius of curve for any small element of the neutral axis.

Hence

$$d^2y = -dx^2/R$$

or

$$\frac{d^2y}{dx^2} = -\frac{1}{R} \tag{1}$$

The expression d^2y/dx^2 of eq. (1) is known as the second differential coefficient of y, the deflection of the beam, with respect to x, the distance from the origin to the point at which the deflection is desired. It is a measure of the rate of change of the slope of the neutral axis.

In Fig. 1c, an element of the beam is enlarged to show the effect of bending deformation. The deformed element is shown by 3–4–5–6. In the undeformed element, the faces 3–B–4 and 5–A–6 are parallel lines. If a line 1–A–2 be drawn parallel to 3–B–4, the deformation of the element is represented by 1–5. If the deformation is small, $1 - 3 = AB = dx$, and we may write $\delta = \frac{f}{E}\,dx$, where f = fiber stress due to a moment M. From eq. (6), p. 23, $f = \frac{Mc}{I}$, and hence

$$\delta = \frac{Mc}{EI}\,dx$$

When the deformations are small, FAB and A–1–5 may be considered as similar triangles, and we have $\frac{\delta}{c} = \frac{dx}{R}$. Solving for δ and equating the resulting expression to the value of δ given above, we have finally $\frac{1}{R} = M/EI$. Substituting this value of $\frac{1}{R}$ in eq. (1), we derive

$$\frac{d^2y}{dx^2} = -\frac{M}{EI} \tag{2}$$

Equation (2) is the general expression for the differential equation of the elastic curve of a beam subjected to bending due to a clockwise or positive moment. To determine the equation of the elastic curve by means of eq. (2), the moment M must be expressed as a function of x, and the resulting expression integrated twice.

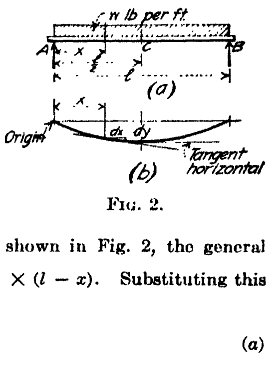


FIG. 2.

2. Application of General Equation of Elastic Curve to Solution of Problems in Deflection of Beams.—In the articles which follow, the equation of the elastic curve will be derived and the maximum deflection will be determined for a few typical cases.

Simple Beam with Uniform Load.—For the conditions shown in Fig. 2, the general expression for moment at any point on the beam is $M = \frac{w}{2} \times (l - x)$. Substituting this value of M in eq. (2) we have

$$\frac{d^2y}{dx^2} = -\frac{wx}{2EI}(l - x) \tag{a}$$

Integrating,

$$\frac{dy}{dx} = -\frac{w}{2EI}\left(\frac{lx^2}{2} - \frac{x^3}{3}\right) + C_1 \tag{b}$$

In this equation, dy/dx is the *slope* of the elastic curve at any point, and C_1 is a constant of integration which depends for its value upon the conditions of the problem. To determine C_1, note that the load on the beam is symmetrical with respect to the beam center. It is therefore evident that the elastic curve will be symmetrical about the beam center, and that the tangent to the elastic curve at the beam center is horizontal. Since slope of a horizontal line is zero, we have as a condition for the determination of C_1 in eq. (b), that $dy/dx = 0$ when $x = \frac{l}{2}$. Substituting these values in eq. (b) and solving for C_1, we have $C_1 = \frac{wl^3}{24EI}$. Eq. (b) then becomes

$$\frac{dy}{dx} = +\frac{w}{24EI}(4x^3 - 6lx^2 + l^3) \tag{3}$$

Integrating eq. (3) we have

$$y = +\frac{w}{24EI}(x^4 - 2lx^3 + l^3x) + C_2$$

To determine C_2 the constant of integration, note from Fig. 2b that $y = 0$ when $x = 0$ Hence from the above equation $C_2 = 0$ and we have finally

$$y = \frac{wx}{24EI}(x^3 - 2lx^2 + l^3) \tag{4}$$

which is the general equation of the elastic curve for a simple beam carrying a uniform load.

The maximum deflection of the beam of Fig. 2 evidently occurs at the point where the tangent to the elastic curve is horizontal. As stated above, the tangent is horizontal when $x = \frac{l}{2}$. Substituting this value of x in eq.(4) we have

$$y_{max} = \frac{5}{384}\frac{wl^4}{EI} \tag{4a}$$

Illustrative Problem.—A simple beam 16 ft. long supports a uniform load of 600 lb. per ft. Determine the maximum deflection of this beam in inches. Assume that the moment of inertia of the beam is 100 in^4., and that the material is steel for which $E = 30{,}000{,}000$ lb. per sq in.

The maximum deflection is given by eq. (4a). In substituting in eq. (4a) attention must be paid to the units in which the several terms are expressed. Since the deflection in inches is desired, all values must be expressed in inch units. Thus $w = {}^{600}\!/_{12}$ lb. per in., and $l = 16 \times 12$ in. Values of E and I are given directly in inch units. Substituting these values in eq. (4a) we have

$$y_{max} = \frac{(5)(600)(16)^4(12)^3}{384EI} = \frac{884{,}736{,}000}{EI} \text{ in.}$$

Compare this result with the problem of p. 52. Substituting values of E and I,

$$y_{max} = \frac{884{,}736{,}000}{(30{,}000{,}000)(100)} = 0.295 \text{ in.}$$

Illustrative Problem.—Determine the deflection of the above beam in feet, using foot units.

Here $w = 600$, $l = 16$, $E = (30{,}000{,}000)\ (144)$ lb. per sq. ft. and $I = \frac{100}{(12)^4}$ ft.^{4}

Hence

$$y_{max} = \frac{(5)(600)(16)^4(12)^4}{(384)(144)(30{,}000{,}000)(100)} = 0.0246 \text{ ft.}$$

Illustrative Problem.—Determine the deflection at a point 5 ft. from the left end of the beam of the above problems. Use inch units.

Here $w = {}^{600}\!/_{12} = 50$ lb. per in.; $x = 5$ ft. $= 60$ in.; $l = 16$ ft. $= 192$ in. Values of E and I are as given above. Substituting in eq. (4) we have

$$y = \frac{(50)(60)}{(24)(30{,}000{,}000)(100)}[(60)^3 - 2(192)(60)^2 + (192)^3]$$

$$y = 0.246 \text{ in.}$$

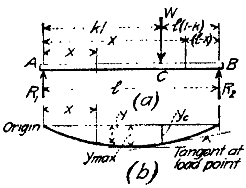


FIG. 3.

Simple Beam with a Single Concentrated Load.—For the conditions shown in Fig. 3, the law of variation of moments on section AC differs from that on section CB. Hence two substitutions must be made in eq. (2). After integrating each equation twice, four constants of integration will appear which must be determined subject to conditions shown in Fig. 3. The detail work is as follows:

For the portion of the beam from A to C, $M_x = R_1x = W(1 - k)x$. Then from eq. (2)

$$\frac{d^2y}{dx^2} = -\frac{W}{EI}(1 - k)x$$

Integrating twice, we have

$$\frac{dy}{dx} = -\frac{W}{EI}(1 - k)\frac{x^2}{2} + C_1 \tag{a}$$

and

$$y = -\frac{W}{EI}(1 - k)\frac{x^3}{6} + C_1x + C_2 \tag{b}$$

For the portion of the beam from C to B, $M_x = R_2(l - x) = Wk(l - x)$. Then from eq. (2)

$$\frac{d^2y}{dx^2} = -\frac{W}{EI}k(l - x)$$

Integrating twice, we have

$$\frac{dy}{dx} = -\frac{Wk}{EI}\left(lx - \frac{x^2}{2}\right) + C_3 \tag{c}$$

and

$$y = -\frac{Wk}{EI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + C_3x + C_4 \tag{d}$$

Equations (*b*) and (*d*) are general expressions for the equations of the elastic curves for the portions of the beam on either side of the load W. However, these equations cannot be used until the values of the constants of integration C_1, C_2, C_3, and C_4 are known. Since there are four unknown terms, an equal number of independent equations must be set up before the unknowns can be determined. The required independent equations may be derived from the necessary relations which must exist between the elastic curves on the two sides of the load W in order that the two elastic curves may be joined to form a single continuous curve.

The four independent equations from which the values of the constants of integration may be determined are derived from conditions shown in Fig. 3*b*. From this figure it is evident that y in eq. (*b*) is zero only when $x = 0$. Also, y in eq. (*d*) is zero only when $x = l$. At point C, where the elastic curves to the right and left of the applied load are joined, it is evident that values of the slope given by eqs. (*a*) and (*c*) must be equal, and also that values of y given by eqs. (*b*) and (*d*) must be equal. Hence we have the four conditions that

(1) $y = 0$ when $x = 0$ in eq. (*b*).
(2) $y = 0$ when $x = l$ in eq. (*d*).
(3) $\frac{dy}{dx}$ from eq. (*a*) $= \frac{dy}{dx}$ from eq. (*c*) when $x = kl$ in these equations.
(4) y from eq. (*b*) $= y$ from eq. (*d*) when $x = kl$ in these equations.

Performing the operations indicated and reducing the resulting expressions to their simplest form, we derive the following condition equations:

(1) $0 = C_2$

(2) $0 = -\frac{Wkl^3}{3EI} + C_3l + C_4$

(3) $C_1 - C_3 = -\frac{Wk^2l^2}{2EI}$

(4) $(C_1 - C_3)kl = -\frac{Wk^3l^3}{3EI} + C_4$

Solving these four equations simultaneously, the values of the constants of integration are found to be

$$C_1 = \frac{Wl^2}{6EI}(2k + k^3 - 3k^2) \qquad C_2 = 0$$

$$C_3 = \frac{Wkl^2}{6EI}(2 + k^2) \qquad C_4 = -\frac{Wk^3l^3}{6EI}$$

Substituting these constants in eqs. (*b*) and (*d*), the equation of the elastic line is found to be as follows:

From A to C

$$y = \frac{Wx}{6EI}(1 - k)[(2 - k)kl^2 - x^2] \tag{5}$$

From C to B

$$y = \frac{W}{6EI}k(l - x)[x(2l - x) - k^2l^2] \tag{6}$$

The general equations for slope of the tangent to the elastic curve at any point, as given by eqs. (*a*) and (*c*) with values of C_1 and C_3 substituted, are as follows

From A to C

$$\frac{dy}{dx} = \frac{W}{6EI}(1 - k)[(2 - k)kl^2 - 3x^2] \tag{5a}$$

From C to B

$$\frac{dy}{dx} = \frac{Wk}{6EI}[3(l - x)^2 - (1 - k^2)l^2] \tag{6a}$$

On substituting values of x in the proper equation, the deflection at any point may be determined. At point C, where the load is located, the deflection may be determined from eqs. (5) or (6) by substituting $x = kl$, and we have

$$y_c = \frac{Wl \cdot k^2}{3EI}(1 - k)^2 \tag{7}$$

The maximum deflection for the beam under consideration can be seen from Fig. 4 to be at the point where the curve becomes horizontal—that is, where $dy/dx = 0$. To locate this point, note from Fig. 3 that the tangent is horizontal on the portion of the curve between points A and C. From eq. (5a) we have

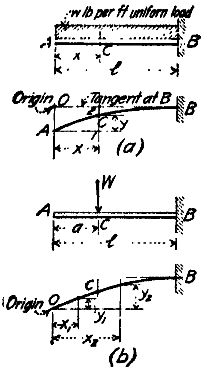


FIG. 4.

$$\frac{dy}{dx} = 0 = \frac{W}{6EI}(1 - k)[(2 - k)kl^2 - 3x^2]$$

Solving this expression for x, we find that the deflection is a maximum when

$$x = l\left[\frac{k}{3}(2 - k)\right]^{1/2} \tag{8}$$

Substituting this value of x in eq. (5), the maximum deflection is found to be

$$y_{max} = \frac{Wl^3}{3EI}(1 - k)\left[\frac{k}{3}(2 - k)\right]^{3/2} \tag{9}$$

Equations (8) and (9) give the position and the amount of the maximum deflection when the load W is located at a distance kl from the left end of the beam.

From eq. (9) it can be seen that the maximum deflection depends upon the position of the load W. Evidently there is some position of the load for which the deflection will be greater than for any other point in the beam. To determine the position of the load for greatest deflection, place $\frac{dy_{max}}{dk}$ from eq. (9) equal to zero and solve for k, from which it will be found that $k = \frac{1}{2}$, or the load should be placed at the beam center. Substituting $k = \frac{1}{2}$ in eq. (9) and denoting the resulting deflection by Δ, we have

$$\Delta = \frac{1}{48}\frac{Wl^3}{EI} \tag{10}$$

Equation (10) gives the deflection at the beam center for a load W placed at that point. This is the greatest deflection for the beam under consideration.

Illustrative Problem.—A 2- × 1-in. piece of wood laid flatwise spans a 24-in. opening. The beam carries a 60-lb. load at a distance of 18 in. from the left end of the beam. Determine the deflection under the load and the maximum deflection of the beam. Assume $E = 1{,}500{,}000$ lb. per sq. in.

The deflection under the load is given by eq. (7) with $W = 60$ lb., $l = 24$ in., $k = {}^{18}\!/_{24} = 0.75$, $E = 1{,}500{,}000$ lb per sq. in., and $I = \frac{1}{12}bd^3 = (\frac{1}{12})(2)(1)^3 = \frac{1}{6}$ in^4.

Thus $$y = \frac{(60)(24)^3(0.75)^2(1 - 0.75)^2}{(3)(1{,}500{,}000)(\frac{1}{6})} = 0.0389 \text{ in.}$$

The maximum deflection for the given loading is found from eq. (9) with values as above, from which

$$y_{max} = \frac{(60)(24)^3(1 - 0.75)\left[\frac{0.75}{3}(2 - 0.75)\right]^{3/2}}{(3)(1{,}500{,}000)(\frac{1}{6})} = 0.0484 \text{ in.}$$

The point at which this deflection occurs is found from eq. (8) to be

$$x = \left[\frac{0.75}{3}(2 - 0.75)\right]^{1/2}24 = (0.558)(24) = 13.42 \text{ in.}$$

from the left end of the beam.

Compare these results with those given on p. 49.

Illustrative Problem.—Determine the angle between the horizontal and the tangent to the elastic curve at the left end of the beam and at the load point for the beam given in the preceding problem.

The slope of the tangent at the left end of the beam is given by eq. (5a) with $x = 0$ from which

$$\text{Slope} = \frac{W}{6EI}(1 - k)(2 - k)kl^2$$

For the values given in the above problem,

$$\text{Slope} = \frac{60(1 - 0.75)(2 - 0.75)(0.75)(24)^2}{(6)(1{,}500{,}000)(\frac{1}{6})} = 0.0054 \text{ radians}$$

In circular measure, a radian is 57° 18′. Hence the required slope is (0.0054)(57.3) = 0.309° = 18.55′. This angle is measured in a clockwise direction about point 0 of Fig. 1 with OL as a horizontal axis.

The slope of the tangent at the load point is given by eq. (5a) and eq. (6a) with $x = kl = 18$ in. With values of the several terms as above, we have from eq. (6a).

$$\text{Slope} = \frac{60}{(6)(1{,}500{,}000)(\tfrac{1}{6})}[3(24-18)^2 - (1-0.75^2)(24)^2](0.75) = 0.00432 \text{ radians} = -0.242 \text{ deg.} = -14.52 \text{ min.}$$

The slope of this tangent is in the direction shown for similar conditions in Fig. 3.

Cantilever Beams.—Assuming the origin of coordinates to be located at point O of the cantilever beam of Fig. 4a, the moment at any point distance x from the origin is $M_x = -\frac{wx^2}{2}$ Substituting this value of M in eq. (2), we have

$$\frac{d^2y}{dx^2} = +\frac{wx^2}{2EI}$$

Integrating

$$\frac{dy}{dx} = +\frac{wx^3}{6EI} + C_1$$

To determine C_1, note from Fig. 4a that the tangent is horizontal when $x = l$. Hence substituting $\frac{dy}{dx} = 0$ when $x = l$ in the above equation, we find $C_1 = -\frac{wl^3}{6EI}$ and

$$\frac{dy}{dx} = +\frac{w}{6EI}(x^3 - l^3) \tag{11}$$

Integrating again

$$y = +\frac{w}{6EI}\left(\frac{x^4}{4} - l^3x\right) + C_2$$

To determine C_2, note from Fig. 4a that $y = 0$ when $x = l$. Therefore $\frac{wl^4}{24} - \frac{wl^4}{6EI} + C_2 = 0$, from which $C_2 = +\frac{1}{8}wl^4$, and we have

$$y = +\frac{w}{24EI}(x^4 - 4l^3x + 3l^4) \tag{12}$$

which is the general equation of the elastic curve for the beam of Fig. 4a. The positive value indicates that y is measured upward from 0. From Fig. 4a it can be seen that the maximum value of y occurs when $x = 0$. Placing $x = 0$ in eq. (12), we have

$$y_{max} = +\frac{wl^4}{8EI} \tag{13}$$

Figure 4b shows a cantilever beam with a single concentrated load at a distance a from the free end. Since the law of variation of moments differs for the two portions of the beam shown by AC and CB of Fig. 4b two substitutions must be made in eq. (2). For an origin at the deflected position of the free end of the beam, the moment at any point in the beam is

From A to C. $M = 0$
From C to B. $M = -W(x - a)$

Substituting in eq. (2), the integrations are as follows:

From A to C

$$EI\frac{d^2y}{dx^2} = 0$$

$$EI\frac{dy}{dx} = C_1 \tag{a}$$

$$EI\,y = C_1x + C_2 \tag{b}$$

From C to B

$$EI\frac{d^2y}{dx^2} = +W(x - a)$$

$$EI\frac{dy}{dx} = W\left(\frac{x^2}{2} - ax\right) + C_3 \tag{c}$$

$$EI\,y = W\left(\frac{x^3}{6} - \frac{ax^2}{2}\right) + C_3x + C_4 \tag{d}$$

The constants of integration may be determined from the following conditions: $y = 0$ when $x = 0$ in eq. (b); $\frac{dy}{dx}$ from eq. (a) $= \frac{dy}{dx}$ from eq. (c) when $x = a$; y from eq. (b) $= y$ from eq. (d) when $x = a$; and $\frac{dy}{dx} = 0$ in eq. (c) when $x = l$. These conditions are evident

from a study of Fig. 4b. Performing the operations indicated, the values of the constants of integration are found to be

$$C_1 = -\frac{W}{2}(l-a)^2 \qquad C_2 = 0$$

$$C_3 = -\frac{Wl}{2}(l-2a) \qquad C_4 = -\frac{Wa^3}{6}$$

Substituting these values in eqs. (b) and (d), the equation of the elastic curve is found to be
From A to C,

$$y_1 = -\frac{Wx}{2EI}(l-a)^2 \tag{14}$$

From C to B

$$y_2 = \frac{W}{6EI}[x^3 - 3ax^2 - 3lx(l-2a) - a^3] \tag{15}$$

The slope of the tangent to the elastic curve, as given by eqs. (a) and (c), is
From A to C.

$$\frac{dy}{dx} = -\frac{W}{2EI}(l-a)^2 \tag{16}$$

From C to B,

$$\frac{dy}{dx} = \frac{W}{2EI}[x^2 - 2ax - l(l-2a)] \tag{17}$$

From eq. (14) it can be seen that the portion OC of the elastic curve is a straight line.

Illustrative Problem.—A 4- × 8-in. wooden member, placed with the 8-in. side vertical, forms a cantilever beam 6 ft. long. At a point 2 ft. from the free end, a 1,000-lb. load is placed. Determine the deflection in inches at the free end of the beam, assuming $E = 1{,}500{,}000$ in.-lb.

The formula for deflection at the free end is given by eq. (15) with $x = l$, from which

$$-\frac{W}{6EI}(l-a)^2(2l+a)$$

The minus sign in this equation indicates that the deflection is upward with respect to the free end of the beam. The value of I for the given beam is $I = \frac{bd3}{12} = \frac{(4)(8)^3}{12} = 170\frac{2}{3}$ in⁴.

Substituting values as given above, using distances in feet and multiplying by 1,728 to reduce to inches, we have

$$y_{max} = -\frac{(1{,}000)(1{,}728)}{(6)(1{,}500{,}000)(170\frac{2}{3})}(6-2)^2(12+2) = -0.252 \text{ in.}$$

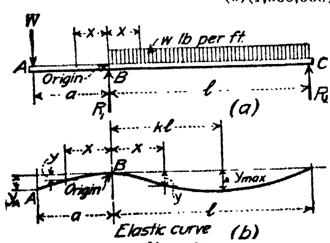


FIG. 5.

Beam with an Overhanging End.—Figure 5a shows a beam supported at points B and C and with an overhanging end AB. Between supports the beam carries a uniform load of w lb. per ft. and at the free end of the overhanging arm the beam supports a single concentrated load of W lb. The complete equation of the elastic curve will be determined for the given conditions.

The reactions and moments are as follows:

$$R_1 = \frac{wl}{2} + W\left(\frac{l+a}{l}\right) \qquad R_2 = \frac{wl}{2} - \frac{Wa}{l}$$

$$M_{AB} = -W(a-x) \qquad M_{BC} = \left(\frac{wx}{2} - \frac{Wa}{l}\right)(l-x)$$

where M_{AB} and M_{BC} denote respectively, the moment at any point on AB or BC. The value of x is positive when measured as shown in Fig. 5a.

Substituting in eq. (2) we have

From A to B

$$EI\frac{d^2y}{dx^2} = +W(a-x)$$

$$EI \quad = +W\left(ax - \frac{x^2}{2}\right) + C_1 \tag{a}$$

$$EI\,y = +W\left(\frac{ax^2}{2} - \frac{x^3}{6}\right) + C_1x + C_2 \tag{b}$$

From B to C

$$EI\frac{d^2y}{dx^2} = +\frac{Wa}{l}(l - x) - \frac{w}{2}(lx - x^2)$$

$$EI\frac{dy}{dx} = +\frac{Wa}{l}\left(lx - \frac{x^2}{2}\right) - \frac{w}{2}\left(\frac{lx^2}{2} - \frac{x^3}{3}\right) + C_3 \qquad \text{(c)}$$

$$EI\,y = +\frac{Wa}{l}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) - \frac{w}{2}\left(\frac{lx^3}{6} - \frac{x^4}{12}\right) + C_3x + C_4 \qquad \text{(d)}$$

The constants of integration are to be determined subject to the following conditions (see Fig. 5b):

$y = 0$ in (b) when $x = 0$; $y = 0$ in (d) when $x = 0$;

$y = 0$ in (d) when $x = l$; and $\frac{dy}{dx}$ in (c) $= -\frac{dy}{dx}$ in (a) when $x = 0$.

The minus sign in this last condition is necessary because of the change in positive directions at point B. Subject to the above conditions, the constants of integration are found to have the following values:

$$C_1 = \frac{Wal}{3} - \frac{wl^3}{24} \qquad C_3 = -\frac{Wal}{3} + \frac{wl^3}{24}$$

$$C_2 = 0 \qquad C_4 = 0$$

Substituting these values in the above equations, the general equations for the elastic curve and slope of the tangent at any point are found to be:

From A to B

$$y = \frac{1}{EI}\left[\frac{Wx}{6}(2al + 3ax - x^2) - \frac{wl^3x}{24}\right] \qquad (18)$$

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{W}{6}(2al + 6ax - 3x^2) - \frac{wl^3}{24}\right] \qquad (19)$$

From B to C

$$y = \frac{1}{EI}\left[-\frac{Wax}{6l}(2l - x) + \frac{wx}{24}(l^2 + lx - x^2)\right](l - x) \qquad (20)$$

$$\frac{dy}{dx} = \frac{1}{EI}\left[-\frac{Wa}{6l}(2l^2 - 6lx + 3x^2) + \frac{w}{24}(l^3 - 6lx^2 + 4x^3)\right] \qquad (21)$$

Figure 5b shows the form of the elastic curve plotted from these equations.

The maximum deflection in the cantilever arm occurs at the free end, point A of Fig. 5. Placing $x = a$ in eq. (18), we have

$$y_A = \frac{a}{EI}\left[\frac{Wa}{3}(l + a) - \frac{wl^3}{24}\right] \qquad (22)$$

The maximum deflection in the span BC occurs where the tangent to the elastic curve is horizontal. This point may be located by placing $\frac{dy}{dx}$ from eq. (21) equal to zero and solving for x. Let kl be this value of x, where k is the fractional part of the span between the left support and the point of maximum deflection. Performing the operation indicated, the value of k is given by the cubic equation

$$k^3 - \left(\frac{3}{2} + \frac{3W}{w}\frac{a}{l^2}\right)k^2 + \frac{6Wa}{wl^2}k + \left(\frac{1}{4} - \frac{2Wa}{wl^2}\right) = 0 \qquad (23)$$

To determine the maximum deflection in any given beam, solve eq. (23) for k by the methods given in Art. **70**b, p. 537, and calculate the corresponding value of x. Substitute this value of x in eq. (20). This procedure is advisable in this case because a general expression for maximum deflection is too complicated and cumbersome.

Illustrative Problem.—A 10-in 25.4-lb. steel I-beam is used to form a beam of the type shown in Fig. 5. Calculate the deflection at the free end of the cantilever arm and the maximum deflection in the span BC. Let $W = 6{,}000$ lb., $w = 1{,}200$ lb. per ft., $a = 5$ ft., and $l = 15$ ft. The moment of inertia of the given I-beam section is 122.1 in⁴. and $E = 30{,}000{,}000$ lb. per sq. in.

The deflection at the free end of the cantilever arm is given by eq. (22). Substituting values given above in this equation, we have, using inch units

$$y_A = \frac{60}{(30{,}000{,}000)(122.1)}\left[\left(\frac{1}{3}\right)(6{,}000)(60)(180 + 60) - \frac{(1{,}200)(180)^3}{(12)(24)}\right]$$

from which

$$y_A = 0.0738 \text{ in.}$$

The point in span BC at which the maximum deflection occurs, is found from eq. (23). Substituting the given values in this equation we derive the following cubic equation for k,

$$k^3 - 1.833k^2 + 0.666k + 0.0278 = 0$$

Solving this equation by the method given in Art. **70b**, p. 537, we find $k = 0.565$. Hence the distance from point B, Fig. 5, to the point of maximum deflection is

$$x = kl = (0.565)(15) = 8.47 \text{ ft.} = 102 \text{ in.}$$

Substituting $x = 102$ in., and other values as given above, in eq. (20), the maximum deflection is found to be

$$y_{max} = \frac{1}{(30,000,000)(122.1)}\left\{-\frac{(6,000)(60)(102)}{(6)(180)}(360 - 102) + \frac{(1,200)(102)}{(12)(24)}[(180)^2 + (180)(102) - (102)^2]\right\}(180 - 102)$$

from which

$$y_{max} = 0.189 \text{ in.}$$

Unit Load Method

3. Derivation of General Formula.—The deflection of a beam due to any given loading may be determined by placing the external work done by this loading during the deflection of the beam equal to the internal work done on the fibers of the beam, for it is evident that a body can be at rest and in a state of elastic equilibrium only when the work of applied loads is balanced by work done within the body. In Art. **37**, p. 9, it has been shown that the internal work, or elastic resilience of a body, is given by the expression

$$\text{Elastic resilience} = K = \frac{1}{2}\frac{f^2}{E}(\text{Volume of body})$$

where f = fiber stress due to applied loading and E = modulus of elasticity of the material.

Let the simple beam of Fig. 6a be acted upon by a 1 lb. or unit load placed at point C, a distance a from the left end of the beam. At any cross-section of the beam at a distance x from any convenient origin, let the moment due to the unit load be m and let the stress on any fiber at a distance c from the neutral axis, Fig. 6b, be denoted by f. If the length of any fiber parallel to the beam axis is dx, and the area of that element is dA, as shown in Fig. 6b, the volume of that element is $dA(dx)$, and we have

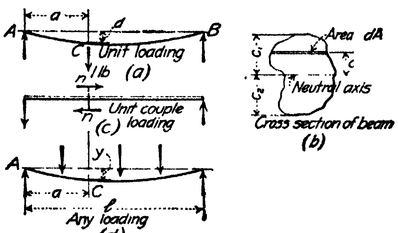


Fig. 6.

$$K = \frac{1}{2}\frac{f^2}{E}dA\,dx$$

From eq. 6, p. 23, $f = \frac{mc}{I}$ and for the entire cross-section of Fig. 6b.

$$K = \frac{1}{2E}\int_{c_2}^{c_1}\frac{m^2c^2}{I^2}dA\,dx$$

But $\int_{c_2}^{c_1} c^2 dA = I$ = moment of inertia of the section. Therefore

$$K = \frac{1}{2E}\frac{m^2}{I}dx$$

For all such sections over the entire beam from A to B,

$$K = \text{Total average internal work} = \frac{1}{2}\int_B^A \frac{m^2}{EI}dx \tag{1}$$

Let the deflection of the beam at point C where the unit load is applied be denoted by d and assume that the load is gradually applied to the beam so that the deflection varies from zero to its maximum value. The average external work done by the unit load is one-half the deflection times the load causing that deflection, or

$$\text{Average external work} = \frac{d}{2} \tag{2}$$

From eqs. (1) and (2) we derive

$$\int_B^A \frac{m^2}{EI}dx \tag{3}$$

Equation (3) gives the deflection of point C, Fig. 6a, due to a 1-lb. load placed at that point. The deflection will be in the direction of the 1-lb. load.

It is sometimes desired to determine the angular rotation of any plane of a beam—as, for example, the plane n-n of Fig. 6c. This may be done by applying a unit couple at the plane in question and calculating the external work due to the rotation. On equating this expression to the internal work, as given by eq. (1), the angular rotation may be determined.

It can be shown that the work done by a couple is equal to the moment of the couple times the angular rotation. Let α_1 denote the angular rotation of plane n-n of Fig. 6a. The average work done by a unit couple during a rotation α_1 is then

$$\text{Average external work} = \tfrac{1}{2}\,\alpha_1 \tag{4}$$

Hence from eqs. (1) and (4)

$$\int_B^A \frac{m^2}{EI}\,dx \tag{5}$$

Equation (5) gives the angular rotation in *radians* of any plane due to a unit couple applied to that plane.

The deflection of any point C, Fig. 6d, or the angular rotation of a plane at that point due to any set of applied loads, such as those shown in Fig. 6d, may be determined by proportion from the corresponding values given by eqs. (3) or (5). In Art. 67, p. 82, it is shown that the deflection is directly proportional to the fiber stress. Since fiber stresses are proportional to moment, it is evident that the deflection, or rotation, due to any set of applied loads is to the corresponding value due to unit loading as the moment due to the applied loads is to the moment due to the unit loading. If y = deflection due to applied loads, and M = moment due to applied loads, we have the proportion

$$y:d::M:m$$

or

$$y = \frac{M}{m}\,d$$

Substituting the value of d from eq. (3) we have

$$y = \int_B^A \frac{Mm}{EI}\,dx \tag{6}$$

If α = angular rotation due to applied loads, a similar proportion gives

$$\alpha = \int_B^A \frac{Mm}{EI}\,dx \tag{7}$$

In these equations, y = deflection of any desired point; α = angular rotation in radians of a plane at any desired point; M = moment due to applied loads; E = modulus of elasticity of material composing the beam; I = moment of inertia of beam section; and m = a quantity of linear dimensions which is equal to the moment at any section due to a unit load, or unit couple, applied at the point whose deflection, or angular rotation, is desired and in the direction of the desired deflection, or rotation.

In solving problems in deflection and angular rotation by the method given above, it is not necessary that the direction of the deflection or rotation be known beforehand. Proper attention paid to the algebraic sign of the product Mm will show whether the correct direction has been assumed for the unit loading. If M and m are alike in character (both positive or negative moments) it is evident from the above discussion that they cause deflections or rotations in the same direction, while if they are unlike in character they cause deflections or rotations in opposite directions. Denoting positive moments by plus and negative moments by minus, the product of like moments carries a plus sign and the product of unlike moments carries a minus sign. Therefore, assume any convenient direction for the unit load or couple and pay careful attention to the sign of the product Mm. If the final result is positive, the deflection or rotation is in the direction assumed for the unit loading. If the final result has a negative sign, the deflection or rotation is in a direction opposite to that assumed for the unit loading.

4. Application of Unit Load Method to Problems in Deflection and Angle of Rotation— *Beam with Uniform Load* (*Moment of Inertia Constant*).—To find the general equation for the vertical deflection of point C at a distance x from the left end of the beam of Fig. 7a

due to a uniform load, apply a 1-lb. load at point C. Evidently the deflection is downward so the 1-lb. load is to be applied as shown in Fig. 7c.

The moment diagram for the uniform loading is shown in Fig. b and the moment diagram for the 1-lb. load is shown in Fig. c. Moment equations, expressed in terms of z, the distance from any point to the left end of the beam, are given on the diagrams. From these diagrams it can be seen that values of M for the entire beam are given by a single equation while values of m are given by two equations, for the law of variation of m changes at the point of application of the unit load. Therefore in substituting in eq. (6), the general expression must be made up for the portions of the beam where the law of variation of moments changes.

Substituting in eq. (6), we have

$$y = \int_0^x \frac{1}{EI}\left[\frac{wz}{2}(l-z)\right]\left[\frac{(l-x)}{l}z\right]dz + \int_x^l \frac{1}{EI}\left[\frac{wz}{2}(l-z)\right]\left[\frac{(l-z)}{l}x\right]dz$$

The first integral is for the portion of the beam from A to C and the second integral is for the portion from C to B. Performing the operations indicated, noting that x is a constant and that z is variable, we have

$$y = \frac{wx}{24EI}(l^3 - 2lx^2 + x^3) \tag{8}$$

Note that eq. (8) is exactly the same as eq. (4), p. 584. Hence eq. (8) gives the equation of the elastic curve for the given beam.

The angular rotation of a vertical plane through point C will now be determined, using eq. (7). Since the tangent to the elastic curve at any point is perpendicular to any normal section of a beam, substitution in eq. (7) will give the slope of the tangent to the elastic curve at any point. Applying a unit couple at point C, assuming a positive or clockwise rotation, the resulting moment diagram is shown in Fig. 7d. Substituting in eq. (7), we have

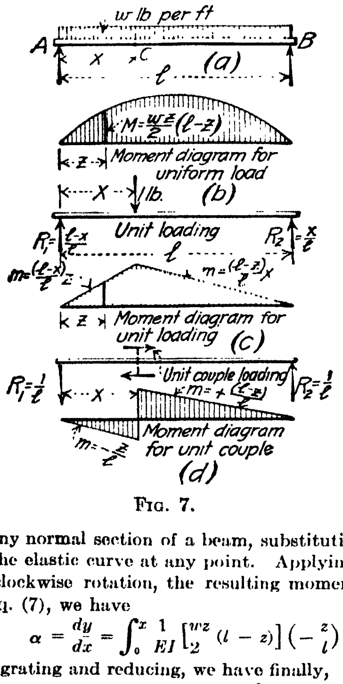


FIG. 7.

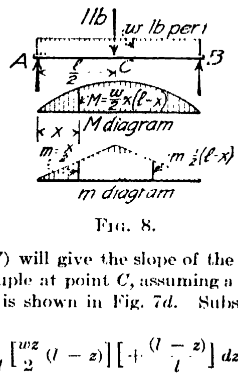


FIG. 8.

$$\alpha = \frac{dy}{dx} = \int_0^x \frac{1}{EI}\left[\frac{wz}{2}(l-z)\right]\left(-\frac{z}{l}\right)dz + \int_x^l \frac{1}{EI}\left[\frac{wz}{2}(l-z)\right]\left[+\frac{(l-z)}{l}\right]dz$$

Integrating and reducing, we have finally,

$$\alpha = \frac{dy}{dx} = \frac{w}{24EI}(l^3 - 6lx^2 + 4x^3) \tag{9}$$

This expression is the same as given by eq. (3) p. 583.

To determine the deflection at the center of a beam uniformly loaded, place the 1-lb. load as shown in Fig. 8. Substituting in eq. (6) values of M and m as shown on Fig. 8, we have

$$y = \int_0^{\frac{l}{2}} \frac{1}{EI}\left[\frac{wx}{2}(l-x)\right]\frac{x}{2}dx + \int_l^l \frac{1}{EI}\left[\frac{wx}{2}(l-x)\right]\left[\frac{l-x}{2}\right]dx$$

Integrating

$$y = \frac{5}{384}\frac{wl^4}{EI} \tag{10}$$

Since the m diagram is symmetrical about the beam center, as shown in Fig. 8, and since M for the entire beam is given by the same equation, the substitution in eq. (6) might have been written

$$y = 2\int_0^{\frac{l}{2}} \frac{1}{EI}\left[\frac{wx}{2}(l - x)\right]\frac{x}{2}\,dx$$

On integrating this expression the result will be the same as given above.

Illustrative Problem.—A simple beam 16 ft. long supports a uniform load of 600 lb. per ft. Determine the maximum deflection of this beam in inches. Assume that the moment of inertia of the beam is 100 in.^{4}, and that the material is steel for which $E = 30{,}000{,}000$ lb. per sq. in.

From conditions of symmetry, it is evident that the maximum deflection occurs at the center of the beam. The loading conditions are as shown in Fig. 8. Substituting given values in the formulas for M and m, using foot units, we have

$$M = 300x(16 - x)\text{ ft.-lb., and } m = \frac{x}{2}\text{ ft.-lb.}$$

The equation for m holds for the left side of the beam. Noting from Fig. 8 that the M and m diagrams are symmetrical about the center of the beam, we have

$$y = \frac{2}{EI}\int_C^A M\,m\,dx = \frac{(2)(1{,}728)}{(30{,}000{,}000)(100)}\int_0^8 [300x(16 - x)]\frac{x}{2}\,dx$$

The multiplier 1,728 in the right-hand member of the above equation is made necessary by the fact that the moment equations are expressed in foot-pound units while the deflection is desired in inch units. Now, as stated in Art. 3, p. 591, M is expressed in linear-force units, m is expressed in linear units, and dx is also in linear units. The quantity $\int M\,m\,dx$ therefore has the dimensions of force times linear units to the third power. Hence to reduce $\int M\,m\,dx$ to inch units when the moments are expressed in feet units, we must multiply by $(12)^3 = 1{,}728$.

Performing the operations indicated by the above equation, we have

$$y = 0.295\text{ in.}$$

Beam with Uniform Load (Moment of Inertia Not Constant).—Assume that the beam is a built-up girder, such as a plate girder with cover plates, as shown in Fig. 9. Let the moment of inertia of the end-quarters be I_1 and the moment of inertia of the middle half be I_2. Required the deflection of the center point. The M and m diagrams are the same as given in Fig. 8.

Since the moment of inertia is not constant for the entire girder, substitution in eq. (6) must be made for the sections for which the moment of inertia differs as well as for the sections for which the law of variation of M and m change.

Substitution in eq. (6) gives

$$y = \frac{2}{E}\left[\frac{1}{I_1}\int_0^{\frac{l}{4}}\left[\frac{wx}{2}(l - x)\right]\frac{x}{2}\,dx + \frac{1}{I_2}\int_{\frac{l}{4}}^{\frac{l}{2}}\left[\frac{wx}{2}(l - x)\right]\frac{x}{2}\,dx\right]$$

Integrating and reducing, we have finally

$$y = \frac{wl^4}{E}\left[\frac{5}{384I_2} + 13\frac{(I_2 - I_1)}{6{,}144I_1I_2}\right] \tag{11}$$

When the moment of inertia is constant, or when $I_1 = I_2$, this expression reduces to $y = \frac{5}{384}\frac{wl^4}{EI}$, as on p. 584.

The above solution for deflection of a girder with varying moment of inertia is very convenient when there are a limited number of changes of moment of inertia and when the loading is comparatively simple. However, since a substitution must be made in eq. (6) for each change in moment of inertia and for each change in the law of variation of M and m, the determination of the deflection of a long plate girder due to a set of concentrated loads becomes a long and tedious process. For such cases the Area Moment Method or the Elastic Weight Method are more convenient.

In some cases, the moment of inertia varies from section to section. When the moment of inertia can be expressed as a function of x, it may be placed in the general equation and

the integration performed. Such integrations are generally very complicated. The above case represents the problem in the form usually encountered in practice.

Illustrative Problem.—Assume the following data for the beam of Fig. 9 and calculate the deflection in inches at the center of the beam. $w = 1{,}200$ lb. per ft., $l = 10$ ft., $I_1 = 36$ in⁴., $I_2 = 48$ in⁴., and $E = 30{,}000{,}000$ lb. per sq. in.

For the conditions shown, $M = 600\,x\,(10 - x)$ ft.-lb., and $m = \frac{x}{2}$ ft.-lb. Noting that values of M and m are symmetrical about the beam center, and the deflection in inches is required, we have from eq. (6)

$$y = \frac{(2)(1{,}728)}{30{,}000{,}000}\left[\int_0^{2.5} \frac{600x(10 - x)}{36}\,\frac{x}{2}\,dx + \int_{2.5}^{5} \frac{600x(10 - x)}{48}\,\frac{x}{2}\,dx\right]$$

from which

$$y = 0.197 \text{ in.}$$

Direct substitution in eq. (11) gives the same result.

Simple Beam with Single Concentrated Load

Illustrative Problem.—A 2- × 1-in. piece of wood laid flatwise spans a 24-in. opening. The beam carries a 60-lb. load at a distance of 18 in. from the left end of the beam. Determine the deflection under the load and the maximum deflection of the beam in inches. Assume $E = 1{,}500{,}000$ lb. per sq. in.

Figure 10*a* shows the given beam and Fig. 10*b* shows the M diagram. The moments are given in inch-pound units.

To determine the deflection under the load apply a 1-lb. load downward at point C of Fig. 10*a*. The m diagram is shown in Fig. 10*c*.

From eq. (6)

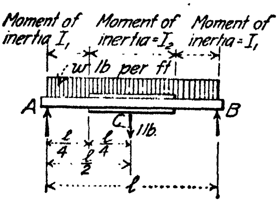


Fig. 9.

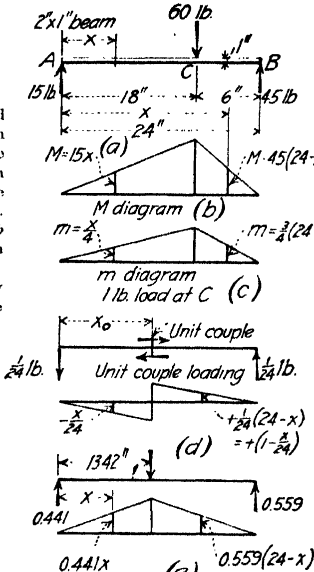


Fig. 10.

$$y_c = \frac{1}{EI}\left[\int_0^{18} (15x)\left(\frac{x}{4}\right) dx + \int_{18}^{24} [45(24 - x)]\left[\frac{3}{4}\,(24 - x)\right] dx\right]$$

For the given conditions, $I = \frac{1}{12}bd^3 = (\frac{1}{12})(2)(1)^3 = \frac{1}{6}$ in⁴. and hence $EI = 250{,}000$.

Performing the integrations indicated above, we have

$$y_c = 0.0389 \text{ in.}$$

The maximum deflection occurs at the point where the tangent to the elastic curve is horizontal. Assume this point to be located at a distance x_0 from the left end of the beam. The value of x_0 may be determined by placing a unit couple at this point, as shown in Fig. 10*d*, substituting values of M and m in eq. (7) and solving for x_0 subject to the condition that α = slope of tangent to elastic curve = 0.

Figure 10*d* shows the values of m in inch-pound units. Substituting in eq. (7) we have

$$\alpha = \frac{1}{EI}\left[\int_0^{x_0} (15x)\left(-\frac{x}{24}\right) dx + \int_{x_0}^{18} (15x)\left(1 - \frac{x}{24}\right)dx + \int_{18}^{24} 45(24 - x)\left(1 - \frac{x}{24}\right) dx\right] = 0$$

Integrating and solving for x_0, we have

$$x_0 = 13.42 \text{ in.}$$

That is, the maximum deflection occurs at a point 13.42 in. from the left end of the beam. Figure 10*e* shows the unit load in position at the point of maximum deflection and the resulting m diagram.

Substituting in eq. (6), we have

$$y_{max} = \frac{1}{EI}\left[\int_0^{13.42}(15x)(0.441x)dx + \int_{13.42}^{18}(15x)[0.559(24-x)]dx\right] + \int_{18}^{24}45(24-x)[0.559(24-x)]dx$$

from which

$$y_{max} = 0.0484 \text{ in.}$$

The above results check those given on p. 71.

Cantilever Beams.—Figure 11 shows a cantilever beam supporting a uniform load. Let it be required to determine the general formula for deflection of the free end. Assuming the deflection of the free end to be downward, the unit load acts as shown in Fig. 11. The M and m diagrams for the applied and unit load are as shown on Fig. 11. Substituting in eq. (6), we have

$$y_{max} = \frac{1}{EI}\int_0^l\left(-\frac{wx^2}{2}\right)(-x)dx = \frac{1}{EI}\int_0^l \frac{wx^3}{2}\,dx$$

Integrating,

$$y_{max} = \frac{wl^4}{8EI}$$

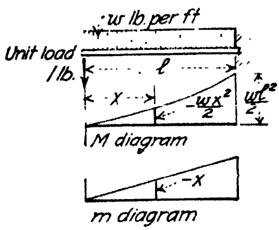


FIG. 11.

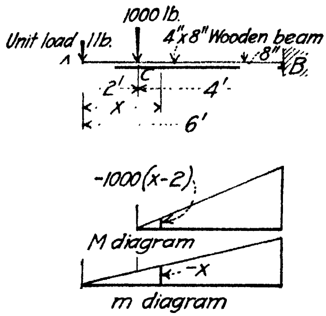


FIG. 12.

Illustrative Problem.—A 6-ft. cantilever beam supports a 1,000-lb. load placed as shown in Fig. 12. The beam consists of a 4- × 8-in. member with the 8-in. side vertical. Determine the deflection of the free end in inches. Assume E for timber as 1,500,000 lb. per sq. in.

Assume the deflection to be downward. For an origin at the free end, the M and m diagrams are as shown in Fig. 12. The moments are expressed in foot-pounds. Substituting in eq. (6), we have

$$y = \frac{1}{EI}\int_2^6[-1{,}000(x-2)(-x)]\,dx = \frac{1}{EI}\int_2^6 1{,}000(x^2-2x)dx = \frac{1{,}000(37\frac{1}{3})}{EI}$$

For a 4- × 8-in. rectangular section

$$I = \frac{bh^3}{12} = \frac{(4)(8)^3}{12} = 170\tfrac{2}{3}$$

Substituting values of E and I in the above equation, remembering that the deflection in inches is desired, we have

$$y = \frac{(1{,}000)(37\frac{1}{3})(1{,}728)}{(1{,}500{,}000)(170\frac{2}{3})} = 0.252 \text{ in.}$$

Beam with Overhanging End.

Illustrative Problem.—A 10-in. 25.4-lb. steel I-beam is used to form a beam of the type shown in Fig. 13. For the dimensions and loadings shown on Fig. 13 calculate the deflection at the free end of the cantilever arm and the maximum deflection in the span BC. The moment of inertia of the given beam section is 122.1 in.^{4} and E = 30,000,000 lb. per sq. in.

The moment diagram for the given loading is shown in Fig. 13a. The values shown on the diagram are expressed in foot-pound units.

To determine the deflection of the free end of the cantilever arm, apply a 1-lb. load acting downward

at point A, of Fig. 13b. The resulting m diagram, expressed in foot-pound units, is as shown in Fig. 13b. Substituting in eq. (6), we have

$$\frac{1{,}728}{(30{,}000{,}000)(122.1)}\left[\int_0^5[-6{,}000(5-x)][-(5-x)]dx + \int_0^{15}[1{,}000(11x-0.6x^2-30)][-\tfrac{1}{3}(15-x)]dx\right]$$

from which finally

$$y = +0.0738 \text{ in. downward deflection}$$

The maximum deflection in the span BC occurs at the point where the slope of the tangent to the elastic curve is horizontal. Assume this point to be located at a distance $15k$ ft. from the left support, where k is a fraction which is less than unity. Apply at this point a unit couple, as shown in Fig. 13c. The m diagram for this couple is shown on Fig. 13c.

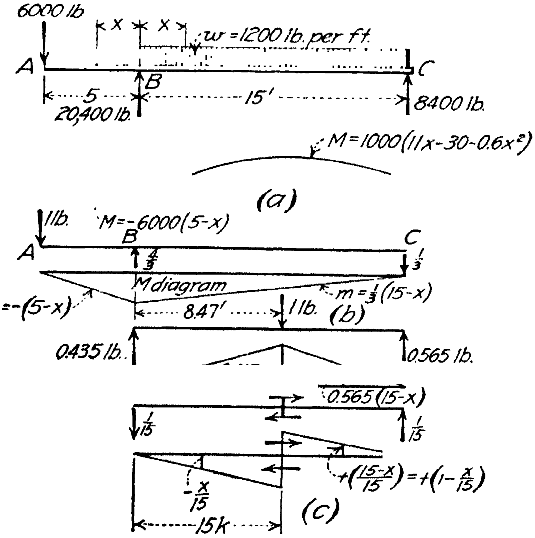


FIG. 13.

Substituting in eq. (7) we have

$$\alpha = \frac{1}{EI}\int_0^{15k}[1{,}000(11x-0.6x^2-30)]\left[-\frac{x}{15}\right]dx + \int_{15k}^{15}\left[1{,}000(11x-0.6x^2-30)\left(1-\frac{x}{15}\right)\right]dx$$

Integrating this expression, placing the result equal to zero, and solving for k, we have the cubic equation

$$k^3 - 1.833k^2 + 0.666k + 0.0278 = 0$$

Solving this equation by the method given in Art. **70**b, p. 537, we find $k = 0.565$. Hence the distance from the left support to the point of maximum deflection is $(15)(0.565) = 8.47$ ft. To determine the maximum deflection, apply a 1-lb. load as shown in Fig. 13d and determine the corresponding m values. Substituting in eq. (6), we have, noting that the deflection in inches is desired,

$$y_{max} = \frac{1{,}728}{(30{,}000{,}000)(122.1)}\left[\int_0^{8.47}[1{,}000(11x-0.6x^2-30)](0.435x)dx + \int_{8.47}^{15}[1{,}000(11x-0.6x^2-30)][0.565(15-x)]dx\right]$$

from which

$$y_{max} = 0.180 \text{ in.}$$

These values check the results given on p. 589

APPENDIX D

WORKING UNIT STRESSES FOR STRUCTURAL TIMBER

Adopted by the American Railway Engineering Association

The working unit stresses given in the table are intended for railroad bridges and trestles. For highway bridges and trestles, the unit stresses may be increased 25 per cent. For buildings and similar structures, in which the timber is protected from the weather and practically free from impact, the unit stresses may be decreased 50 per cent. To compute the deflection of a beam under long continued loading instead of that when the load is first applied, only 50 per cent of the corresponding modulus of elasticity given in the table is to be employed.

UNIT STRESSES IN POUNDS PER SQUARE INCH

Kind of timber	Bending			Shearing				Compression					
	Extreme fiber stress		Modulus of elasticity	Parallel to the grain		Longitudinal shear in beams		Perpendicular to the grain		Parallel to the grain		Working stresses for columns	
	Average ultimate	Working stress	Average	Average ultimate	Working stress	Average ultimate	Working stress	Elastic limit	Working stress	Average ultimate	Working stress	Length under $15 \times d$	Length over $15 \times d$ $\left(1 - \frac{l}{60d}\right)$
Douglas fir	6,100	1,200	1,510,000	690	170	270	110	630	310	3,600	1,200	900	1,200
Longleaf pine	6,500	1,300	1,610,000	720	180	300	120	520	260	3,800	1,300	975	1,300
Shortleaf pine	5,600	1,100	1,480,000	710	170	330	130	340	170	3,400	1,100	825	1,100
White pine	4,400	900	1,130,000	400	100	180	70	290	150	3,000	1,000	750	1,000
Spruce	4,800	1,000	1,310,000	600	150	170	70	370	180	3,200	1,100	825	1,100
Norway pine	4,200	800	1,190,000	590*	130	250	100	...	150	2,600*	800	600	800
Tamarack	4,600	900	1,220,000	670	170	260	100	...	220	3,200*	1,000	755	1,000
Western hemlock	5,800	1,100	1,480,000	630	160	270*	100	440	220	3,500	1,200	900	1,200
Redwood	5,000	900	800,000	300	80	...	...	400	150	3,300	900	675	900
Bald cypress	4,800	900	1,150,000	500	120	...	...	340	170	3,900	1,100	825	1,100
Red cedar	4,200	800	800,000	...	...	...	...	470	230	2,800	900	675	900
White oak	5,700	1,100	1,150,000	840	210	270	110	920	450	3,500	1,300	975	1,300

Unit stresses are for green timber and are to be used without increasing the live load stresses for impact. Values noted,* are for partially air dry timbers.
In the formulas given for columns, l = length of column, and d = least side or diameter, in inches.

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